

Application of Tropical Geometry on Subsemigroup of Order-preserving Full Transformation

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Abstract

This work ties together the tropical geometry and semigroup. We applied tropical geometry on subsemigroup of order-preserving full transformation (OT_n) by degenerating the elements of classical algebra into tropical algebra in which the multiplicities were deduced through their tropical curves.

Keywords: Semigroup, Subsemigroup, Full transformation, Order-preserving, Tropical polynomial, Tropical curve, Multiplicity and Height.

Introduction

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation while a transformation of X is a function from X to itself. The transformation semigroup is one of the most fundamental mathematical objects that occur in theoretical computer sciences, where the properties of language depend on the algebraic properties of various transformation semigroup related to them.

Let $X = \{1, 2, \dots, n\}$ be a finite set ordered in standard way. A map $\alpha: \text{Dom}(\alpha) \subseteq X_n \rightarrow \text{Im}(\alpha) \subseteq X_n$ is said to be full (or total), if $\text{Im}(\alpha) = X_n$ denoted by T_n

A Subsemigroup of $(S, \cdot)$ is a nonempty subset T of S which is closed under the multiplication of S . In otherwords, T is a subsemigroup under multiplication.

The name 'tropical' was coined by some French mathematician which is denoted by $\mathbb{T}$. As in classical algebra, a tropical polynomial expression $P(x) = \sum_{i=0}^d a_i x^i$ induces a tropical polynomial function denoted by P on $\mathbb{T}$;

$$P: \mathbb{T} \rightarrow \mathbb{T} \tag{1}$$

A strange space with mysterious properties hide behind the enigmatic name of tropical geometry. This strange object is a line because of the geometric properties it satisfied.

The set of tropical numbers is defined as $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ endowed with the operations called tropical addition and multiplication:

$$"x + y" = \max\{x, y\}; \quad (2)$$

$$"x \times y" = x + y \quad (3)$$

with the usual conventions

$$\forall x \in \mathbb{T}, "x + (-\infty)" = \max(x, -\infty) = x \quad \text{and} \quad "x \times (-\infty)" = x + (-\infty) = -\infty \quad (4)$$

The tropical numbers along with these two operations form a semi-field, that is, they satisfies all the axioms of a field except the existence of an inverse for the law of addition. In tropical operations, $"2x" \neq "x + x"$ but $"2x" = x + 2$ and $"0x" = x$ but not equal to 0 . In this paper, tropical operations will be placed under quotation marks. A tropical polynomial is a convex piecewise affine function and each piece has an integer slope.

Tropical geometry has exploded as an area of research in the last decade with many new connections and applications appearing each year. These include enumerate geometry and arithmetic geometry. It builds on the older area of tropical mathematics more commonly known as max-plus algebra which arises in semigroup theory, computer science and optimization. Brugalle (2012) studied some aspects of tropical geometry, Maclagan (2012) looked into introduction to tropical algebraic geometry. Also, Shaw (2013) worked on tropical intersection product in matroidal fans. Furthermore, Brugalle and Shaw (2015) analysed obstructions to approximating tropical curves in surfaces via intersection theory. Recently, Usamot et. al (2018) studied tropicalization of idempotent elements on full transformation semigroup. For standard concepts and terms in semigroup and tropical geometry theory see [Howie (1995), Umar(2010), Brugalle et.al (2015)].

Our work ties together the tropical geometry and semigroup namely the study of the tropical polynomials and their multiplicities. The elements in OT_n were degenerated into tropical polynomial and some lemma as well as theorem were established with relevant examples to validate the results.

Basic Definitions

Definition (Brugalle et. al. 2015)

A tropical number r is a root of a Polynomial $P(x)$ in one variable in which the points x_0 on the graph $P(x)$ has a corner at x_0 for $-\infty \leq r \leq \infty$.

Definition (Brugalle et. al. 2015)

The Multiplicity of a tropical root r denoted by $M(r)$ defined as

$$M(r_n) = |m_i - m_{i+1}| \quad i = 1, 2, \dots, n$$

where m_i are the slopes of the lines in the tropical curve of $P(x)$ intersecting above r .

Proposition (Brugalle et. al. 2015)

The tropical semi-field is algebraically closed. That is, every tropical polynomial of degree $d > 0$ has exactly d roots when counted with multiplicities.

Definition (Umar 2010)

Let $S = OT_n$ and $\alpha \in OT_n$. Then, the height of $\alpha = |Im\alpha|$

Definition (Umar 2010)

Let $\alpha \in OT_n$ then, the order-preserving full transformation semigroup on X_n is

$$OT_n = \{\alpha \in OT_n : (\forall x, y \in Dom\alpha) x \leq y \Rightarrow x\alpha \leq y\alpha\}.$$

Main Results

Theorem 1: Let $S = OT_n$. Then, every elements in OT_n has the same multiplicity for each n .

Proof: Let $S = OT_n$ and $\alpha, \beta \in S$ such that $\alpha = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ b_1 & b_2 & b_3 & \dots & b_n \end{pmatrix}$ and

$$\beta = \begin{pmatrix} c_1 & c_2 & c_3 & \dots & c_n \\ d_1 & d_2 & d_3 & \dots & d_n \end{pmatrix} \text{ for } a_n = c_n \in dom(\alpha, \beta) \text{ and } b_n \neq d_n \in Im(\alpha, \beta).$$

Consider the polynomials defined from α and β i.e

$$\alpha(x) = a_i x^n + a_{i+1} x^{n-1} + a_{i+2} x^{n-2} + \dots + a_{i+k} x^{n-k} + b_i x^{n-1} + b_{i+1} x^{n-2} + \dots + b_{i+k} x^{n-k}$$

and

$$\beta(x) = c_i x^n + c_{i+1} x^{n-1} + c_{i+2} x^{n-2} + \dots + c_{i+k} x^{n-k} + d_i x^{n-1} + d_{i+1} x^{n-2} + \dots + d_{i+k} x^{n-k}$$

having the tropical polynomials;

$$\top \alpha(x) = "a_i x^n + a_{i+1} x^{n-1} + a_{i+2} x^{n-2} + \dots + a_{i+k} x^{n-k} + b_i x^{n-1} + b_{i+1} x^{n-2} + \dots + b_{i+k} x^{n-k} "$$

$$\top \beta(x) = "c_i x^n + c_{i+1} x^{n-1} + c_{i+2} x^{n-2} + \dots + c_{i+k} x^{n-k} + d_i x^{n-1} + d_{i+1} x^{n-2} + \dots + d_{i+k} x^{n-k} "$$

Which gives;

$$\begin{aligned} & \max\{a_i + nx, a_{i+1} + (n-1)x, \dots, a_{i+k} + (n-k)x, b_i + (n-1)x, b_{i+1} + (n-2)x, b_{i+k} + (n-k)x\} \\ & \max\{c_i + nx, c_{i+1} + (n-1)x, \dots, c_{i+k} + (n-k)x, d_i + (n-1)x, d_{i+1} + (n-2)x, d_{i+k} + (n-k)x\}. \end{aligned}$$

Then, from the tropical curve we observe that, α and β have different roots but the same multiplicities. i.e, $MT_{\alpha(x)} = MT_{\beta(x)}$.

