

On the use of analysis of variance under unequal group variances

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ABSTRACT

In this study, we imposed Analysis of variances test (ANOVA) which use when we have more than two treatments or different levels of a single factors that we wish to compare then we assume homogeneity of variances across the groups being compared although most of the earlier works that have addressed the problem of testing equality of mean variance overestimates the appropriate variance and the test statistic becomes conservative. This is the well-known Behrens – Fisher problem. Then we are interested in comparing several treatments means in this work, we made use the analysis of variance under unequal variances when the groups variances differ. It will be very inappropriate to use the pooled sample variance (S_p^2) as a single value for the variances, instead the sample harmonic mean of variances (S_h^2) is proposed as an alternative to the pooled sample variance when there is heterogeneity of variances. The distribution theoretically and confirmed using simulation studies and this proposed harmonic mean of variance was, examined in this work and found useful for unequal variances. Data set from Kwara State Ministry of Health on the incidence of diabetes diseases for male patients was used to illustrate the relevance of our proposed test statistic.

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Introduction

Analysis of Variance is one of the most popular models in statistics. In general, interest is in testing the homogeneity of the different group means using the classical Analysis of Variance (ANOVA). However, the standard assumption of homogeneous error variances which is crucial in ANOVA is seldomly met in statistical practice. In such a case one has to assume a model with heteroscedastic error variances. Analysis of variance assumes that the sample data sets have been drawn from populations that follow a normal distribution. One-way analysis of variance for example, evaluates the effect of a single factor on a single response variable. For example, a clinician may be interested in determining whether or not there are differences in the age distribution of patients enrolled in different study groups. To satisfy the assumptions, the patients must be selected randomly from each of the population groups, a value for age for the response variable is recorded for each sampled patient, the distribution of the response variable can then be assumed to be normally distributed. See (Ott, 1984 and Abidoye 2012).

Often in the study of behavioral ecology, and more widely in science, we require to statistically test whether the central tendencies (mean or median) of 2 groups are different from each other on the basis of samples of the 2 groups. In surveying recent issues of *Behavioral Ecology* (Volume 16, issues 1–5), I found that, of the 130 papers, 33 (25%) used at least one statistical comparison of this sort. Three different tests were used to make this comparison: Student's *t*-test (67 occasions; 26 papers), Mann–Whitney *U* test (43 occasions; 21 papers), and the *t*-test for unequal variances. See (Zimmerman 2004 and Neuhauser 2002).

The analysis of variance uses a linear regression approach and consequently supports unequal sample sizes. This is important because designers of experiments seldom have complete control over the ultimate sample sizes in their studies. However, two-way analysis of variance for example does not support empty cells (factor levels with no sample data points). Each of the factors must have two or more levels and the factor combination must have one or more sample observations.

Sometimes differences in variances are systematic or predictable. For some populations the magnitude of the variance or standard deviation may be proportional to the magnitude of the mean. For example, for many biological organisms, populations with larger means also have larger variances. This type of variance inequality may be handled by making “transformations” on the data, which employ the analysis of some function of the *y*'s, such as $\log y$, rather than the original values. The transformed data may have equal variances and the pooled *t* test can then be used. Not all problems with unequal variances are amenable to this type of analysis; hence we need alternate procedures for performing inferences on the means of two populations based on data from independent samples. (Sawilowsky et. al 2002 and Delacre, Lakens and Leys 2017)

The conventional analysis of variance (ANOVA) is also based on the assumption of normality, independence of errors and equality of the error variances. Studies have shown that the *F*-test is not robust under the violation of equal error variances, especially if the sample sizes are not equal and some authors have developed an exact Analysis of variance for testing the means of *g* independent normal populations by using one or two stage procedures. See (Jonckheere 1954), (Dunnett 1964), (Montgomery 1981), (Dunnet and Tamhane 1997), (Yahya and Jolayemi 2003).

This test procedure having a specified type one error rate of α should be powerful to determine whether the differences among the sample means are large enough to imply that the corresponding population means are different. See (Abidoye et.al 2015a, 2015b).

It is well known that widely – used statistical significance tests, including the two – sample student *t* test and the ANOVA *F* test, are derived under an assumption of homogeneity of variance. Violation of this assumption when sample sizes are unequal substantially alters Type 1 error probabilities. When a larger variance is associated with a larger sample size, the probability of a Type 1 error declines below the nominal significance. See (Ruxton 2006, Zimmerman 2004, Neuhauser 2002 and Quinn and Keough 2002)

Methodology

We are interested in developing a suitable test procedure to test the hypothesis:

$H_0 : \mu_1 = \mu_2 = \dots = \mu_g$ against non-directional alternative, $H_1: \mu_1 \neq \mu$, for at least one i ,

$$i = 1, 2, \dots, g \dots\dots\dots(2.1)$$

where the error term $e_{ij} \sim N(0, \sigma_i^2) \quad i = 1, 2, \dots, g$.

