

Chapter 1

ON $H_3(1)$ HANKEL DETERMINANT FOR SOME CLASSES OF UNIVALENT FUNCTIONS

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Abstract

Focus in this paper is on the Hankel determinant, $H_3(1)$, for the well-known classes of bounded-turning, starlike and convex functions in the open unit disk $E = \{z \in \mathbb{C}: |z| < 1\}$. The results obtained complete the series of research works in the search for sharp upper bounds on $H_3(1)$ for each of these classes.

Keywords: Hankel determinant, functions of bounded turning, starlike and convex univalent functions.

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1. Introduction

Let A the class of functions

$$f(z) = z + a_2 z^2 + \cdots \quad (1.1)$$

which are analytic in E . A function $f \in A$ is said to be of bounded turning, starlike and convex respectively if and only if, for any $z \in E$, $\operatorname{Re} f'(z) > 0$, $\operatorname{Re} z f'(z)/f(z) > 0$ and $\operatorname{Re} (1 + z f''(z)/f'(z)) > 0$. By usual notations we denote these classes of functions respectively by R , S^* and C . Let $n \geq 0$ and $q \geq 1$. Then the q -th Hankel determinant is

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defined as:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2(q-1)} \end{vmatrix}$$

(see [11] for example). The determinant has been investigated by several authors with the subject of inquiry ranging from rate of growth of $H_q(n)$ as $n \rightarrow \infty$ [11, 12] to the determination of precise bounds on $H_q(n)$ for specific q and n for some favored classes of functions [4, 5, 10]. In particular, sharp upper bounds on $H_2(2)$ were obtained by the authors of articles [4, 5, 10] for various classes of functions. In the present investigation, our focus is on the Hankel determinant, $H_3(1)$, for the well-known classes of bounded-turning, starlike and convex functions in E .

By the definition, $H_3(1)$ is given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

For $f \in A$, $a_1 = 1$ so that

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$$

and, by triangle inequality, we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \quad (1.2)$$

Incidentally, all of the functionals on the right side of the inequality (1.2) have known (and sharp) upper bounds in the classes of functions which are of interest in this paper, except $|a_2a_3 - a_4|$. The last one is the well-known Fekete-Szegő functional. For R , sharp bound $2/3$ was reported in [1] (with R corresponding to $n = \alpha = 1$, $\beta = 0$ in the classes $T_n^\alpha(\beta)$ studied there) while for S^* and C , sharp bounds 1 and $1/3$, respectively, were given in [6]. Janteng et-al [4, 5] obtained for the functional $|H_2(2)| \equiv |a_2a_4 - a_3^2|$ sharp bounds $4/9$, 1 and $1/8$, respectively, for R , S^* and C . Furthermore, it is known that for $k = 2, 3, \dots$, $|a_k| \leq 2/k$, $|a_k| \leq k$ and $|a_k| \leq 1$ also respectively for R , S^* and C (see [2, 9]). Thus, finding the best possible bounds on $|a_2a_3 - a_4|$ for each of the classes and using those known inequalities, then the sharp upper bounds on $H_3(1)$ follow as simple corollaries.

Our investigation follows a method of classical analysis devised by Libera and Zlotki-ewicz [7, 8]. The same has been employed by many authors in similar works (see also [4, 5, 10]). In the next section, we state the necessary lemmas while, in Section 3, we present our main results.

2. Preliminary Lemmas

Let P denote the class of functions $p(z) = 1 + c_1z + c_2z^2 + \dots$ which are regular in E and satisfy $\operatorname{Re} p(z) > 0$ for any $z \in E$. To prove the main results in the next section, we shall require the following two lemmas:

Lemma 2.1 ([2]). Let $p \in P$, then $|c_k| \leq 2$, $k = 1, 2, \dots$, and the inequality is sharp.