Example 1: (a) Let $\alpha, \beta \in OT_{10}$, then consider the transformations $\alpha = (3)(4)(8)(9)(7\ 5](10\ 9](1\ 2\ 3](6\ 5\ 4]$ and $\beta = (1)(5)(6)(10)(3\ 1](4\ 2\ 1](9\ 8\ 7\ 6]$ with polynomial functions;

$$\alpha(x) = x^{10} + 4x^9 + 6x^8 + 7x^7 + 9x^6 + 10x^5 + 12x^4 + 13x^3 + 17x^2 + 19x + 9$$

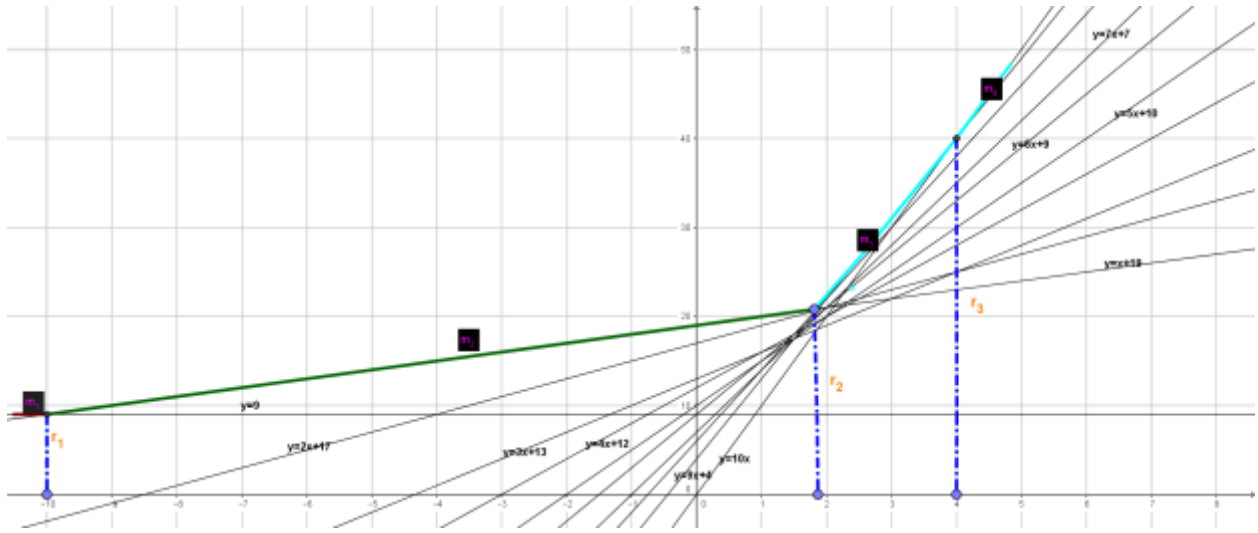
$$\beta(x) = x^{10} + 3x^9 + 4x^8 + 5x^7 + 7x^6 + 11x^5 + 13x^4 + 14x^3 + 16x^2 + 18x + 10$$

Then, by the virtue of (2) and (3) we have;

$$\top \alpha(x) = \max\{10x, 9x+4, 8x+6, 7x+7, 6x+9, 5x+10, 4x+12, 3x+13, 2x+17, x+19, 9\}$$

$$\top \beta(x) = \max\{10x, 9x+3, 8x+4, 7x+5, 6x+7, 5x+11, 4x+13, 3x+14, 2x+16, x+18, 10\}$$

Tropical curve of $\alpha(x)$



GRAPH I

From graph I, the roots are; $r_1 = -10, r_2 = 1.8$ and $r_3 = 4$ with slopes $m_1 = 0, m_2 = 1, m_3 = 9$ and $m_4 = 10$ having multiplicities

