

BOOLEAN RING WITH UNDERLYING POWER RHOTRIX SET

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ABSTRACT

This paper introduces 'the notion of power rhotrix set ring' and its method of construction. The ring is a form of Boolean ring which uses operations of symmetric difference and intersection for addition and multiplication respectively. The basic properties of the power rhotrix set ring are discussed and a number of its subrings are uncovered. It is hope that this idea would serve as a foundation for extending many other concepts in ring theory to power rhotrix set ring theory.

Keywords: Ring, rhotrix set ring, power rhotrix set ring, Boolean ring.

INTRODUCTION

A set is a well-defined collection of objects and these objects are called elements. A rhotrix set is a set whose elements are called rhotrices (rhotrix for singular). A rhotrix ring is a ring whose underlying set is called rhotrix set. A power rhotrix set ring is a ring whose underlying set is a power rhotrix set.

In the literature of rhotrix theory, the idea of rhotrix ring was introduced by Mohammed (2007) as a result of classification of rhotrices into algebraic structures. In the work, the set of all rhotrices of size three together with the operations of rhotrix addition and heart based (or commutative) method for rhotrix multiplication were presented. The notion of ideals for commutative real rhotrix ring of size three was equally presented by Mohammed (2009). Following this, Tudunkaya and Makanjuola (2010) constructed finite field of all integer hearty rhotrices of size three over finite field Z_p of integers modulo prime p . In 2011a, Usaini and Tudunkaya generalized this field to all integer hearty rhotrices of size n over finite field Z_p .

Mohammed (2009) defined quotient rhotrix as a rhotrix of the form $R/S = S^{-1}R$, provided the heart of the rhotrix S is non-zero. Consequent upon the definition of quotient rhotrix, Usaini and Tudunkaya (2011b) adopted it to present finite field of fractions of all integer hearty rhotrix of size 3 over finite field Z_p . The idea in (Usaini and Tudunkaya, 2011b) was extended by Usaini and Tudunkaya (2012) to the construction of finite field of fractions of all integer hearty rhotrices of size n over Z_p .

Thereafter, Mohammed and Shettima (2017a) discussed the commutative rhotrix ring of size n over the ring of all real numbers. Following this, Mohammed and Shettima (2017b) identified a number of subrings of the ring of all real rhotrices of size 3, which were confirmed valid for rhotrices of size n through the extension in size. Recently, Mohammed and Shettima (2018) considered ideals and their respective quotient rings in commutative full rhotrix ring and denoted it as the triple $(R_n(R), +, \circ)$, where the set $R_n(R)$ contains all rhotrices of size n , with entries from arbitrary ring of numbers, together with

Now, considering all the works aforesaid in the literature of rhotrix ring theory, the underlying rhotrix sets contain elements called rhotrices. In this work, the notion of a rhotrix ring having underlying set as power rhotrix set will be constructed. The basic properties of this power rhotrix set ring will be presented and some interesting results will be uncovered. Towards achieving this goal, first a

THE GENERAL RHOTRIX SET

$$R_n(\mathbf{R}) = \left\{ \left\langle \begin{array}{cccccccc} & & & & r_1 & & & \\ & & & & r_2 & r_3 & r_4 & \\ & & \dots & \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & r_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots & \dots & r_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} \\ & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & \dots & \dots & \dots & \dots & \dots & \\ & & & r_{t-3} & r_{t-2} & r_{t-1} & & \\ & & & & r_t & & & \end{array} \right\rangle : r_i \in \mathbf{R} \right\}, \quad (1)$$

THE POWER RHOTRIX SET

$$P(R_n(\mathbb{R})) = \{S_n(\mathbb{R}): S_n(\mathbb{R}) \subseteq R_n(\mathbb{R})\} \quad (2)$$

OPERATIONS ON POWER RHOTRIX SET

$$A_n(\mathbb{R}) + B_n(\mathbb{R}) := A_n(\mathbb{R}) \triangle B_n(\mathbb{R}) = (A_n(\mathbb{R}) \cup B_n(\mathbb{R})) \setminus (A_n(\mathbb{R}) \cap B_n(\mathbb{R})). \quad (3)$$

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$$A_n(\mathbb{R}) B_n(\mathbb{R}) = A_n(\mathbb{R}) \cap B_n(\mathbb{R}). \quad (4)$$

THE POWER RHOTRIX SET RING

This section presents the main result as a theorem below. The result proves that given a rhotrix set as defined by (1). The power of this rhotrix set as defined by (2) forms a power rhotrix set ring under operations of addition as defined by (3) and multiplication as defined by (4).

