



Bayesian Linear Restricted Least Squares Estimators

Isiaka Oloyede¹

Accepted: 26 November 2022

© The Indian Society for Probability and Statistics (ISPS) 2022

Abstract

Bayesian Statistics had been outshined classical Statistics over the years due to its probabilistic approach to modelling and data analytics. In recent times, much of model complexity and data ambiguity emerged of which the existing methods have their drawbacks. Model complexity and data uncertainty most times exhibit multicollinearity which render the inferences of least squares invalid. This then call for improvement in the existing methods so as to capture the trend in model complexity and data uncertainty. The Monte Carlo simulations was adopted to compare the restricted least squares family of estimators (RLSFE) with the presence of multicollinearity and two sets of linear restrictions were imposed on the model. The risk values and relative efficiency of the RLSFE under quadratic loss function were estimated to compare the estimators. Bayesian restricted least squares estimator was derived, the posterior mean and Bayes estimate were deprioritised with prior mean. The study found out that posterior mean and Bayes estimates outperformed classical restricted least squares estimators due to their lower risk values. The relative efficiency pointed to the fact that posterior mean and Bayes estimates are most preferred and efficient. The study further observed monotonic increase in the risk values as the ρ increases which proved the negative impact of multicollinearity of the inferences of regression parameters.

Keywords Restricted · Bayesian · Stein · Simulation · Prior · Posterior

1 Introduction

Stein (1956) proposed Stein estimator, and its generalized form was proposed in James and Stein (1960). Chaturvedi and Shalabh (1990) adopted Stein estimator in modelling and estimating parameters of regression with autocorrelated errors term. Chaturvedi and Shalabh (2014) examined Bayesian approach to regression under extended balance loss function (EBLF). Ozbay et al. (2017) extended the work of

✉ Isiaka Oloyede
oloyede.i@unilorin.edu.ng

¹ University of Ilorin, Ilorin, Nigeria

Chaturvedi et al. (2001) modelling under multicollinearity, they proposed feasible generalized restricted ridge regression estimator (FGSRRRE) to examine multicollinearity and autocorrelation. Monte Carlo simulation was performed in an attempt to compare their estimator with existing estimators using predictive mean squares errors (PMSE). They claimed that their estimator has smaller pmse. Chaturvedi et al. (2006) claimed that Stein-rule estimator led to improvement in the risk properties of the ordinary least squares (OLS) in their simulation study, they opined that the restricted Stein-rule is preferred to feasible restricted generalized least squares estimator.

Chaturvedi and Shukala (1990) adopted Stein-rule estimator to obtain regression parameters with non-spherical disturbances and claimed its superiority over feasible generalized least squares under a quadratic loss function. They added that stein-rule estimator performed better than traditional unbiased least squares estimator having lower risk under quadratic loss. Charles (2015) claimed that X is of full rank thereby ignoring a situation whereby $X'X$ is singular, so far estimation is possible due to restriction. David and Michael (1981) claimed that if restricted least squares (RLS) is truly imposed on regression equation which may frequently due to the problem of multicollinearity, thus restricted least squares (RLS) estimator of (β_r) is unbiased and efficient than Ordinary least squares estimator of β . They added that RLS estimator may be preferred to OLS if mean square error (mse) is used to make comparison. Toro-Vizcarrondo and Wallace (1968) under generalized mean squares error (GMSE), the null hypothesis was developed which showed that RLS outperformed OLS. Giles and Rayner (1979) used GMSE to compare OLS with Bayesian estimator. David and Michael (1981) reported that it is not feasible to compare two RLS estimators using GMSE criterion, they added that neither of the estimators will depict dominance of the other for β and σ^2 . Due to the insensitivity of GMSE to compare two RLS as reported in the literature, this study therefore adopted pitman closeness criterion under quadratic loss function. Jibo (2014) reported that mr-k estimator is superior to OLS under pitman closeness criterion.