$$\text{If we define } \delta_i = \mu_i - \mu, \dots\dots\dots(2.2)$$

then,

$$H_0 : \delta_i = \mu_i - \mu = 0 \quad \forall_i \text{ vs } H_1 : \delta_i \neq 0 \text{ for at least one } i$$

The unbiased estimate of δ_i is

$$\hat{\delta}_i = \bar{X}_i - \bar{X} \dots\dots\dots(2.3)$$

Therefore

$$\hat{\delta}_i \sim N[\delta_i, V(\delta_i)]$$

where

$$V(\delta_i) = V(Y_i)$$

$$= V(\bar{X}_i - \bar{\bar{X}})$$

$$= V(\bar{X}_i) + V(\bar{\bar{X}}) - 2 \text{cov}(\bar{X}_i, \bar{\bar{X}})$$

$$= V(\bar{X}_i) + V(\bar{\bar{X}}) - 2 \text{cov}(\bar{X}_i, \frac{\sum \bar{X}_i}{g})$$

$$= V(\bar{X}_i) + V(\bar{\bar{X}}) - 2 \text{cov}(\bar{X}_i, \frac{\bar{X}_i}{g})$$

$$\begin{aligned}
 &= \frac{\sigma_i^2}{n_i} + \frac{\sigma_H^2}{n} - \frac{2}{g} V(\bar{X}_i) \\
 &= \frac{\sigma_i^2}{n_i} + \frac{\sigma_H^2}{n} - \frac{2\sigma_i^2}{gn_i}, \text{ see (Abidoye 2012 and Abidoye et.al 2016)} \\
 &= \frac{\sigma_H^2}{n} + \left(1 - \frac{2}{g}\right) \frac{\sigma_i^2}{n_i} \dots\dots\dots(2.4)
 \end{aligned}$$

Therefore,

$$V(\delta_i) = \frac{\sigma_i^2}{n_i} + \frac{\sigma^2}{n} \dots\dots\dots(2.5)$$

$$= \sigma_H^2 \left(\frac{1}{n} + \frac{1}{n_i} \left(1 - \frac{2}{g}\right) \right) \dots\dots\dots(2.6)$$

$$\hat{\sigma}_H^2 = S_H^2$$

$$= \left[\frac{1}{g} \left(\frac{1}{s_1^2} + \frac{1}{s_2^2} + \dots + \frac{1}{s_g^2} \right) \right]^{-1} \dots\dots\dots(2.7)$$

where S_i^2 is the variance of the i^{th} group. S_H^2 has r degrees of freedom, where $r = 22.096 + 0.266(n-g) - 0.000029(n-g)^2$ as defined in (Abidoye et. al 2015).

S_H^2 is likened to the common variance as defined in one – way Analysis of variance ANOVA. The one – way ANOVA is presented in Table 1 below. In the table one the sum of squares between groups and adjusted sum of squares do not necessarily sum to total sum of squares, also the degrees of freedom.

Analysis of variance table

Source variation	d.f	SS	MS
Treatment (between groups)	t-1	$SSt = \frac{\sum Y_i^2}{t} - \frac{(\sum Y_{ij})^2}{n}$	$SSt/t-1 = MSS_t$
Adjusted error	r	$r \times S_H^2$	S_H^2
Total	n-1	$SST = \sum Y_{ij}^2 - \frac{(\sum Y_{ij})^2}{n}$	

where $r = \Omega = [r + 1/2]$

Small simulations show that without any loss of generality nearest integer values can be used to approximate the r degrees of freedom.

Application

The data used in this study is applicable in Medicine; the data used were secondary data, collected primarily by Kwara State Ministry of Health, Ilorin, Kwara State, Nigeria. They were extracts from incidence of diabetes diseases for male patients for ten consecutive years, covering the period 2001 – 2010.

Table 2: Showing the incidence of diabetes diseases for male patients in Kwara State for ten years (2001- 2010).

Years	1	2	3	4	5	6	7	8	9	10
Zone A	37	80	58	48	35	46	53	39	64	76
Zone B	14	19	12	21	23	13	15	16	11	14
Zone C	15	18	11	19	22	14	13	15	10	13
Zone D	11	19	10	18	23	12	14	16	12	15

We need to verify the equality of the variances between these four zones. That is, testing the hypothesis;

$$H_0 : \sigma_A^2 = \sigma_B^2 = \sigma_C^2 = \sigma_D^2 \text{ vs } H_1 : \sigma_i^2 \neq \sigma_j^2 \text{ for atleast } (i, j), i = A, B, C, D, j = A, B, C, D$$

Table 3: Levene test for variance equality.

	Levene Statistic	df1	df2	P-value
Response	10.975	3	36	0.000

Since P- value < 0.05 we therefore reject H_0 and therefore conclude that the variances are not equal.