It may be worthy of note to state a number of basic properties about this new Boolean ring as follows:

- (i) the empty rhotrix set ϕ is the zero element in this power rhotrix set ring.
- (ii) the universal rhotrix set $R_n(\mathbb{R})$ is the unity and the only unit in the ring of power rhotrix set.
- (iii) All elements in the power rhotrix set ring are idempotent.
- (iv) The only nilpotent element in the power rhotrix set ring is the empty set ϕ .

Theorem (Full ring of power rhotrix ring)

Let $\mathbb{R}$ be the set of all real numbers. Let $R_n(\mathbb{R})$ be the rhotrix set over $\mathbb{R}$. Let $P(R_n(\mathbb{R}))$ be the power of $R_n(\mathbb{R})$. Let the operations of addition and multiplication on $P(R_n(\mathbb{R}))$ be define respectively by (*the symmetric difference as addition*)

$$A_n(\mathbb{R}) + B_n(\mathbb{R}) := A_n(\mathbb{R}) \triangle B_n(\mathbb{R}) = (A_n(\mathbb{R}) \cup B_n(\mathbb{R})) \setminus (A_n(\mathbb{R}) \cap B_n(\mathbb{R}))$$

and (*the intersection as product*)

$$A_n(\mathbb{R}) B_n(\mathbb{R}) = A_n(\mathbb{R}) \cap B_n(\mathbb{R}) \text{ for all } A_n(\mathbb{R}), B_n(\mathbb{R}) \in P(R_n(\mathbb{R})).$$

Then $P(R_n(\mathbb{R}))$ is a ring, named as full power rhotrix set ring.

Proof

It will be shown that the power rhotrix set together with operations of addition and multiplication as defined satisfies ring axioms as below.

- (i) *Addition is commutative*

The operation of addition is commutative on $P(R_n(\mathbb{R}))$ since union and intersection are commutative operations. So $\forall A_n(\mathbb{R}), B_n(\mathbb{R}) \in P(R_n(\mathbb{R}))$, it follows that

$$\begin{aligned} A_n(\mathbb{R}) + B_n(\mathbb{R}) &:= A_n(\mathbb{R}) \triangle B_n(\mathbb{R}) = (A_n(\mathbb{R}) \cup B_n(\mathbb{R})) \setminus (A_n(\mathbb{R}) \cap B_n(\mathbb{R})) \\ &= (B_n(\mathbb{R}) \cup A_n(\mathbb{R})) \setminus (B_n(\mathbb{R}) \cap A_n(\mathbb{R})) \\ &= B_n(\mathbb{R}) \triangle A_n(\mathbb{R}) \\ &:= B_n(\mathbb{R}) + A_n(\mathbb{R}) \end{aligned}$$

- (ii) *Addition is associative*

The operation of addition is associative on $P(R_n(\mathbb{R}))$ since the operations of union and intersection of any three sets satisfies associative property. So $\forall A_n(\mathbb{R}), B_n(\mathbb{R}), C_n(\mathbb{R}) \in P(R_n(\mathbb{R}))$,