Rao et al. (1986), Keating and Mason (1988) adopted generalized pitman nearness (GPN) criterion and proved that stein rule estimator outperformed maximum likelihood estimator. Sen et al. (1989) showed that stein rule estimator outperformed maximum likelihood estimator (MLE) under quadratic loss function of GPN criterion (Mason et al. 1990). Tran and Chaturvedi (1997) claimed that two stage hierarchical information (2SHI) estimator outperformed stein rule and OLS estimators when quadratic loss function pitman nearness criterion is adopted.

Bayesian restricted least estimation has not been thoroughly examined in the literature, the recent literature is the work of Lewis et al. (2021) which studied Bayesian restricted likelihood methods within the realm of outliers datasets and model misspecification. They provided alternative approach to solve the problem identified from the drawbacks of previous studies. They concluded that their proposed estimator proffered better predictive solution compared with competing methods. This study examined and proposed Bayesian restricted least in the realm of non-spherical disturbances (multicollinearity) both in the dataset and models and compared it with classical approaches. The sections of the study are arranged as follow: Sect. 1 contains introduction while Sect. 2 examined methodical design that thoughtfully

examining the statistical framework of the study. Data analysis and interpretation were in Sect. 3 whereas Sect. 4 dealt with conclusion of the study.

1.1 Multicollinearity Effects on Inferences

Multicollinearity has significant effect on the inferences of regression estimates, there may be interchange of positive and negative sign of the parameters affected leading to misleading judgement or decision. Standard error most times become large and thereby leading to failure of rejecting null hypothesis on parameter significant. Some parameters are significant wrongly thereby render p-value and confident interval invalid. More importantly, the idea of holding some variables constant in order to interpret the increase in one unit of a particular regressor effect on dependent variable y can no longer hold due to relationship amongst the variables (Oloyede 2021).

2 Methodical Design of Restricted Family of Estimators

Definition Let $y = X\beta + u$ be linear regression model where y is the $n \times 1$ vector of dependent variable, X is $n \times p$ matrix of independent variables, β' is a set of unknown parameters with u as the error term while R is the restricted matrix of the β coefficients. The restricted least squares estimator is expressed as in Chaturvedi and Shukla (1990)

$$L\hat{\beta}_R = \beta - (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}(R\beta - r) \quad (1)$$

where β is the estimate of ordinary least squares.

If $R\beta - r = 0$ then $\hat{\beta}_R = \hat{\beta}$.

Considering (Chaturvedi and Shukla 1990), the stein rule of β is

$$\hat{\beta}_s = \left[1 - \frac{a}{n} \frac{(y - X\beta)'(y - X\beta)}{(\beta'X'X\beta)} \right] \beta \quad (2)$$

The Restricted version of stein-rule can be derived as

$$\hat{\beta}_{RS} = \left[1 - \frac{a}{n} \frac{(y - X\beta_R)'(y - X\beta_R)}{(\beta_R'X'X\beta_R)} \right] \beta_R \quad (3)$$

2.1 Bayesian Restricted Least Squares Estimator

Theorem The restricted posterior density of restricted β and σ is obtained through marginalising of the conjugate of normal-inverse gamma and restricted likelihood since both are of the same family of distribution.

The linear model is express as:

$$y = X\beta_R + u \quad (4)$$

The likelihood function of β, X and y , where $\theta = (\beta, \lambda)$ given sample vectors $X_1, X_2 = (1, 2, \dots, n)'$ and $y = (y_1, y_2, \dots, y_n)'$ and incorporating restricted β is expressed as

$$L(\beta_R, \sigma^2 | X, y) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp \left[-\frac{1}{2\sigma^2} (y - X\beta_R)' (y - X\beta_R) \right] \quad (5)$$

Simplify further.

$(2\pi)^{-\frac{n}{2}}$ is omitted since it is a constant which ordinary contribute no significant effect.