$$M(r_1) = |m_1 - m_2|$$

$$= 1$$

$$M(r_2) = |m_2 - m_3|$$

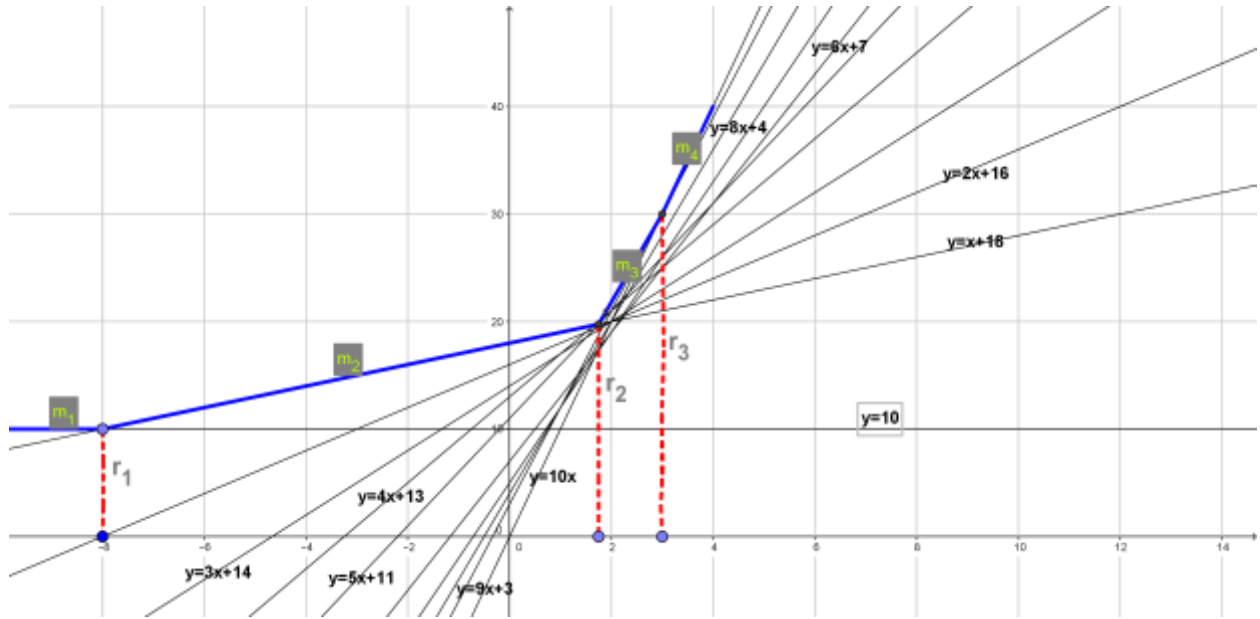
$$= 8$$

$$M(r_3) = |m_3 - m_4|$$

$$= 1$$

Hence, the multiplicities of $T_{\alpha(x)} = (1 \ 8 \ 1)$.

Tropical curve of $\beta(x)$



GRAPH II

From the tropical curve of $\beta(x)$, the roots are; $r_1 = -8, r_2 = 1.8$ and $r_3 = 3$ with slopes $m_1 = 0, m_2 = 1, m_3 = 9$ and $m_4 = 10$ having multiplicities

$$M(r_1) = |m_1 - m_2|$$

$$= 1$$

$$M(r_2) = |m_2 - m_3|$$

$$= 8$$

$$M(r_3) = |m_3 - m_4|$$

$$= 1$$

Hence, the multiplicities of $T_{\beta(x)} = (1 \ 8 \ 1)$.

(b). Consider $\gamma, \zeta \in OT_{17}$ with transformations $\gamma = (1)(4)(7)(12)(15)(17) (2 \ 1) (3 \ 1) (16 \ 15) (6 \ 5 \ 4) (10 \ 8 \ 7) (11 \ 9 \ 7) (14 \ 13 \ 12)$ and $\zeta = (2)(3)(6)(9)(12)(17) (1 \ 2) (5 \ 4 \ 3) (8 \ 7 \ 6) (11 \ 10 \ 9) (16 \ 15 \ 14 \ 13 \ 12)$

having polynomial functions:

$$\gamma(x) = x^{17} + 3x^{16} + 4x^{15} + 5x^{14} + 9x^{13} + 10x^{12} + 12x^{11} + 15x^{10} + 16x^9 + 17x^8 + 19x^7 + 21x^6$$

$$+ 25x^5 + 26x^4 + 28x^3 + 31x^2 + 32x + 17$$

and

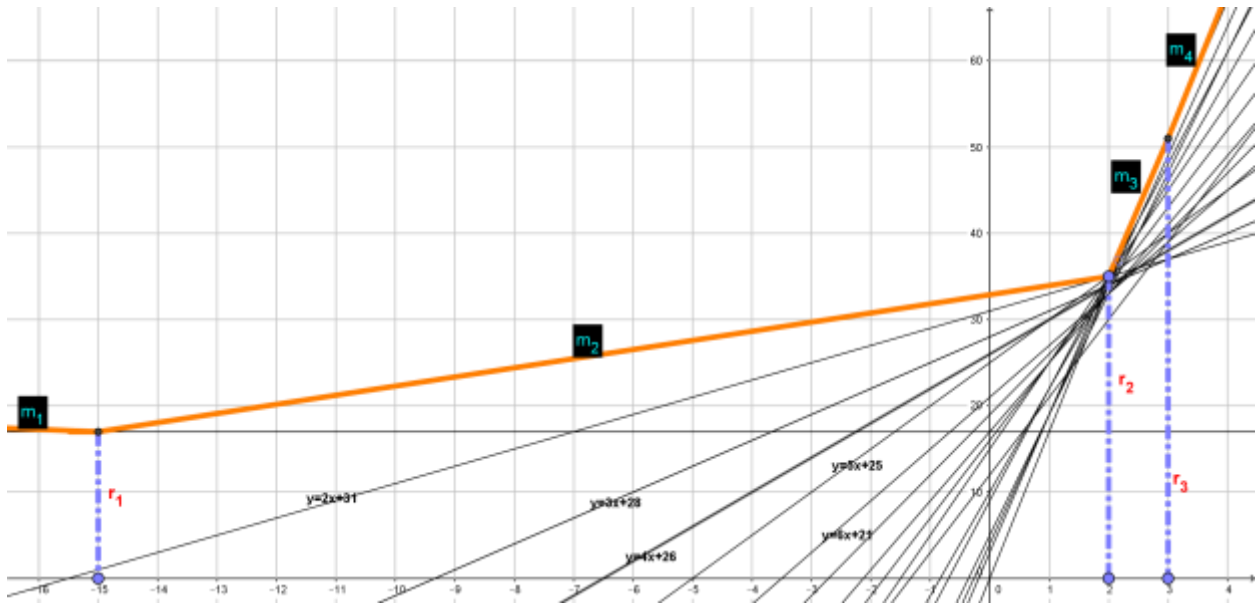
$$\zeta(x) = x^{17} + 4x^{16} + 5x^{15} + 7x^{14} + 8x^{13} + 10x^{12} + 13x^{11} + 14x^{10} + 16x^9 + 19x^8 + 20x^7 + 22x^6 + 25x^5 + 26x^4 + 28x^3 + 30x^2 + 33x + 17$$

Then, by conditions (2) and (3) we have;

$$T\gamma(x) = \max \left\{ \begin{array}{l} 17x, 16x + 3, 15x + 4, 14x + 5, 13x + 9, 12x + 10, 11x + 12, 10x + 15, 9x + 16, 8x + 17, \\ 7x + 19, 6x + 21, 5x + 25, 4x + 26, 3x + 28, 2x + 31, x + 32, 17 \end{array} \right\}$$

$$T\zeta(x) = \max \left\{ \begin{array}{l} 17x, 16x + 4, 15x + 5, 14x + 7, 13x + 8, 12x + 10, 11x + 13, 10x + 14, 9x + 16, 8x + 19, \\ 7x + 20, 6x + 22, 5x + 25, 4x + 26, 3x + 28, 2x + 30, x + 33, 17 \end{array} \right\}$$

