

SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY THE INVERSES OF CERTAIN INTEGRAL OPERATORS

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ABSTRACT. In this paper we introduce and study new classes of analytic and univalent functions defined by the inverses of certain integral operators in the unit disk $E = \{z \in \mathbb{C}: |z| < 1\}$. The new classes of functions have been studied conveniently via corresponding non-iterative integral transformations of the Caratheodory functions.

1. INTRODUCTION

Let A be the class of functions

$$f(z) = z + a_2 z^2 + \cdots \quad (1.1)$$

which are analytic in E . Denote by P the class of functions

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (1.2)$$

which are also analytic in the unit disk E and satisfy $\operatorname{Re} p(z) > 0$, $z \in E$. Furthermore, for $0 \leq \beta < 1$, let $P(\beta)$ denote the subclasses of P consisting of analytic functions of the form $p_\beta(z) = \beta + (1 - \beta)p(z)$, $p \in P$. It is well-known that $f \in A$ is a function of bounded turning of order β if $f'(z) \in P(\beta)$ and that all such $f \in A$ are univalent in the open unit disk and we denote their class by $R(\beta)$.

Several important and interesting families of integral operators have been defined and studied in the theory of geometric functions, including each of the following

$$\mathcal{J}_{\sigma, \gamma}^j f(z) = \int_0^z \nabla_{\sigma, \gamma}^j(z, t) f(t) dt, \quad j = 1, 2$$

where

$$\nabla_{\sigma, \gamma}^1(z, t) = \binom{\sigma + \gamma}{\gamma} \frac{\sigma}{z^\gamma} t^{\gamma-1} \left(1 - \frac{t}{z}\right)^{\sigma-1},$$

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and

$$\nabla_{\sigma,\gamma}^2(z,t) = \frac{(\gamma+1)^\sigma t^{\gamma-1}}{z^\gamma \Gamma(\sigma)} \left(\log \frac{z}{t} \right)^{\sigma-1}.$$

where Γ denotes the Gamma function, $f \in A$ has the normalization $f(0) = f'(z) - 1 = 0$ as in the series representation (1.1) (see [8, 10]), $\gamma > -1$ and $\sigma > 0$ in both cases. Using the Beta and Gamma functions respectively, noting that

$$\binom{\sigma}{\gamma} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\gamma+1)\Gamma(\gamma+1)}$$

we find that:

$$\mathcal{J}_{\sigma,\gamma}^1 f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(\sigma+\gamma+1)\Gamma(\gamma+k)}{\Gamma(\gamma+1)\Gamma(\sigma+\gamma+k)} a_k z^k$$

and

$$\mathcal{J}_{\sigma,\gamma}^2 f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\gamma+1}{\gamma+k} \right)^\sigma a_k z^k.$$

These integrals generalize many earlier ones especially those in [5, 13]. Li and Srivastava [12] observed that these integrals are closely related to the *multiplier transformation* or (*fractional integral and fractional derivative*) investigated by Flett [7] and Kim et al [11].

In this paper, we consider the following inverses of the above integrals defined as follows:

$$\mathcal{L}_{\sigma,\gamma}^j f(z) = \int_0^z \Delta_{\sigma,\gamma}^j(z,t) f(t) dt, \quad j = 1, 2$$

where

$$\Delta_{\sigma,\gamma}^1(z,t) = \binom{\gamma}{\sigma+\gamma} \frac{-\sigma}{z^{\sigma+\gamma}} \left(1 - \frac{t}{z} \right)^{-(\sigma+1)} t^{\sigma+\gamma-1},$$

and

$$\Delta_{\sigma,\gamma}^2(z,t) = \frac{(\gamma+1)^{-\sigma} t^{\gamma-1}}{z^\gamma \Gamma(-\sigma)} \left(\log \frac{z}{t} \right)^{-\sigma-1},$$

where Γ , γ and σ are as defined above, so that by application of the Beta and Gamma functions again and noting also that $\Gamma(-\sigma) = -\Gamma(1-\sigma)/\sigma$, we find that

$$\mathcal{L}_{\sigma,\gamma}^1 f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(\gamma+1)\Gamma(\sigma+\gamma+k)}{\Gamma(\sigma+\gamma+1)\Gamma(\gamma+k)} a_k z^k$$

and

$$\mathcal{L}_{\sigma,\gamma}^2 f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\gamma+k}{\gamma+1} \right)^\sigma a_k z^k.$$