Lemma 2.2. ([7, 8]). Let $p \in P$, then

$$2c_2 = c_1^2 + x(4 - c_1^2) \quad (2.1)$$

and

$$4c_3 = c_1^3 + 2xc_1(4 - c_1^2) - x^2c_1(4 - c_1^2) + 2z(1 - |x|^2)(4 - c_1^2) \quad (2.2)$$

for some x, z such that $|x| \leq 1$ and $|z| \leq 1$.

3. Main Results

Theorem 3.1. Let $f \in R$. Then

$$|a_2a_3 - a_4| \leq \frac{5}{18} \sqrt{\frac{5}{3}}$$

and the inequality is sharp.

Proof. Let $f \in R$. Then there exists a $p \in P$ such that $f'(z) = p(z)$, wherefrom equating coefficients we find that $2a_2 = c_1$, $3a_3 = c_2$ and $4a_4 = c_3$. Thus we have

$$|a_2a_3 - a_4| = \left| \frac{c_1c_2}{6} - \frac{c_3}{4} \right|. \quad (3.1)$$

Substituting for c_2 and c_3 using Lemma 2.2, we obtain

$$|a_2a_3 - a_4| = \left| \frac{c_1^3}{48} - \frac{c_1(4 - c_1^2)x}{24} + \frac{c_1(4 - c_1^2)x^2}{16} - \frac{(4 - c_1^2)(1 - |x|^2)z}{8} \right|. \quad (3.2)$$

By Lemma 2.1, $c_1 \leq 2$. Then, letting $c_1 = c$, we may assume without restriction that $c \in [0, 2]$. Thus, applying the triangle inequality on (3.2) with $\rho = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{c^3}{48} + \frac{(4 - c^2)}{8} + \frac{c(4 - c^2)\rho}{24} + \frac{(c - 2)(4 - c^2)\rho^2}{16} \\ &= F(\rho). \end{aligned}$$

Now, we have

$$F'(\rho) = \frac{c(4 - c^2)}{24} + \frac{(c - 2)(4 - c^2)\rho}{8} > 0$$

implying that $F(\rho)$ is an increasing function of ρ on the closed interval $[0, 1]$. Hence $F(\rho) \leq F(1)$ for all $\rho \in [0, 1]$. That is,

$$\begin{aligned} F(\rho) &\leq \frac{c(5 - c^2)}{12} \\ &= G(c) \end{aligned}$$

where from we have $G'(c) = (5 - 3c^2)/12$ and $G''(c) = -c/2$. Since $c \in [0, 2]$, by the assumption, it follows that $G(c)$ attains maximum at $c = \sqrt{5/3}$. Thus

$$G(c) \leq \frac{5}{18} \sqrt{\frac{5}{3}}$$

which corresponds to $\rho = 1$ and $c = \sqrt{5/3}$ and it is the desired upper bound.

By setting $c_1 = c = \sqrt{5/3}$ and selecting $x = -1$ in (2.1) and (2.2), we find that $c_2 = -1/3$ and $c_3 = -4/3\sqrt{5/3}$. Using these in (3.1) we see that equality is attained which shows that our result is sharp. This completes the proof. $\square$

Let $f \in R$. Then using the above result in (1.2) together with the known inequalities $|a_3 - a_2^2| \leq 2/3$ [1], $|a_2a_4 - a_3^2| \leq 4/9$ [4] and $|a_k| \leq 2/k$, $k = 2, 3, \dots$ [9], we have the sharp inequality:

Corollary 3.2. *Let $f \in R$. Then*

$$|H_3(1)| \leq \frac{2736\sqrt{3} + 675\sqrt{5}}{4860\sqrt{3}} = 0.742267747509603.$$

Theorem 3.3. *Let $f \in S^*$. Then*

$$|a_2a_3 - a_4| \leq 2$$

and the inequality is sharp. Equality is attained by the Koebe function $k(z) = z/(1 - z)^2$.