Tropical curve of $\gamma(x)$



GRAPH III

From the graph of III, the roots are; $r_1 = -15, r_2 = 2$ and $r_3 = 3$ with slopes $m_1 = 0, m_2 = 1, m_3 = 16$ and $m_4 = 17$ having multiplicities

$$M(r_1) = |m_1 - m_2|$$

$$= 1$$

$$M(r_2) = |m_2 - m_3|$$

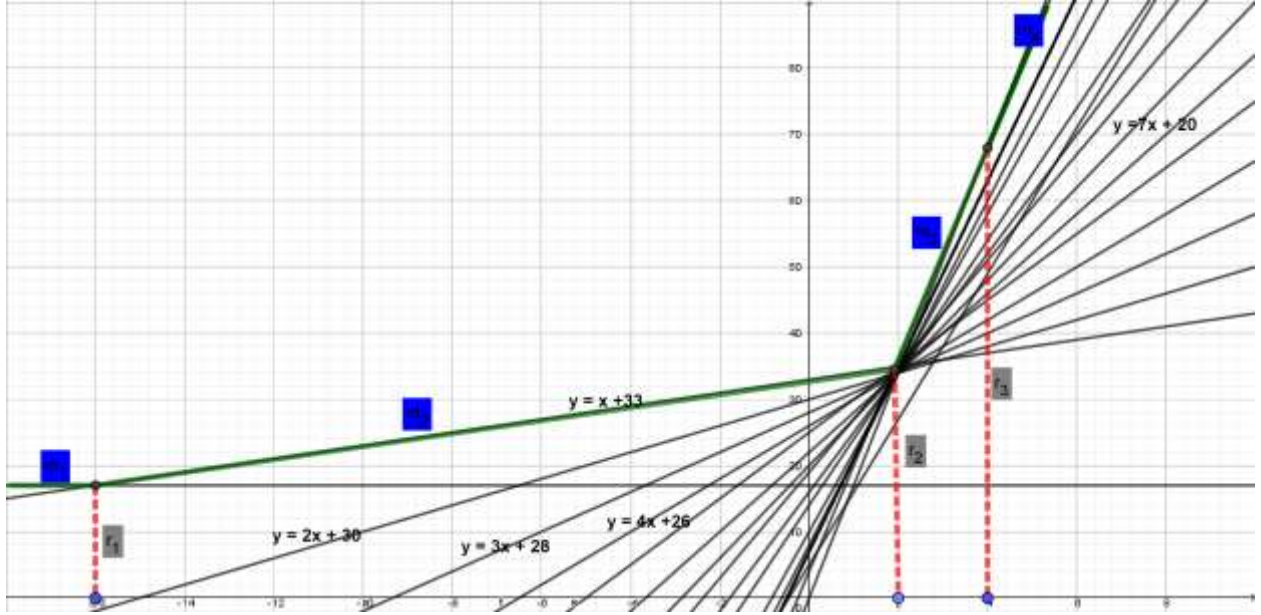
$$= 15$$

$$M(r_3) = |m_3 - m_4|$$

$$= 1$$

Hence, the multiplicities of $T_{\beta(x)} = (1 \ 15 \ 1)$.

Tropical curve of $\zeta(x)$



GRAPH IV

From the above tropical curve of $\zeta(x)$, the roots are; $r_1 = -16, r_2 = 2$ and $r_3 = 4$ with slopes $m_1 = 0, m_2 = 1, m_3 = 16$ and $m_4 = 17$ having multiplicities

$$M(r_1) = |m_1 - m_2|$$

$$= 1$$

$$M(r_2) = |m_2 - m_3|$$

$$= 15$$

$$M(r_3) = |m_3 - m_4|$$

$$= 1$$

Hence, the multiplicities of $T_{\beta(x)} = (1 \ 15 \ 1)$.

Proposition 2: Every transformation in OT_n for $1 \leq n \leq 3$ have multiplicities of height 1.

Proof : Let the elements $\sigma, \rho, \gamma \in OT_n$, where $\sigma \in (OT_1)$, $\rho \in (OT_2)$ and $\gamma \in (OT_3)$ such that $\sigma = \begin{pmatrix} e_1 \\ f_1 \end{pmatrix}$, $\rho = \begin{pmatrix} e_1 & e_2 \\ f_1 & f_2 \end{pmatrix}$ and $\gamma = \begin{pmatrix} e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \end{pmatrix}$ with polynomial defines as follows;

$$\sigma(x) = e_1x + f_1, \quad \rho(x) = e_1x^2 + e_2x + f_1x + f_2 \text{ and}$$

$$\gamma(x) = e_1x^3 + e_2x^2 + e_3x + f_1x^2 + f_2x + f_3$$

Then, their tropical polynomials were given as:

$$\mathbb{T} \sigma(x) = "e_1x+f_1" = \max\{e_1 + x, f_1\},$$

$$\mathbb{T} \rho(x) = "e_1x^2+e_2x+f_1x+f_2" = \max\{2x + e_1, x + e_2, x + f_1, f_2\}$$

$$\begin{aligned} \mathbb{T} \gamma(x) &= "e_1x^3+e_2x^2+e_3x+f_1x^2+f_2x+f_3" \\ &= \max\{3x + e_1, 2x + e_2, x + e_3, 2x + f_1, x + f_2, f_3\}. \end{aligned}$$

From the curves, we obtained their multiplicities as

$$|MT_{\sigma(x)}| = |MT_{\rho(x)}| = |MT_{\gamma(x)}| = 1$$

Hence the result.

Theorem 3: For $n \geq 4$, the height of the multiplicities in OT_n have the same height of order 2.