Hence we have the relation

$$\mathcal{L}_{\sigma,\gamma}^j(\mathcal{J}_{\sigma,\gamma}^j f(z)) = \mathcal{J}_{\sigma,\gamma}^j(\mathcal{L}_{\sigma,\gamma}^j f(z)) = f(z) \quad (1.3)$$

and note, in particular, that

$$\mathcal{L}_{1,0}^j f(z) = z f'(z).$$

Now using the inverse operators $\mathcal{L}_{\sigma,\gamma}^j$ we now isolate the following classes of functions.

Definition 1.1. For real numbers σ , γ and β as defined above, we say a function $f \in A$ is in the class $\mathcal{B}_{\sigma,\gamma}^j(\beta)$ if and only if

$$\operatorname{Re} \frac{\mathcal{L}_{\sigma,\gamma}^j f(z)}{z} > \beta, \quad z \in E.$$

In [3] we studied the classes $B_n^\sigma(\beta)$ of functions $f \in A$ satisfying

$$\operatorname{Re} \frac{L_n^\sigma f(z)}{z} > \beta, \quad z \in E$$

where σ is any fixed real number satisfying $\sigma - (n-1) > 0$ for $n \in \mathbb{N}$, $0 \leq \beta < 1$ and $L_n^\sigma : A \rightarrow A$ are convolution operators defined as:

$$L_n^\sigma f(z) = (\tau_\sigma * \tau_{\sigma,n}^{(-1)} * f)(z)$$

where also $\tau_{\sigma,n}$ is defined as $\tau_{\sigma,n}(z) = z/(1-z)^{\sigma-(n-1)}$ with $\sigma - (n-1) > 0$ and $\tau_{\sigma,n}^{(-1)}$ such that $(\tau_{\sigma,n} * \tau_{\sigma,n}^{(-1)})(z) = z/(1-z)$; and $\tau_{\sigma,0}$ and $\tau_{\sigma,0}^{(-1)}$ are written simply as τ_σ and $\tau_\sigma^{(-1)}$ respectively.

In series forms we find that for $\sigma = 1$,

$$\frac{\mathcal{L}_{1,\gamma}^j f(z)}{z} = 1 + \sum_{k=1}^{\infty} \frac{\gamma + k + 1}{\gamma + 1} a_{k+1} z^k$$

and

$$\frac{L_1^\sigma f(z)}{z} = 1 + \sum_{k=1}^{\infty} \frac{\sigma + k}{\sigma} a_{k+1} z^k$$

which establishes the equivalence

$$\mathcal{B}_{1,\gamma}^j(\beta) \equiv B_1^{\gamma+1}(\beta) \quad (1.4)$$

of classes of functions. Thus $\mathcal{B}_{1,\gamma}^j(\beta)$ consists of univalent functions as is known of $B_1^\sigma(\beta)$ [3]. It can also be shown further that $\mathcal{B}_{1,0}^j(\beta) \equiv R(\beta)$. By the relations (3) and the associated geometric conditions, functions in $\mathcal{B}_{\sigma,\gamma}^j(\beta)$ can be represented in terms of functions in $P(\beta)$ as $f(z) = \mathcal{J}_{\sigma,\gamma}^j[zp_\beta(z)]$. The remaining parts of this paper is organised thus: in Section 2 we present some transformations of $P(\beta)$ which we would need in the study of $\mathcal{B}_{\sigma,\gamma}^j(\beta)$ as presented in Section 3.

2. INTEGRAL TRANSFORMATION OF THE CLASS $P(\beta)$

The $P(\beta)$ family of functions of the form (2) of order β is well known for its central role in geometric functions theory. In earlier works [1, 3] we have developed the following integral transformations of functions in $P \equiv P(0)$.