$$L(\beta_R, \sigma^2 | X, y) \propto \sigma^{-n} \exp \left[-\frac{1}{2\sigma^2} (y'y - 2\beta_R' X'y + \beta_R' X' X \beta_R) \right] \quad (6)$$

$$L(\beta_R, \sigma^2 | X, y) \propto \sigma^{-n} \exp$$

$$\left[-\frac{1}{2\sigma^2} \left(y'y - 2\beta_R' X'y + \beta_R' X' X \beta_R - 2 \left((X'X)^{-1} X'y \right)' X'y + 2 \left((X'X)^{-1} X'y \right)' X'X \left((X'X)^{-1} X'y \right) \right) \right] \quad (7)$$

$$= \sigma^{-n} \exp \left[-\frac{1}{2\sigma^2} (y - Xb_R)' (y - Xb_R) + b_R' X' X b_R + \beta_R' X' \beta_R - 2\beta' X' X b \right] \quad (8)$$

$$= \sigma^{-n} \exp \left[-\frac{1}{2\sigma^2} \left(\hat{\sigma}(n-k) + (\beta_R - b_R)' X' X (\beta_R - b_R) \right) \right] \quad (9)$$

Setting the priors

$$p(\beta_R | \sigma^2) = (2\pi)^{-\frac{k}{2}} |\Sigma|^{-\frac{1}{2}} \exp \left[-\frac{1}{2} (\beta_R - \mathbb{B})' \Sigma^{-1} (\beta_R - \mathbb{B}) \right] \quad (10)$$

$$p(\sigma^2) \propto \sigma^{-(a-k)} \exp \left[-\frac{b}{\sigma^2} \right] \quad (11)$$

Note, normal-inverse gamma priors are conjugate priors and were selected because the prior and posterior densities are of the same family of distributions.

$$p(\beta_R | \sigma^2) p(\sigma^2) = (2\pi)^{-\frac{k}{2}} |\Sigma|^{-\frac{1}{2}} \exp \left[-\frac{1}{2} (\beta_R - \mathbb{B})' \Sigma^{-1} (\beta_R - \mathbb{B}) \right] \times \sigma^{-(a-k)} \exp \left[-\frac{b}{\sigma^2} \right] \quad (12)$$

$$p(\beta_R | \sigma^2) p(\sigma^2) = \sigma^{-(a-k)} \exp \left[-\frac{1}{2\sigma^2} (\beta_R - \mathbb{B})' \Sigma^{-1} (\beta_R - \mathbb{B}) + 2b \right] \quad (13)$$

$n \times k$ size of X matrix; β_R unknown parameters; $\mathbb{B}$ prior mean vector of beta (true value); σ^2 prior variance for beta; $\hat{\sigma}^2 = \frac{(y-X\beta)(y-X\beta)'}{n-k}$; $a-k$ first hyper-parameter; b second hyper-parameter; Joint posterior density

$$\pi(\beta_R, \sigma^2 | X, y) \propto \sigma^{-n} \exp \left[-\frac{1}{2\sigma^2} \left(\hat{\sigma}(n-k) + (\beta_R - b_R)' X' X (\beta_R - b_R) \right) \right] \times \sigma^{-(a-k)} \exp \left[-\frac{1}{2\sigma^2} (\beta_R - \mathbb{B})' \Sigma^{-1} (\beta_R - \mathbb{B}) + 2b \right] \quad (14)$$

$$\propto \sigma^{-n-a+k} \exp \left[-\frac{1}{2\sigma^2} \left(\hat{\sigma}(n-k) + (\beta_R - b_R)' X' X (\beta_R - b_R) + 2b + \sigma^2 (\beta_R - \mathbb{B})' \Sigma^{-1} (\beta_R - \mathbb{B}) \right) \right] \quad (15)$$

In an attempt to obtain marginal distribution of β_R , there is need to have Jacobian transformation after σ^2 is replaced with s , $J = \left| \frac{d}{ds} \sigma \right| = \left| \frac{d}{ds} s^{-\frac{1}{2}} \right| = \frac{1}{2} s^{-\frac{3}{2}}$.