it follows that

$$\begin{aligned}(A_n(\mathbb{R}) + B_n(\mathbb{R})) + C_n(\mathbb{R}) &:= (A_n(\mathbb{R}) \triangle B_n(\mathbb{R})) \triangle C_n(\mathbb{R}) \\&= [(A_n(\mathbb{R}) \cup B_n(\mathbb{R})) \setminus (A_n(\mathbb{R}) \cap B_n(\mathbb{R}))] \triangle C_n(\mathbb{R}) \\&= \{[(A_n(\mathbb{R}) \cup B_n(\mathbb{R})) \setminus (A_n(\mathbb{R}) \cap B_n(\mathbb{R}))] \cup C_n(\mathbb{R})\} \\&\quad \setminus \{[(A_n(\mathbb{R}) \cup B_n(\mathbb{R})) \setminus (A_n(\mathbb{R}) \cap B_n(\mathbb{R}))] \cap C_n(\mathbb{R})\} \\&= \{A_n(\mathbb{R}) \cup [(B_n(\mathbb{R}) \cup C_n(\mathbb{R})) \setminus (B_n(\mathbb{R}) \cap C_n(\mathbb{R}))]\} \\&\quad \setminus \{A_n(\mathbb{R}) \cap [(B_n(\mathbb{R}) \cup C_n(\mathbb{R})) \setminus (B_n(\mathbb{R}) \cap C_n(\mathbb{R}))]\} \\&= A_n(\mathbb{R}) + (B_n(\mathbb{R}) + C_n(\mathbb{R}))\end{aligned}$$

(iii) *Existence of zero element*

There exists the empty set ϕ in $P(R_n(\mathbb{R}))$ which serves as zero element under operation of addition such that $\forall A_n(\mathbb{R}) \in P(R_n(\mathbb{R}))$, it will be obtain that

$$\begin{aligned}A_n(\mathbb{R}) + \phi &:= A_n(\mathbb{R}) \triangle \phi = (A_n(\mathbb{R}) \cup \phi) \setminus (A_n(\mathbb{R}) \cap \phi) \\&= A_n(\mathbb{R}) \setminus \phi = A_n(\mathbb{R})\end{aligned}$$

Also,

$$\begin{aligned}\phi + A_n(\mathbb{R}) &:= \phi \triangle A_n(\mathbb{R}) = (\phi \cup A_n(\mathbb{R})) \setminus (\phi \cap A_n(\mathbb{R})) \\&= A_n(\mathbb{R}) \setminus \phi = A_n(\mathbb{R})\end{aligned}$$

(iv) *Existence of negatives*

For each $A_n(\mathbb{R}) \in P(R_n(\mathbb{R}))$, there exists $-A_n(\mathbb{R})$ in $P(R_n(\mathbb{R}))$ as its negative such that

$$\begin{aligned}A_n(\mathbb{R}) + (-A_n(\mathbb{R})) &:= A_n(\mathbb{R}) - A_n(\mathbb{R}) \\&= (A_n(\mathbb{R}) \cap (A_n(\mathbb{R}))^c) \\&= \phi\end{aligned}$$

Notice that the axioms (i) to (iv) above turn the pair $(P(R_n(\mathbb{R})), +)$ into Abelian group.

(v) *Multiplication is associative*

The multiplication of any two sets on $P(R_n(\mathbb{R}))$ is defined as the intersection of the two sets. Moreover, the operation of intersection sets satisfies associative property. So $\forall A_n(\mathbb{R}), B_n(\mathbb{R}), C_n(\mathbb{R}) \in P(R_n(\mathbb{R}))$,

it follows that

$$\begin{aligned} ((A_n(\mathbb{R})B_n(\mathbb{R}))C_n(\mathbb{R})) &= (A_n(\mathbb{R}) \cap B_n(\mathbb{R})) \cap C_n(\mathbb{R}) \\ &= A_n(\mathbb{R}) \cap (B_n(\mathbb{R}) \cap C_n(\mathbb{R})) \\ &= A_n(\mathbb{R}) (B_n(\mathbb{R})C_n(\mathbb{R})) \end{aligned}$$

Also, notice that axiom (v) above turns the pair $(P(R_n(\mathbb{R})), \cdot)$ into a Semigroup.