Definition 2.1. ([1]) Let $p \in P$ and $\alpha > 0$ be real. The n th iterated integral transform of $p(z)$, $z \in E$ is defined as

$$p_n(z) = \frac{\alpha}{z^\alpha} \int_0^z t^{\alpha-1} p_{n-1}(t) dt, \quad n \geq 1$$

with $p_0(z) = p(z)$

and

Definition 2.2. ([3]) Let $p \in P$. Let σ be any fixed real number such that $\sigma - (n-1) > 0$ for $n \in \mathbb{N}$. We define the *sigma-nth* integral iteration of $p(z)$, $z \in E$ as

$$p_{\sigma,n}(z) = \frac{\sigma - (n-1)}{z^{\sigma-(n-1)}} \int_0^z t^{\sigma-n} p_{\sigma,n-1}(t) dt, \quad n \geq 1$$

with $p_{\sigma,0}(z) = p(z)$.

The transformations are iterative and closely associated with certain families analytic and univalent functions involving the well known Salagean and Rucheweyh derivatives (see [1, 3]) and have been used effectively to characterize them. Furthermore we have, in [2], used the transformation in Definition 1 together with a method of Nehari and Netanyahu to determine the best possible coefficient bounds for some classes of functions. New classes of functions have also been studied in [4] via the transformations.

In this paper, two new non-iterative integral transformations of this same important family of functions are presented.

Definition 2.3. Let $p \in P(\beta)$. Let $\sigma > 0$, $\gamma > -1$ be real numbers. Then, for $z \in E$ define

$$\phi_{\sigma,\gamma}^j(p(z)) = \int_0^z \lambda_{\sigma,\gamma}^j(z,t) p(t) dt, \quad j = 1, 2 \quad (2.1)$$

where

$$\lambda_{\sigma,\gamma}^1(z,t) = \sigma \frac{t^\gamma}{z^{\gamma+1}} \binom{\sigma+\gamma}{\gamma} \left(1 - \frac{t}{z}\right)^{\sigma-1}$$

and

$$\lambda_{\sigma,\gamma}^2(z,t) = \frac{(\gamma+1)^\sigma}{\Gamma(\sigma)} \frac{t^\gamma}{z^{\gamma+1}} \left(\log \frac{z}{t}\right)^{\sigma-1}.$$

Economising on symbols, we would simply adopt the notation ϕ^j , $j = 1, 2$ when we do not require to specify the parameters σ, γ . Now if $p(z)$ has series expansion (1.2), then evaluating the integrals (2.1) in terms of the Beta and Gamma functions we obtain

$$\phi^j(p(z)) = 1 + (1-\beta) \sum_{k=1}^{\infty} c_k^j z^k, \quad j = 1, 2,$$

where

$$c_k^1 = \frac{\Gamma(\sigma+\gamma+1)\Gamma(\gamma+k+1)}{\Gamma(\gamma+1)\Gamma(\sigma+\gamma+k+1)} c_k = c_{\sigma,\gamma,k}^1 c_k \quad (2.2)$$

and

$$c_k^2 = \left(\frac{\gamma + 1}{\gamma + k + 1} \right)^\sigma c_k = c_{\sigma, \gamma, k}^2 c_k. \quad (2.3)$$

The transformations are analytic, normalized by $p(0) = 1$ and $\phi^j \neq 0$ for $z \in E$ and are well defined. We quickly make the following

Remark 2.1. (i) The extremal functions for the transformations are the transformations of the Moebius function: $\phi^j(L_{0,\beta}(z))$, $j = 1, 2$, up to rotations, where

$$L_{0,\beta}(z) = \frac{1 + (1 - 2\beta)z}{1 - z}.$$

(ii) The inequalities

$$|c_k^1| \leq 2(1 - \beta)c_{\sigma, \gamma, k}^1; \quad |c_k^2| \leq 2(1 - \beta)c_{\sigma, \gamma, k}^2$$

hold, following from the well-known Caratheodory Lemma ($|c_k| \leq 2$).

(iii) $\phi_{1,\gamma}^1(p(z)) \equiv \phi_{1,\gamma}^2(p(z))$ for all $\gamma > -1$.

(iv) Finally using (2.2) and (2.3) we have the following relations

$$z(\phi_{\sigma, \gamma}^1)' = (\sigma + \gamma)[\phi_{\sigma-1, \gamma}^1 - \phi_{\sigma, \gamma}^1] \quad (2.4)$$

and

$$z(\phi_{\sigma, \gamma}^2)' = (\gamma + 1)[\phi_{\sigma-1, \gamma}^2 - \phi_{\sigma, \gamma}^2]. \quad (2.5)$$

For real numbers $\sigma > 0$, $\gamma > -1$ we denote the classes of transformations ϕ^j by $\Phi^j(\beta)$. Next we characterize the transformations ϕ^j . Proofs of assertions, in some cases, are omitted since they follow, *mutatis mutandis*, as we have done in [1, 3]. We shall need the following definition and lemmas.