Replace σ^2 with s

$$\propto \left(s^{-\frac{1}{2}} \right)^{-n-a+k} \exp \left[-\frac{1}{2}s \left(\hat{\sigma}(n-k) + (\beta_R - b_R)' X' X (\beta_R - b_R) + 2b + s^{-1} (\beta_R - \mathbb{B})' \Sigma^{-1} (\beta_R - \mathbb{B}) \right) \right] \left(\frac{1}{2} s^{-\frac{3}{2}} \right) \quad (16)$$

Integrate w.r.t s –nuisance parameter in order to obtain β

$$= \int_0^\infty \frac{1}{2} s^{\frac{n+a-k-3}{2}} \exp \left[-\frac{1}{2}s \left(\hat{\sigma}(n-k) + (\beta_R - b_R)' X' X (\beta_R - b_R) + 2b + s^{-1} (\beta_R - \mathbb{B})' \Sigma^{-1} (\beta_R - \mathbb{B}) \right) \right] ds \quad (17)$$

Recall

$$1 = \int_0^\infty \frac{q^{p+1}}{\Gamma(p+1)} s^p e^{-qs} ds \quad (18)$$

$$\frac{\Gamma(p+1)}{q^{p+1}} = \int_0^\infty s^p e^{-qs} ds \quad (19)$$

where $p = \frac{n+a-k-3}{2}$.

$$q = \frac{1}{2} \left(\hat{\sigma}(n-k) + (\beta_R - b_R)' X' X (\beta_R - b_R) + 2b + s^{-1} (\beta_R - \mathbb{B})' \Sigma^{-1} (\beta_R - \mathbb{B}) \right)$$

with degree of freedom as $\nu = n - k$.

$\pi(\beta | p, q) \propto q^{-(p+1)} = q^{-\frac{n}{2}}$ then replace it

$$\pi(\beta | p, q) \propto$$

$$\frac{1}{2} \left(\hat{\sigma}(n-k) + (\beta_R - b_R)' X' X (\beta_R - b_R) + 2b + s^{-1} (\beta_R - \mathbb{B})' \Sigma^{-1} (\beta_R - \mathbb{B}) \right)^{-\left(\frac{n+a-k-1}{2} \right)} \quad (20)$$

To obtain marginal posterior densities we have:

$$\pi(\sigma|X, y) \propto \int_0^\infty \sigma^{-n-a+k} \exp \left[-\frac{1}{2\sigma^2} \left(\hat{\sigma}(n-k) + (\beta_R - b)' X' X (\beta_R - B) + 2b + \sigma^2 (\beta_R - \mathbb{B})' \Sigma^{-1} (\beta_R - \mathbb{B}) \right) \right] d\beta_R \quad (21)$$

$$\pi(\sigma|X, y) \propto \int_0^\infty \sigma^{-n-a+k} \exp \left[-\frac{1}{2\sigma^2} (\hat{\sigma}(n-k)) \right] \times \int_0^\infty \exp \left[-\frac{1}{2\sigma^2} \left((\beta_R - b_R)' X' X (\beta_R - b_R) + 2b + \sigma^2 (\beta_R - \mathbb{B})' \Sigma^{-1} (\beta_R - \mathbb{B}) \right) \right] d\beta_R \quad (22)$$

Through simplification we have

$$\pi(\sigma|X, y) \propto \sigma^{-n-a+k} \exp \left[-\frac{1}{2\sigma^2} (\hat{\sigma}(n-k)) \right] (2\pi\sigma^2)^{\frac{k}{2}} \quad (23)$$

$$\pi(\sigma|X, y) \propto (\sigma^2)^{-\frac{1}{2}(n+a-2k)} \exp \left[-\frac{1}{2\sigma^2} (\hat{\sigma}(n-k)) \right] \quad (24)$$

$$\text{Thus } \hat{\beta}_R \sim MVN \left(\hat{\beta}_R, \hat{\sigma}^2 (X'X)^{-1} \left[1 - (X'X)^{-1} R' [R(X'X)^{-1} R']^{-1} R \right] \right) \\ \hat{\sigma}^2 \sim G \left(a_1 - \frac{n}{2}, b_1 + \frac{1}{2} \sum_{i=1}^n (y_i - X\beta_R)^2 + 2 [R(X'X)^{-1} R']^{-1} (R\beta_R - r)' (r - R\beta_R) \right) \quad (25)$$

2.2 Monte Carlo Simulation

The Monte Carlo simulation was adopted to compare the restricted least squares family of estimators, the sample size was specified as 30 with p set at 6. ρ_i was systematically selected from 0.1 to, 0.9, this was used to generate multicollinearity dataset of which we have both moderate and severe multicollinearity data. The covariates X_s were generated using multivariate distribution. The error term U was generated based on $E(UU') \neq 0$ and the restricted parameters were specified as