Proof. Let $f \in S^*$. Then there exists a $p \in P$ such that $zf'(z) = f(z)p(z)$. Equating coefficients, we find that $a_2 = c_1$, $2a_3 = c_2 + c_1^2$ and $6a_4 = 2c_3 + 3c_1c_2 + c_1^3$. Thus we have

$$|a_2a_3 - a_4| = \frac{1}{3}|c_1^3 - c_3|. \quad (3.3)$$

Substituting for c_3 from Lemma 2.2, we obtain

$$|a_2a_3 - a_4| = \frac{1}{12}|3c_1^3 - 2c_1(4 - c_1^2)x + c_1(4 - c_1^2)x^2 - 2(4 - c_1^2)(1 - |x|^2)z|. \quad (3.4)$$

Since $c_1 \leq 2$ by Lemma 2.1, let $c_1 = c$ and assume without restriction that $c \in [0, 2]$. Applying the triangle inequality on (3.4) with $\rho = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{1}{12}[3c^3 + 2(4 - c^2) + 2c(4 - c^2)\rho + (c - 2)(4 - c^2)\rho^2] \\ &= F(\rho). \end{aligned}$$

Differentiating $F(\rho)$, we have

$$F'(\rho) = \frac{1}{12}[2c(4 - c^2) + 2(c - 2)(4 - c^2)] > 0.$$

This implies that $F(\rho)$ is an increasing function of ρ on $[0, 1]$. Hence $F(\rho) \leq F(1)$ for all $\rho \in [0, 1]$. That is,

$$F(\rho) \leq c.$$

Since $c \in [0, 2]$, it follows then that $F(\rho) \leq 2$. Thus the maximum of the functional $|a_2a_3 - a_4|$ corresponds to $\rho = 1$ and $c = 2$.

If $c_1 = c = 2$ in (2.1) and (2.2), then we have $c_2 = c_3 = 2$. Using these in (3.3), we see that equality is attained, which shows that our result is sharp. Furthermore, it is easily seen

that the extremal function in this case is the well known Koebe function $k(z) = z/(1-z)^2$. This completes the proof. $\square$

For $f \in S^*$, using the known inequalities $|a_k| \leq k$, $k = 2, 3, \dots$ [2], $|a_2a_4 - a_3^2| \leq 1$ [5] and $|a_3 - a_2^2| \leq 1$ [6] together with Theorem 3.3, we have the next corollary.

Corollary 3.4. *Let $f \in S^*$. Then*

$$|H_3(1)| \leq 16$$

and the inequality is sharp. Equality is attained by a rotation, $k_1(z) = z/(1+z)^2$, of the Koebe function.

Theorem 3.5. *Let $f \in C$. Then*

$$|a_2a_3 - a_4| \leq \frac{4}{9\sqrt{3}}$$

and the inequality is sharp.

Proof. For any $f \in C$ given by (1.1), there exists a $p \in P$ such that $(zf'(z))' = f'(z)p(z)$. Then equating coefficients we find that $2a_2 = c_1$, $6a_3 = c_2 + c_1^2$ and $24a_4 = 2c_3 + 3c_1c_2 + c_1^3$. Thus we have

$$|a_2a_3 - a_4| = \frac{1}{24}|c_1^3 - c_1c_2 - 2c_3|. \quad (3.5)$$

Substituting for c_2 and c_3 using Lemma 2.2, we obtain

$$|a_2a_3 - a_4| = \frac{1}{48}|-3c_1(4 - c_1^2)x + c_1(4 - c_1^2)x^2 - 2(4 - c_1^2)(1 - |x|^2)z|. \quad (3.6)$$

With $c_1 \leq 2$ from Lemma 2.1, we let $c_1 = c$ and assume also without restriction that $c \in [0, 2]$. Thus, applying the triangle inequality on (3.6), with $\rho = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{(4 - c^2)}{24} + \frac{c(4 - c^2)\rho}{16} + \frac{(c - 2)(4 - c^2)\rho^2}{48} \\ &= F(\rho). \end{aligned}$$