Proof: From Theorem 1, the result follows such that

$$|MT_{(OT_4)}| = |MT_{(OT_5)}| \dots = |MT_{(OT_n)}| = 2$$

Example2: (a) Let $\delta \in OT_4$, so that $\delta = (2)(3)(1 \ 2] (4 \ 3]$

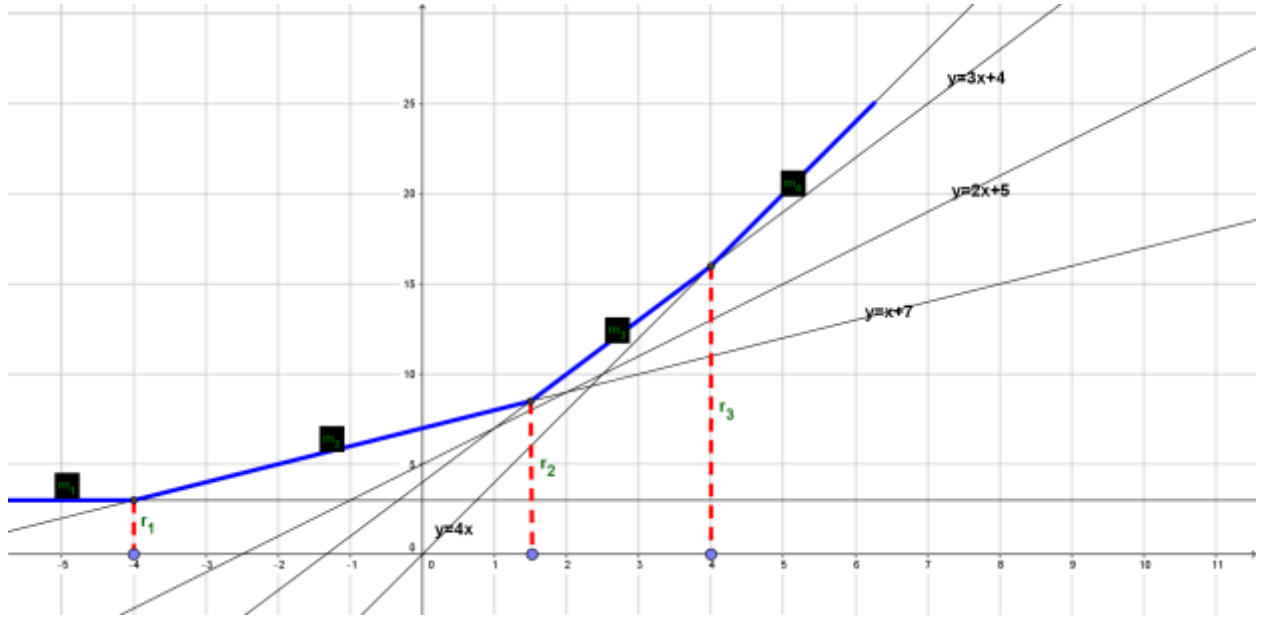
with polynomial function;

$$\delta(x) = x^4 + 4x^3 + 5x^2 + 7x + 3$$

Then, by the equation (2) and (3) we have;

$$\mathbb{T} \delta(x) = \max\{4x, 3x + 4, 2x + 5, x + 7, 3\}$$

Tropical curve of $\delta(x)$



GRAPH V

From the above curve, the roots were; $r_1 = -4, r_2 = 1.5$ and $r_3 = 4$ with slopes $m_1 = 0, m_2 = 1, m_3 = 3$ and $m_4 = 4$ having multiplicities

$$M(r_1) = |m_1 - m_2|$$

$$= 1$$

$$= 2$$

$$M(r_3) = |m_3 - m_4|$$

$$= 1$$

Thus, the multiplicities of $\mathbb{T}_{\delta(x)} = (1 \ 2 \ 1)$.

(b) Consider $\phi \in OT_{11}$ with transformation $\phi = (1)(6)(7)(10) \begin{pmatrix} 3 & 1 \\ & \end{pmatrix} \begin{pmatrix} 5 & 2 \\ & \end{pmatrix} \begin{pmatrix} 8 & 7 \\ & \end{pmatrix} \begin{pmatrix} 9 & 7 \\ & \end{pmatrix} \begin{pmatrix} 11 & 10 \\ & \end{pmatrix} \begin{pmatrix} 4 & 2 \\ & \end{pmatrix} \begin{pmatrix} 1 \\ & \end{pmatrix}$

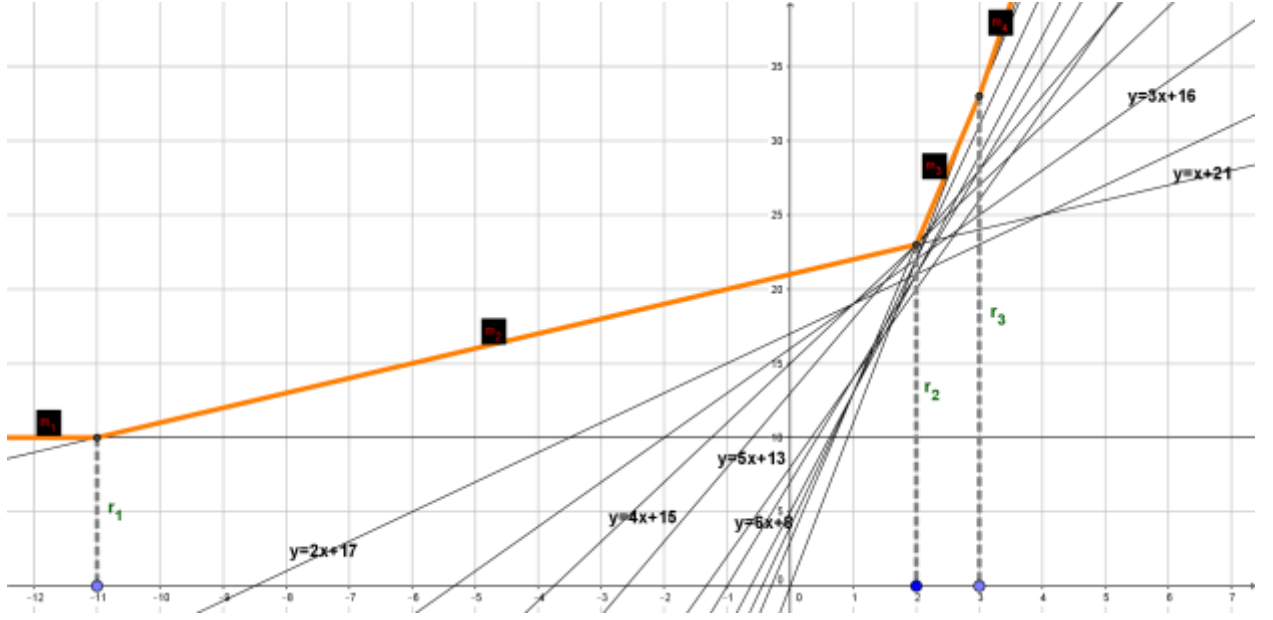
having polynomial function

$$\phi(x) = x^{11} + 3x^{10} + 4x^9 + 5x^8 + 7x^7 + 8x^6 + 13x^5 + 15x^4 + 16x^3 + 17x^2 + 21x + 10$$