$b_3 + b_4 = 1$ and $b_4 + b_6 = 0$ with $R = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$ and $q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The matrix

Fig. 1 Depicts autocorrelation plot for the choice of thinning

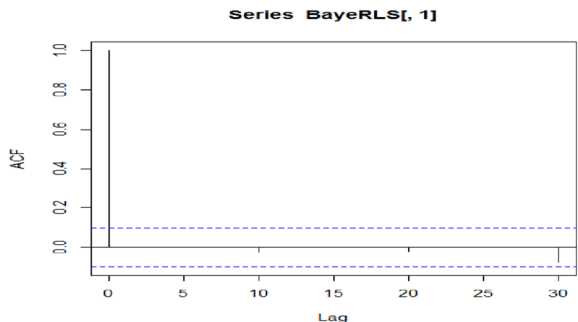


Table 1 showing variance inflation factor of the data generated

ρ	$\hat{\beta}_{01}$	$\hat{\beta}_{02}$	$\hat{\beta}_{03}$	$\hat{\beta}_{04}$	$\hat{\beta}_{05}$
0.1	1.785325	1.217913	1.930596	1.671217	1.452277
0.2	2.771761	1.665373	3.057676	2.636272	1.959127
0.3	4.647451	2.547162	5.018296	4.457057	2.947415
0.4	8.182233	4.217301	8.523845	7.878597	4.803048
0.5	15.10060	7.483754	15.14482	14.56669	8.406863
0.6	29.77650	14.40066	28.82776	28.74441	15.99509
0.7	65.68985	31.29746	61.64241	63.42416	34.44689
0.8	180.4569	85.21176	164.8183	174.2174	93.10047
0.9	868.2574	407.8967	774.7486	838.0588	443.0215

Table 2 showing risk values and relative efficiency of restricted family of Estimators

ρ	$R(\hat{\beta}_R)$	$R(\hat{\beta}_{RS})$	$R(\hat{\beta}_{RPM})$	$R(\hat{\beta}_{RBAYES})$	$E_{FF\hat{\beta}_{RS}}$	$E_{FF\hat{\beta}_{RPM}}$	$E_{FF\hat{\beta}_{RBAYES}}$
0.1	11.8332	18.8338	5.885857	5.919469	0.62829594	2.0104464	1.99903066
0.2	14.3209	18.849	6.053441	6.036585	0.75976975	2.36574537	2.37235125
0.3	15.5009	18.8568	6.491827	6.333024	0.82203237	2.38775617	2.44763007
0.4	16.1252	18.8607	6.091179	6.063398	0.85496297	2.64730358	2.65943288
0.5	16.7196	18.8623	5.942714	5.959828	0.88640304	2.813462	2.80538297
0.6	17.7634	18.8636	6.061424	6.042594	0.94167603	2.93056549	2.93969775
0.7	20.2032	18.864	6.035536	6.024472	1.07099237	3.34737462	3.3535221
0.8	27.6065	18.8646	5.959823	5.970717	1.46340235	4.63210065	4.62364905
0.9	68.859	18.8649	6.360822	6.248666	3.65011211	10.8254876	11.0197921

R has K-columns which is conformable with b with j rows of j restrictions of full row rank, thus $j \leq k$. Thereafter, the parameters β_s (true values) were specified as 0.81, 1.89, 2.97, 1.17, 1.35 and 1.62 respectively to generate variable y. The number of replications of our experiment is set at 5000 with burn-in of 1000 which specified the draws that were discarded to remove the effect of the initial values in MCMC simulation while iteration was set at 1000 for other Monte Carlo simulation. The thinning is set at 5 to ensure the removal of the effect of autocorrelation in the MCMC simulation. The analysis was carried out using (R Core Team 2021) (Fig. 1).

2.3 Performance Metrics

Unweighted quadratic loss function was adopted to compare the restricted least squares family of estimators in a finite sample.