Computation on incidence of diabetes diseases for male patients: From the data above the following summary statistics were obtained:

$$\text{Zone A: } \bar{Y}_A = 53.6, S_A^2 = 250.04, n_A = 10$$

$$\text{Zone B: } \bar{Y}_B = 15.8, S_B^2 = 15.7, n_B = 10$$

$$\text{Zone C: } \bar{Y}_C = 15.0, S_C^2 = 13.9, n_C = 10$$

$$\text{Zone D: } \bar{Y}_D = 15.0, S_D^2 = 16.7, n_D = 10$$

$$\text{In the above, data set, } n_i = 10, g = 4, n = \sum_{i=1}^4 n_i = 40, S_H^2 = \left(\frac{1}{4} \sum_{i=1}^4 \frac{1}{S_i^2} \right)^{-1}, S_H^2 = 20.05$$

The main hypothesis is

$$H_0 : \mu_A = \mu_B = \mu_C = \mu_D = \mu \text{ against } H_1 : \mu_i \neq \mu, \text{ for at least one } i, \text{ i.e. } i = A, B, \dots, D$$

$$SST = \sum Y_{ij}^2 - \frac{(\sum Y_{ij})^2}{n}$$

$$= 37^2 + 80^2 + \dots + 12^2 + 15^2 - 7182.4$$

$$= 31,209.6$$

$$SSt = \frac{\sum Y_i^2}{t} - \frac{(\sum Y_{ij})^2}{n}$$

$$= 35726 - 7182.4$$

$$= 28543.6$$

$$SSE = SST - SSt$$

$$= 31209.6 - 28543.6$$

$$= 2666$$

Analysis of variance table

Source variation	d.f	SS	MS	F
Between groups	3	28543.6	9514.5	474.54
Adjusted Error	31.63	634.18	20.05	
Total	39	31209.6		

$$F_{3,31.63,\alpha} \approx F_{3,32,\alpha} = 8.62$$

Analysis of variance shows that incidence rate of diabetes diseases in the four zones are significantly different at 5% level of significance which support the earlier results (Abidoye *et. al* 2016), even though the sum of squares due to zones plus the adjusted error sum of square did not sum up to total sum of squares.

Conclusion

In this work we have established that ANOVA test may suffice if the adjusted error (the Harmonic mean of the population variances) is known. The sum of square treatments and sum of square of adjusted error may not necessarily sum up to sum of squares; and the adjusted error degrees of freedom need not necessarily be an integer. The sample harmonic variance is proposed as an alternative to the pooled sample variance when there is heterogeneity of variances. The distribution theoretically and confirmed using simulation studies and this proposed harmonic mean of variance was , examined in this work and found useful for unequal variances.

References

- Abidoye, A. O (2012). *Development of hypothesis testing technique for ordered statistics, alternatives under heterogeneous variances* (Unpublished Ph.D Thesis) University of Ilorin, Nigeria
- Abidoye, A. O., Jolayemi, E. T., Sanni, O. O. M., & Oyejola, B. A. (2015). Development of hypothesis testing on type one error and power function. *Ilorin Journal of Science*, 2(1), 68-79.
- Abidoye, A.O, Jolayemi, E.T, Sanni, O.O.M and Oyejola, B. A. (2016). On application of modified f – statistic: an example of sales distribution of pharmaceutical drug. *Science Journal World* 11 (2), 23 – 26.
- Delacre, M., Lakens, D., & Leys, C. (2017). Why psychologists should by default use Welch's t-test instead of Student's t-test. *International Review of Social Psychology*, 30(1), 92-101.
- Dunnett, C. W and Tamhane, A. C (1997): Multiple testing to establish superiority / equivalence of a new treatment compare with k standard treatments. *Statistics in Medicine* 16, 2489 – 2506.
- Dunnett , C. W (1964). New tables for multiple comparison with a control. *Biometric*, 20, 482-491.
- Jonckheere, A.R (1954). A distribution-free k-sample test against ordered alternative. *Biometrical*, 41, 133-145.
- Montgomery, D.C (1981). *Ddesign and analysis of experiment 2nd ed*. New York: John Wiley and sons Inc..
- Neuhauser, M (2002). Two – sample tests when variances are unequal . *Anim Behav* 63: 823 –825.
- Ott, L (1984). An introduction to statistical methods and data analysis 2nd ed. Boston: P.W.S Publisher..
- Quinn, G. P and Keough, M.J (2002) *Experimental design and data analysis for biologists*. Cambridge: Cambridge University Press.
- Ruxton, G. D. (2006). The unequal variance t-test is an underused alternative to Student's t-test and the Mann–Whitney U test. *Behavioral Ecology*, 17(4), 688-690.
- Sawilowsky, S.S, Einstein, F. S and Behrens – Fisher (2002). The probable difference between two means with difference variances. *Journal Modern Applied Statistical Methods*. Vol 8, pp 461 – 472.

- Yahya, W.B and Jolayemi, E.T (2003). Testing ordered means against a control. *JNSA*, 16, 40- 51.
- Zimmerman, D. W. (2004). A note on preliminary tests of equality of variances. *British Journal of Mathematical and Statistical Psychology*, 57(1), 173-181.