Then, by the virtue of (2) and (3) we have;

$$\tau \phi(x) = \max\{11x, 10x + 3, 9x + 4, 8x + 5, 7x + 10, 6x + 7, 5x + 13, 4x + 15, 3x + 16, 2x + 17, x + 21, 10\}$$

Tropical curve of $\phi(x)$



GRAPH VI

From the graph $\phi(x)$, the roots are; $r_1 = -11, r_2 = 2$ and $r_3 = 3$ with slopes $m_1 = 0, m_2 = 1, m_3 = 10$ and $m_4 = 11$ having multiplicities

$$M(r_1) = |m_1 - m_2| = 1$$

$$M(r_2) = |m_2 - m_3| = 9$$

$$M(r_3) = |m_3 - m_4| = 1$$

Similarly, the multiplicities of $T_{\phi(x)} = (1 \ 9 \ 1)$

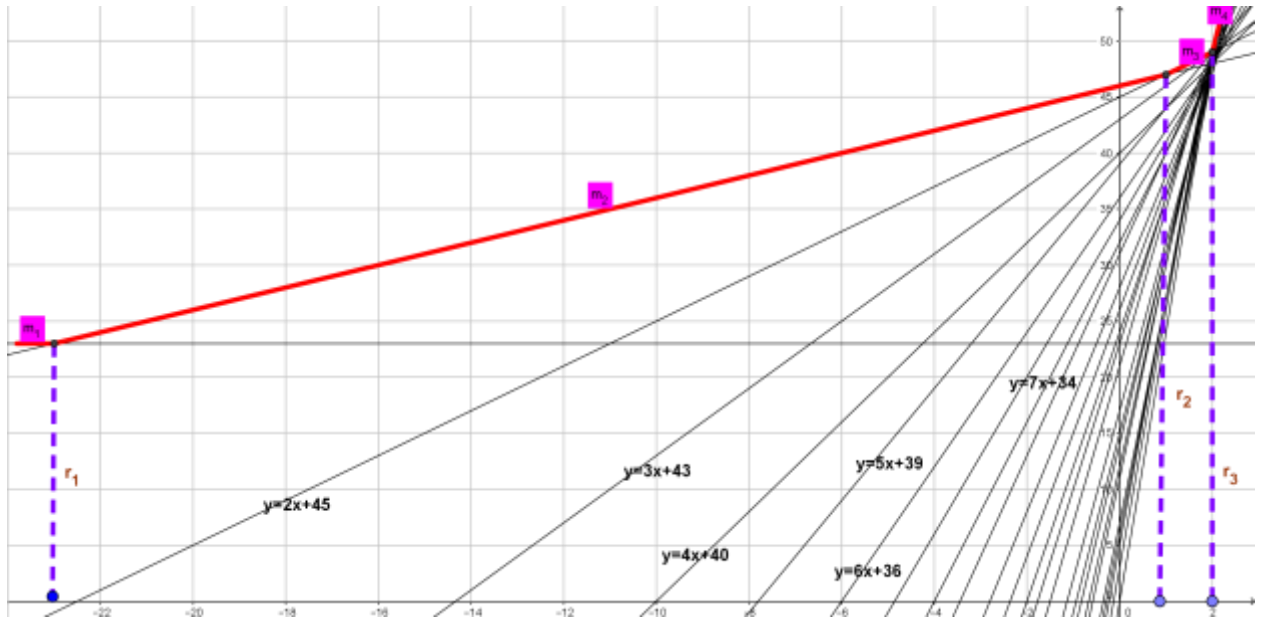
(c) Again let $\mu \in OT_{24}$ with transformation given as $\mu = (1)(2)(5)(10)(11)(15)(19)(21)(22) \begin{pmatrix} 3 & 2 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 8 & 6 \\ 20 & 19 \end{pmatrix} \begin{pmatrix} 9 & 7 & 6 & 5 \\ 18 & 17 & 16 & 15 \end{pmatrix} \begin{pmatrix} 24 & 23 & 22 \end{pmatrix}$ having polynomial function;

$$\mu(x) = x^{24} + 3x^{23} + 5x^{22} + 6x^{21} + 7x^{20} + 11x^{19} + 12x^{18} + 14x^{17} + 15x^{16} + 17x^{15} + 21x^{14} + 23x^{13} + 24x^{12} + 26x^{11} + 28x^{10} + 31x^9 + 32x^8 + 34x^7 + 36x^6 + 39x^5 + 40x^4 + 43x^3 + 45x^2 + 46x + 23$$

By equations (2) and (3) gives;

$$T\mu(x) = \max \left\{ \begin{array}{l} 24x, 23x + 3, 22x + 5, 21x + 6, 20x + 7, 19x + 11, 18x + 12, 17x + 14, 16x + 15, 15x + 17, 14x + 21, \\ 13x + 23, 12x + 24, 11x + 26, 10x + 28, 9x + 31, 8x + 32, 7x + 34, 6x + 36, 5x + 39, 4x + 40, \\ 3x + 43, 2x + 45, x + 46, 23 \end{array} \right\}$$

Tropical curve of $\mu(x)$



GRAPH VII

From the graph VII, the roots were; $r_1 = -23, r_2 = 1$ and $r_3 = 2$ having slopes $m_1 = 0, m_2 = 1, m_3 = 23$ and $m_4 = 24$ with multiplicities

$$M(r_1) = |m_1 - m_2| = 1$$

$$M(r_2) = |m_2 - m_3| = 22$$

$$M(r_3) = |m_3 - m_4| = 1$$

Thus, the multiplicities of $T_{\mu(x)} = (1 \ 22 \ 1)$

From Example2 (a),(b) and (c) it shows that

$$|MT_{(OT_4)}| = |MT_{(OT_{11})}| = |MT_{(OT_{24})}| = 2$$

Hence the result.

Theorem 4: Let $S = OT_n$, then $|MT_{(OT_n)}| = (1, n - 2, 1)$ for all $n \geq 3$.