(vi) *Multiplication is both left and right distributive over addition*

The operation of multiplication of sets on $P(R_n(\mathbb{R}))$ is defined by the intersection of the sets, while the operation of addition on $P(R_n(\mathbb{R}))$ is defined by the symmetric difference of the sets. Furthermore, it is well known in classical set theory that the operation of intersection is both left and right distributive over the operation of union. So it follows that for all $A_n(\mathbb{R}), B_n(\mathbb{R}), C_n(\mathbb{R}) \in P(R_n(\mathbb{R}))$, it follows through computation that

$$\begin{aligned} A_n(\mathbb{R}) (B_n(\mathbb{R}) + C_n(\mathbb{R})) &:= A_n(\mathbb{R}) \cap (B_n(\mathbb{R}) \triangle C_n(\mathbb{R})) \\ &= (A_n(\mathbb{R}) \cap B_n(\mathbb{R})) \triangle (A_n(\mathbb{R}) \cap C_n(\mathbb{R})) \\ &:= (A_n(\mathbb{R})B_n(\mathbb{R})) + (A_n(\mathbb{R})C_n(\mathbb{R})) \end{aligned}$$

Also, right distribution shows that for all $A_n(\mathbb{R}), B_n(\mathbb{R}), C_n(\mathbb{R}) \in P(R_n(\mathbb{R}))$, it follows from computation that

$$\begin{aligned} (B_n(\mathbb{R}) + C_n(\mathbb{R}))A_n(\mathbb{R}) &:= (B_n(\mathbb{R}) \triangle C_n(\mathbb{R})) \cap A_n(\mathbb{R}) \\ &= (B_n(\mathbb{R}) \cap A_n(\mathbb{R})) \triangle (C_n(\mathbb{R}) \cap A_n(\mathbb{R})) \\ &:= (A_n(\mathbb{R})B_n(\mathbb{R})) + (A_n(\mathbb{R})C_n(\mathbb{R})) \end{aligned}$$

Thus, axioms (i) to (vi) make the triple $(P(R_n(\mathbb{R})), +, \cdot)$ a commutative ring.

(vii) *Existence of unity*

For all $A_n(\mathbb{R}) \in P(R_n(\mathbb{R}))$, there exists a unity which is the universal set $R_n(\mathbb{R})$ on $P(R_n(\mathbb{R}))$ such that

$$\begin{aligned} A_n(\mathbb{R})R_n(\mathbb{R}) &= A_n(\mathbb{R}) \cap R_n(\mathbb{R}) = R_n(\mathbb{R}) \cap A_n(\mathbb{R}) \\ &= R_n(\mathbb{R})A_n(\mathbb{R}) \\ &= A_n(\mathbb{R}) \end{aligned}$$

Hence axioms (i) to (vii) show that $(P(R_n(\mathbb{R})), +, \cdot)$ is a commutative ring with unity.

Example

Let $X_n(\mathbb{R}) = \{U_n, D_n\}$, where U_n is an upper triangular rhotrix and D_n is a down triangular rhotrix. Let $P(X_n(\mathbb{R})) = \{\phi, \{U_n\}, \{D_n\}, X_n(\mathbb{R})\}$ be the power of $X_n(\mathbb{R})$. It follows $P(X_n(\mathbb{R}))$ is a ring with respect to operations define by (3) and (4).

Before the corollary below, it is vital to record the definitions of upper triangular and lower triangular rhotrix sets given in (Mohammed and Shettima, 2017) as below.

Definition (Subring of power rhotrix set ring)

A non-empty subset or class $P(S_n(\mathbb{R}))$ of a power rhotrix set ring $P(R_n(\mathbb{R}))$ is called its subring when $P(S_n(\mathbb{R}))$ is a ring under the operations defined on $P(R_n(\mathbb{R}))$.

Definition (Upper Triangular Rhotrix Set)

The rhotrix set of the form:

$$U_n(\mathbb{R}) = \left\{ \begin{array}{ccccccc} & & & & u_1 & & \\ & & & & u_2 & u_3 & u_4 \\ & & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & u_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \dots \\ & & 0 & 0 & 0 & \dots & \dots & \dots & 0 \\ & & & \dots & \dots & \dots & \dots & \\ & & & 0_{t-3} & 0_{t-2} & 0_{t-1} & \\ & & & & 0_t & & \end{array} \right\} : u_i \in \mathbb{R}, \quad (5)$$

where $1 \leq i \leq t$, $t = \frac{1}{2}(n^2 + 1)$, $n \in 2\mathbb{Z}^+ + 1$, $n \setminus 2$ is the integer value on division of n by 2 is called upper triangular rhotrix set.