$l(\hat{\beta}_R, \hat{\beta}_R^*) = (\hat{\beta}_R - \hat{\beta}_R^*)'(\hat{\beta}_R - \hat{\beta}_R^*)$ where $\hat{\beta}_R$, $\hat{\beta}_{RS}$, $\hat{\beta}_{RPM}$ and $\hat{\beta}_{RBAYES}$ are the restricted least squares, restricted stein least squares, restricted posterior mean and Bayes estimates respectively. Risk values for Posterior and Bayes estimates were

Fig. 2 Depicts the risk values of restricted least squares family of estimators

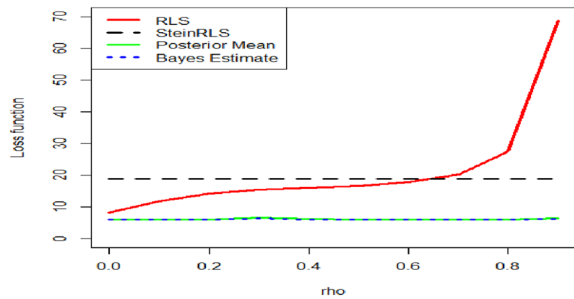
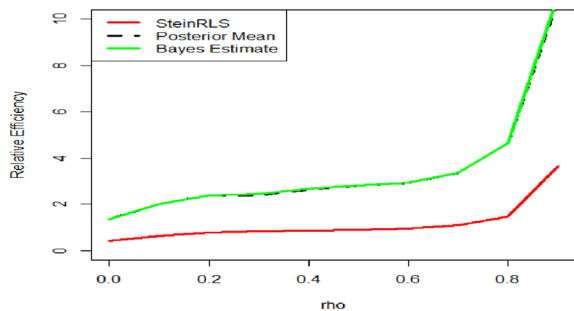


Fig. 3 Depicts the relative efficiency of restricted least squares family of estimators



deprioritised by averaging it with prior mean, this remove the dependency of Bayesian on prior and bias over classical, thus afford equivalent comparison. The performances of model evaluation and parameters learning through the minimization of loss function showcase a good prediction and how well an estimate will be, this attests to its widely choice.

3 Data Analysis and Interpretation

The outcome of the study were presented in Tables 1 and 2 specifying variance inflation factor of the generated dataset, risk values and relative efficiency of restricted least squares family of estimators.

The above table depicts the variance inflation factors that were used to detect the presence of multicollinearity in the data generated. $vif \leq 1$ implies no multicollinearity, the vif between 5 and 10 indicates high correlation which may be problematic, while the vif between 1 and 5 is moderately multicollinearity.

The study adopted (Chaturvedi et al. 2006) restricted Stein-rule estimator and considered multicollinearity being one of the non-spherical disturbances in the data generated process. The presence of multicollinearity in the data was verified with VIF, this is depicted in Table 1 above. The result obtained in Table 2 is in concordance with the work of Chaturvedi et al. (2006). The restricted posterior mean and Bayes estimates were outperformed both restricted least squares and restricted stein-rule estimator. Relative efficiency being the ratio of restricted least squares and other estimators were depicted in Table 2, thus the efficiency in excess of 1 indicates

that $\hat{\beta}_{RS}$ is more efficient and preferred to $\hat{\beta}_R$ and vice versa. Analogously, $\hat{\beta}_{RPM}$ and $\hat{\beta}_{RBAYES}$ are more efficient and preferred to $\hat{\beta}_R$. From the above table, the study observed that the effect of multicollinearity is severe as the ρ for all the estimators considered in the study (Figs. 2 and 3).

4 Contributions

The study has opened the limelight with which data and model that are with the presence of multicollinearity could be solved and have inferences that will improve the judgmental decision. The novel Bayesian version of restricted least squares when multicollinearity is present has vividly shown less risk values and high relative efficiency. Further works will be done in the other areas of non-spherical disturbances as well as the intertwine of two or more non-spherical disturbances.

5 Conclusion

The study proposed novel Bayesian restricted least squares with multicollinearity and made the comparison with classical restricted family of estimators. The study concluded that both restricted posterior mean and Bayes estimates that are probabilistic in nature are more efficient and preferred to other restricted estimators. The study found out that posterior mean and Bayes estimates outperformed classical restricted least squares estimators due to their lower risk values. The relative efficiency pointed to the fact that posterior mean and Bayes estimates are most preferred and efficient. The study therefore recommends restricted Bayesian estimator in the presence of multicollinearity. It is crystal clear and visually observed that there exist upward trend monotonically in the risk values as the ρ increases.