Proof: From Theorem 1 and Proposition 2 the slopes m_i where $1 \leq i \leq 4$. We deduce that $m_1 = 0, m_2 = 1, m_3 = n - 1$ and $m_4 = n \ \forall n \geq 3$. And also By definition 2.2. we have,

$$|MT_{(OT_n)}| = \begin{cases} 1, & i = 1 \\ n - 2, & 2 \leq i \leq 3 \\ 1, & 3 \leq i \leq 4 \end{cases}$$

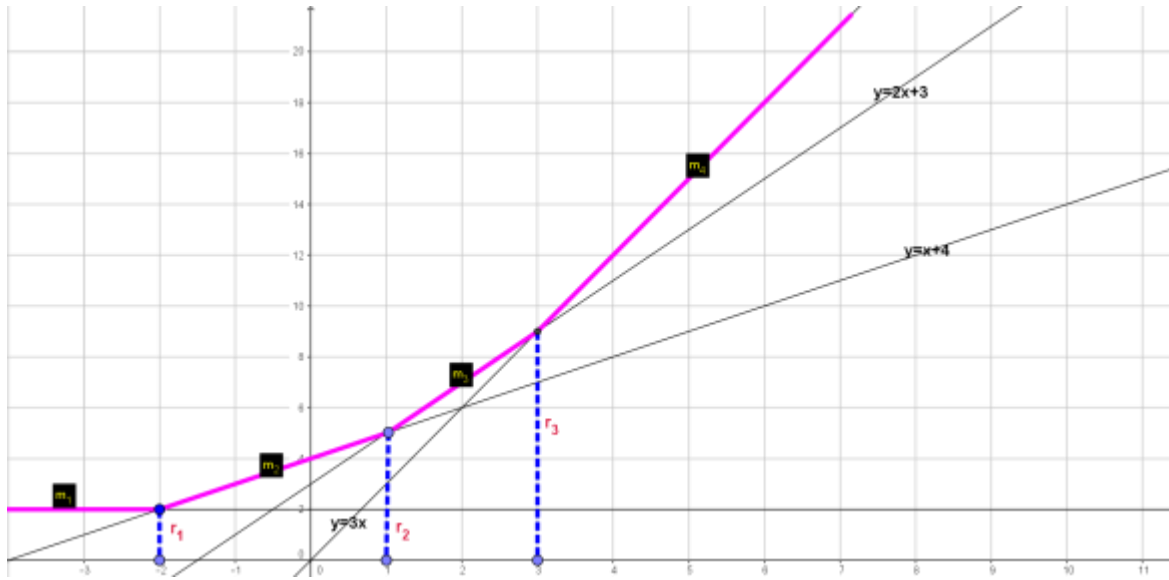
Example3: (i) Given $\rho = (1)(2 \ 1)(3 \ 2) \in OT_3$ having polynomial function

$$\rho(x) = x^3 + 3x^2 + 4x + 2$$

and tropical polynomial

$$T\rho(x) = \max\{3x, 2x + 3, x + 4, 2\}.$$

Tropical curve of $\rho(x)$



GRAPH VIII

By the curve $\rho(x)$, the roots are; $r_1 = -2, r_2 = 1$ and $r_3 = 3$ and slopes $m_1 = 0, m_2 = 1, m_3 = 2$, $m_4 = 3$ having multiplicities

$$M(r_1) = |m_1 - m_2|$$

$$= 1$$

$$M(r_2) = |m_2 - m_3|$$

$$= 1$$

$$M(r_3) = |m_3 - m_4|$$

$$= 1$$

Hence, the multiplicities of $T_{\rho(x)} = (1 \ 1 \ 1)$

(ii) Let $\theta \in OT_8$, then $\theta = (2) \ (1 \ 2 \] \ (4 \ 3 \ 2] (8 \ 7 \ 6 \ 5 \ 3]$

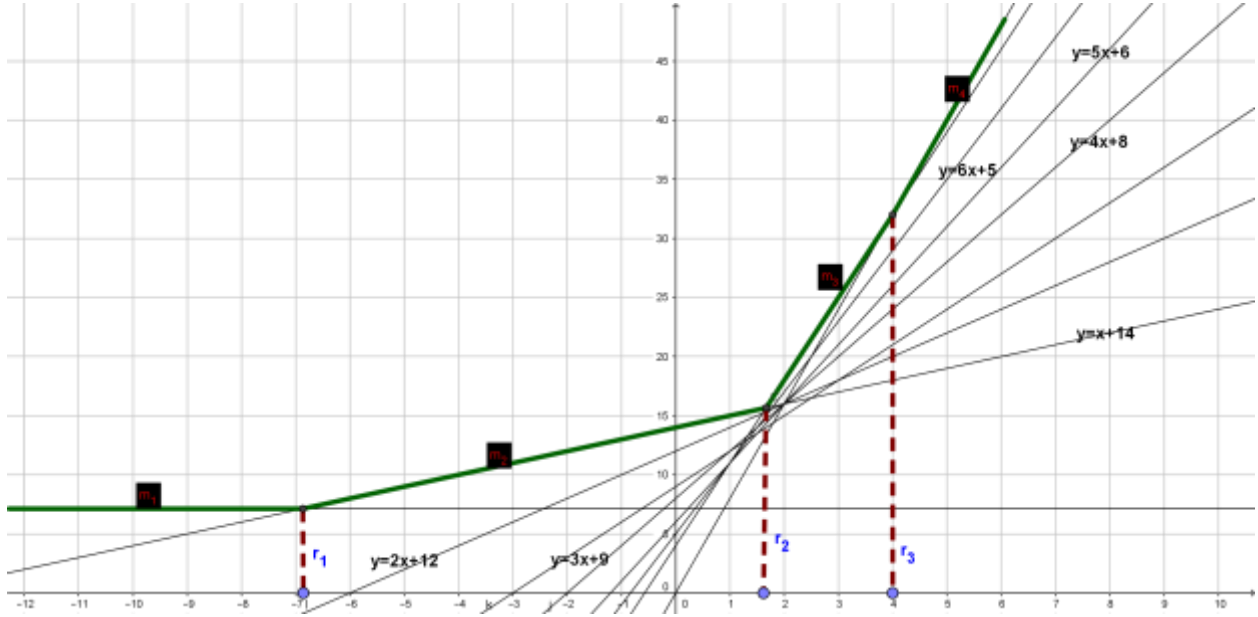
with polynomial function;

$$\theta(x) = x^8 + 4x^7 + 5x^6 + 6x^5 + 8x^4 + 9x^3 + 12x^2 + 14x + 7$$

Then, by the virtue of (2) and (3) we have;

$$\mathbb{T}\theta(x) = \max\{8x, 7x + 4, 6x + 5, 5x + 6, 4x + 8, 3x + 9, 2x + 12, x + 14, 7\}$$