Differentiating $F(\rho)$, we get

$$F'(\rho) = \frac{c(4 - c^2)}{16} + \frac{(c - 2)(4 - c^2)\rho}{24} > 0,$$

which implies that $F(\rho)$ is an increasing function of ρ on $[0, 1]$. Hence $F(\rho) \leq F(1)$ for all $\rho \in [0, 1]$. That is,

$$\begin{aligned} F(\rho) &\leq \frac{c(4 - c^2)}{12} \\ &= G(c) \end{aligned}$$

wherefrom we have $G'(c) = (4 - 3c^2)/12$ and $G''(c) = -c/2$. Since $c \in [0, 2]$ by our assumption, it follows that $G(c)$ attains maximum at $c = 2/\sqrt{3}$. That is

$$G(c) \leq \frac{4}{9\sqrt{3}}$$

corresponding to $\rho = 1$ and $c = 2\sqrt{3}$, which is the desired upper bound.

If we set $c_1 = c = 2\sqrt{3}$ and take $x = -1$ in (2.1) and (2.2) we find that $c_2 = -2/3$ and $c_3 = -10/(3\sqrt{3})$. Using these in (3.5) we see that equality is attained which shows that our result is sharp. This completes the proof. $\square$

Finally for $f \in C$ if we use the known inequalities $|a_k| \leq 1$, $k = 2, 3, \dots$ [2], $|a_2 a_4 - a_3^2| \leq 1/8$ [5] and $|a_3 - a_2^2| \leq 1/3$ [6] together with the last result, we obtain the following sharp inequality:

Corollary 3.6. *Let $f \in C$. Then*

$$|H_3(1)| \leq \frac{32 + 33\sqrt{3}}{72\sqrt{3}} = 0.714933452973167.$$

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References

- [1] K. O. Babalola and T. O. Opoola, *On the coefficients of certain analytic and univalent functions*, Advances in Inequalities for Series, (Edited by S. S. Dragomir and A. Sofo) Nova Science Publishers (<http://www.novapublishers.com>) (2008), 5–17.
- [2] P. L. Duren, *Univalent functions*, Springer Verlag. New York Inc. 1983.
- [3] U. Grenander and G. Szego, *Toeplitz forms and their application*, University of California Press, Berkeley and Los Angeles, 1958.
- [4] A. Janteng, S. Abdulhalim and M. Darus, Coefficient inequality for a function whose derivative has positive real part, *J. Ineq. Pure and Applied Math.* (<http://jipam.vu.edu.au/>) 7(2) Art. 50 (2006), 1–5.
- [5] A. Janteng, S. Abdulhalim and M. Darus, Hankel determinant for starlike and convex functions, *Int. Journal of Math. Anal.* 1(13) (2007), 619–625.
- [6] F. R. Keogh and E. P. Merkes, A coefficient inequality for certain classes of analytic functions, *Proc. Amer. Math. Soc.* 20 (1969), 8–12.
- [7] R. J. Libera and E. J. Zlotkiewicz, Early coefficient of the inverse of a regular convex function, *Proc. Amer. Math. Soc.* 85(2) (1982), 225–230.
- [8] R. J. Libera and E. J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in P , *Proc. Amer. Math. Soc.* 87(2) (1983), 251–257.

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- [9] T. H. Macgegor, Functions whose derivative has a positive real part, *Trans. Amer. Math. Soc.* **104** (1962), 532–537.
- [10] A. K. Mishra and P. Gochhayat, Second Hankel determinant for a class of analytic functions defined by fractional derivative, *Int. Journal of Math. and Math. Sci.* 2008 Art. 153280 (doi:10.1155/2008/153280), 1–10.
- [11] J. W. Noonan and D. K. Thomas, Second Hankel determinant of areally mean p-valent functions, *Trans. Amer. Math. Soc.* **223** (1976), 337–346.
- [12] K. I. Noor, Hankel determinant problem for the class of functions with bounded boundary rotation, *Rev. Roum. Math. Pures et Appl.* **28(8)** (1983), 731–739.