Acknowledgements The author expressed sincere appreciation to the editor-in-chief, associate editor and anonymous reviewers who indefatigably made necessary corrections and suggestions on the previous version of this article. Their contributions were in no small measure made this version as it is.

Funding The author single handedly developed and wrote the contents of the manuscript and there is no financial interest to report. I certify that the submission is my original work and it is not under review at any other publications.

Declaration

Conflict of interest The author has no conflicts of interest to declare.

References

Charles BM (2015) Mathematical statistics for applied econometrics. CRS Press, Taylor & Francis Group. A Chapman and Hall book, pp 253–256

- Chaturvedi A, Shalabh S (2014) Bayesian estimation of regression coefficients under extended balanced loss function. *Commun Stat Theory Methods* 43:4253–4264
- Chaturvedi A, Shukla G (1990) Stein rule estimation in linear model with non-scalar error covariance matrix. *Sankhya B* 52:293–304
- Chaturvedi A, Wan ATK, Singh SP (2001) Stein-rule restricted regression estimator in a linear regression model with nonspherical disturbances. *Commun Statist Theory Methods* 30(1):55–68
- Chaturvedi A, Alan TKW, Singh SP (2006) Stein-rule restricted regression estimator in a linear regression model with non-spherical disturbances. *Commun Stat Theory Methods* 30(1):55–68. <https://doi.org/10.1081/STA-100001558>
- David KG, Michael JP (1981) On comparing restricted least squares estimators. *J Econ* 15:397404
- Giles DEA, Rayner AC (1979) The mean squared errors of the maximum likelihood and natural-conjugate Bayes regression estimators. *J Econ* 11:319–334
- James W, Stein C (1960) Estimation with quadratic loss. In: *Proceedings of Fourth Berkeley symposium on mathematical statistics and probability*. Vol. 1, Univ. of Calif. Press, 361–379, Berkeley
- Jibo W (2014) Comparison of some estimators under the pitman's closeness criterion in linear regression model. *J Appl Math*. <https://doi.org/10.1155/2014/654949>
- Keating JP, Mason RL (1988) James stein estimation from an alternative perspective. *Am Stat* 42:160–164
- Lewis JR, MacEachern SN, Lee Y (2021) Bayesian restricted likelihood methods: conditioning on insufficient statistics in bayesian regression (with discussion). *Bayesian Anal* 16(4):1393–1462
- Mason RL, Keating JP, Sen PK, Blaylock NW (1990) Comparison of linear estimators using pitman's measure of closeness. *J Am Stat Assoc* 85:579–581
- Oloyede I (2021) Bayesian hetero-elasticnet (a gibbs sampler approach). *Tanzan J Sci* 47(3):981–987
- Ozbay N, Dawoud I, Kaciranlar S (2017) Feasible generalized stein-rule restricted ridge regression estimators. *JAMSI* 13(1):77–97
- R Core Team (2021) R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. URL <https://www.R-project.org/>
- Rao CR, Keating JP, Mason RL (1986) The Pitman nearness criterion and its determination. *Commun Stat* 15:3173–3191
- Sen PK, Kubokawa T, Saleh AK (1989) The stein paradox in the sense of pitman closeness. *Ann Stat* 17:1375–1386
- Stein C (1956) Inadmissibility of usual estimator for the mean of a multivariate normal distribution. *Proceedings of Third Berkeley symposium on mathematical statistics and probability*. Vol. 1, Univ. of Calif. Press, 197–206, Berkeley
- Toro-Vizcarrondo C, Wallace TD (1968) A test of the mean square error criterion for restrictions in linear regressions. *J Am Stat Assoc* 63:5588572
- Tran VH, Chaturvedi A (1997) Performance of the 2SHI estimator under the generalized pitman nearness criterion. *Commun Statist Theory Methods* 26(5):1227–1238. <https://doi.org/10.1080/03610929708831978>

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.