Tropical curve of $\theta(x)$



GRAPH VIII

From the above tropical curve of $\theta(x)$, the roots are; $r_1 = -6.9, r_2 = 1.6$ and $r_3 = 4$ with slopes $m_1 = 0, m_2 = 1, m_3 = 7$ and $m_4 = 8$ having multiplicities

$$M(r_1) = |m_1 - m_2|$$

$$= 1$$

$$M(r_2) = |m_2 - m_3|$$

$$= 6$$

$$M(r_3) = |m_3 - m_4|$$

$$= 1$$

Hence, the multiplicities of $\mathbb{T}_{\theta(x)} = (1 \ 6 \ 1)$

(iii) Let $\varpi \in OT_{30}$, then $\varpi = (1)(6)(10)(11)(25)(30)(16)(20)(21)(27) \ (2 \ 1 \](5 \ 4 \ 3 \ 1) \ (9 \ 8 \ 7 \ 6) \ (15 \ 14 \ 13 \ 12 \ 11) \ (19 \ 18 \ 17 \ 16) \ (24 \ 23 \ 22 \ 21) \ (26 \ 25) \ (29 \ 28 \ 27)$

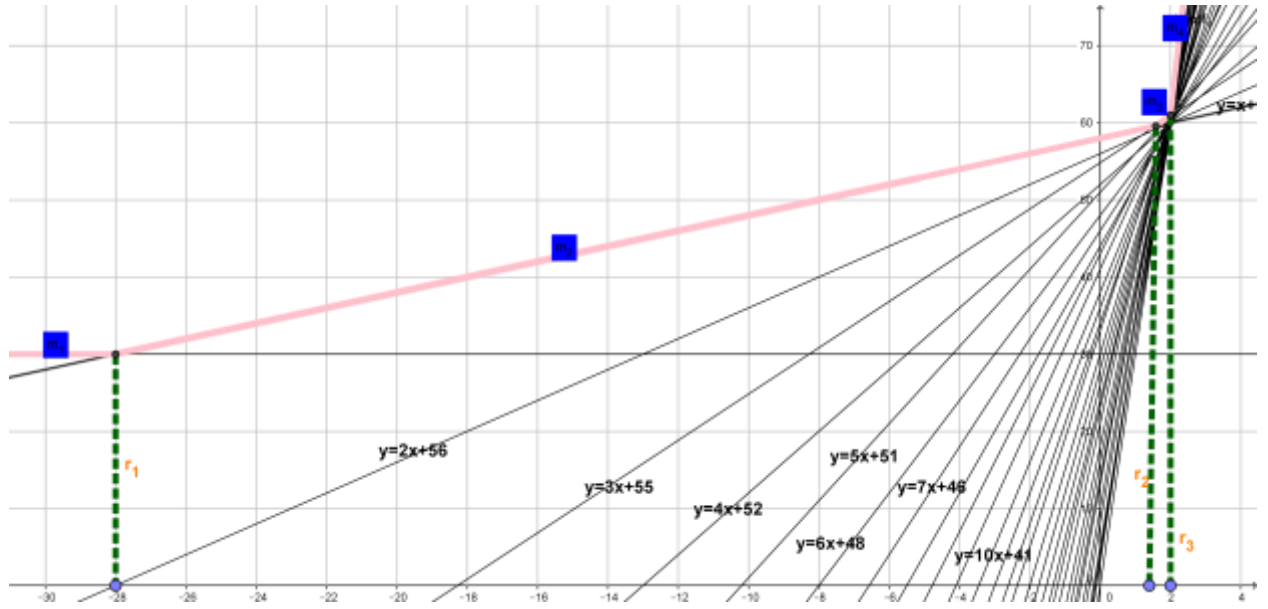
with polynomial function;

$$\begin{aligned}\varpi(x) = & x^{30} + 3x^{29} + 4x^{28} + 5x^{27} + 8x^{26} + 10x^{25} + 13x^{24} + 14x^{23} + 16x^{22} + 18x^{21} + 21x^{20} + 23x^{19} + \\ & 24x^{18} + 26x^{17} + 28x^{16} + 30x^{15} + 33x^{14} + 34x^{13} + 36x^{12} + 38x^{11} + 41x^{10} + 43x^9 + 44x^8 + \\ & 46x^7 + 48x^6 + 51x^5 + 52x^4 + 55x^3 + 56x^2 + 58x + 30\end{aligned}$$

Then, by the virtue of (2) and (3) we have;

$$\mathcal{T}\varpi(x) = \max \left\{ \begin{array}{l} 30x, 29x + 3, 28x + 4, 27x + 5, 26x + 8, 25x + 10, 24x + 13, 23x + 14, 22x + 16, 21x + 18, 20x + 21, \\ 19x + 23, 18x + 24, 17x + 26, 16x + 28, 15x + 30, 14x + 33, 13x + 34, 12x + 36, 11x + 38, 10x + 41, \\ 9x + 43, 8x + 44, 7x + 46, 6x + 48, 5x + 51, 4x + 52, 3x + 55, 2x + 56, x + 58, 30 \end{array} \right\}$$

Tropical curve of $\varpi(x)$



GRAPH IX

From the above tropical curve of $\varpi(x)$, the roots are; $r_1 = -28, r_2 = 1.6$ and $r_3 = 2$ with slopes $m_1 = 0, m_2 = 1, m_3 = 29$ and $m_4 = 30$ having multiplicities

$$\begin{aligned}M(r_1) &= |m_1 - m_2| \\ &= 1\end{aligned}$$

$$M(r_2) = |m_2 - m_3|$$

$$= 28$$

$$M(r_3) = |m_3 - m_4|$$

$$= 1$$

Hence, the multiplicities of $T_{\rho(x)} = (1 \ 28 \ 1)$

Remark: The sum of multiplicities on every transformation in OT_n is equal to the degree of its transformation.

Conclusion

In this paper, we have shown that for each n , the multiplicity of elements in subsemigroup of order-preserving OT_n are the same. Furthermore, for $1 \leq n \leq 3$ and $n \geq 4$ the height of their multiplicities are 1 and 2 respectively and finally for all $n \geq 3$, their multiplicities are of the form $(1, n - 2, 1)$.

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