Definition (Down Triangular Rhotrix Set)

The rhotrix set of the form:

$$D_n(\mathbb{R}) = \left\{ \begin{array}{ccccccc} & & & & 0_1 & & \\ & & & & 0_2 & 0_3 & 0_4 \\ & & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & d_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \\ & & & d_{t-3} & d_{t-2} & d_{t-1} & \\ & & & & d_t & & \end{array} \right\} : d_i \in \mathbb{R}, \quad (6)$$

where $1 \leq i \leq t$, $t = \frac{1}{2}(n^2 + 1)$, $n \in 2\mathbb{Z}^+ + 1$, $n \setminus 2$ is the integer value on division of n by 2 is called down triangular rhotrix set.

Corollary 1 (Power upper triangular rhotrix set ring)

Let $\mathbb{R}$ be the set of all real numbers. Let $U_n(\mathbb{R})$ be the set of all upper triangular rhotrices with entries from $\mathbb{R}$. Let $P(R_n(\mathbb{R}))$ be the power of $U_n(\mathbb{R})$. Let the operations of addition and multiplication on $P(U_n(\mathbb{R}))$ be defined respectively by the symmetric difference as addition:

$$A_n(\mathbb{R}) + B_n(\mathbb{R}) := A_n(\mathbb{R}) \triangle B_n(\mathbb{R}) = (A_n(\mathbb{R}) \cup B_n(\mathbb{R})) \setminus (A_n(\mathbb{R}) \cap B_n(\mathbb{R}))$$

and the intersection as product:

$$A_n(\mathbb{R}) B_n(\mathbb{R}) = A_n(\mathbb{R}) \cap B_n(\mathbb{R})$$

for all $A_n(\mathbb{R}), B_n(\mathbb{R}) \in P(U_n(\mathbb{R}))$. Then $P(U_n(\mathbb{R}))$ is power upper triangular rhotrix set ring. In particular, $P(U_n(\mathbb{R}))$ is a subring of $P(R_n(\mathbb{R}))$.

Proof

Substituting $R_n(\mathbb{R}) = U_n(\mathbb{R})$ and $P(R_n(\mathbb{R})) = P(U_n(\mathbb{R}))$ in the theorem above. The proof follows.

Corollary 2 (Ring of power down triangular rhotrix set)

Let $\mathbb{R}$ be the set of all real numbers. Let $D_n(\mathbb{R})$ be the set of all down triangular rhotrices with entries from $\mathbb{R}$. Let $P(R_n(\mathbb{R}))$ be the power of $D_n(\mathbb{R})$. Let the operations of addition and multiplication on $P(D_n(\mathbb{R}))$ be defined respectively by the symmetric difference as addition:

$$A_n(\mathbb{R}) + B_n(\mathbb{R}) := A_n(\mathbb{R}) \triangle B_n(\mathbb{R}) = (A_n(\mathbb{R}) \cup B_n(\mathbb{R})) \setminus (A_n(\mathbb{R}) \cap B_n(\mathbb{R}))$$

and the intersection as product:

$$A_n(\mathbb{R}) B_n(\mathbb{R}) = A_n(\mathbb{R}) \cap B_n(\mathbb{R})$$

for all $A_n(\mathbb{R}), B_n(\mathbb{R}) \in P(U_n(\mathbb{R}))$. Then $P(D_n(\mathbb{R}))$ is a power down triangular rhotrix set ring. In particular, $P(D_n(\mathbb{R}))$ is a subring of $P(R_n(\mathbb{R}))$.

Proof

Substituting $R_n(\mathbb{R}) = D_n(\mathbb{R})$ and $P(R_n(\mathbb{R})) = P(D_n(\mathbb{R}))$ in the theorem above. The proof follows.

CONCLUSION

The notion of ring having power rhotrix set as an underlying set had been presented. This form of Boolean ring used set operations known as symmetric difference as its addition operation and that of intersection as its multiplication operation. The basic properties of the power rhotrix set ring as well as a number of its subrings had been discussed. Based on our knowledge of literature of rhotrix ring theory, this is the first time a rhotrix ring had 'rhotrix class' as an underlying set. So it is hoped that this work would serve as a foundation for future studies of Boolean ring with power rhotrix set as an underlying set.

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