An infinite sequence $a_0, a_1, \dots, a_k, \dots$ of nonnegative numbers is said to be a *convex null sequence* if $a_k \rightarrow 0$ as $k \rightarrow \infty$ and $a_0 - a_1 \geq a_1 - a_2 \geq \dots \geq a_k - a_{k+1} \geq \dots \geq 0$ (see [9] for example).

Lemma 2.1 ([6]). *Let $\{c_k\}_{k=0}^\infty$ be a convex null sequence. Then the function $p(z) = c_0/2 + c_1z + c_2z^2 + \dots$, $z \in E$, is analytic in E and $\operatorname{Re} p(z) > 0$.*

Lemma 2.2 (see [9]). *If $p(z)$ is analytic in E , $p(0) = 1$ and $\operatorname{Re} p(z) > 1/2$, $z \in E$, then for any function $q(z)$ analytic in E , the convolution $p * q$ takes its values in the convex hull of $q(E)$.*

Theorem 2.3. $\Phi^j(\beta) \subset P(\beta)$.

Proof. For $j = 1$, let $z = re^{i\theta}$ and $t = \rho e^{i\theta}$, $0 < \rho \leq r < 1$ so that

$$\phi^1(p(re^{i\theta})) = \binom{\sigma + \gamma}{\gamma} \frac{\sigma}{r^{\gamma+1}} \int_0^r \left(1 - \frac{\rho}{r}\right)^{\sigma-1} \rho^\gamma p(\rho e^{i\theta}) d\rho,$$

which gives

$$\operatorname{Re} \phi^1(p(re^{i\theta})) = \binom{\sigma + \gamma}{\gamma} \frac{\sigma}{r^{\gamma+1}} \int_0^r \left(1 - \frac{\rho}{r}\right)^{\sigma-1} \rho^\gamma \operatorname{Re} p(\rho e^{i\theta}) d\rho.$$

Since $\operatorname{Re} p(z) > \beta$, we have

$$\operatorname{Re} \phi^1(p(re^{i\theta})) > \beta \binom{\sigma + \gamma}{\gamma} \frac{\sigma}{r^{\gamma+1}} \int_0^r \left(1 - \frac{\rho}{r}\right)^{\sigma-1} \rho^\gamma d\rho.$$

Evaluating the above integral in terms of Beta function, noting again that

$$\binom{\sigma}{\gamma} = \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma - \gamma + 1)\Gamma(\gamma + 1)}$$

we shall obtain

$$\operatorname{Re} \phi^1(p(re^{i\theta})) > \beta.$$

This proves that $\Phi^3(\beta) \subset P(\beta)$. For $j = 2$ the proof is similar, using the Gamma function. $\square$

Theorem 2.4. *If $\sigma_2 > \sigma_1 > 0$, then*

$$\Phi_{\sigma_2, \gamma}^j(\beta) \subset \Phi_{\sigma_1, \gamma}^j(\beta).$$

Proof. Write $\sigma_2 = \sigma_0 + \sigma_1$, where $\sigma_0 > 0$. Then for $j = 1$, we shall prove that $\operatorname{Re} \phi_{\sigma_2, \gamma}^1 > \beta$ given that $\operatorname{Re} \phi_{\sigma_1, \gamma}^1 > \beta$. From (2.2) we have

$$\begin{aligned} c_{\sigma_2, \gamma, k}^1 &= \frac{\Gamma(\sigma_2 + \gamma + 1)\Gamma(\gamma + k + 1)}{\Gamma(\gamma + 1)\Gamma(\sigma_2 + \gamma + k + 1)} \\ &= \frac{\Gamma(\sigma_0 + \sigma_1 + \gamma + 1)\Gamma(\gamma + k + 1)}{\Gamma(\gamma + 1)\Gamma(\sigma_0 + \sigma_1 + \gamma + k + 1)} \\ &= \frac{(\sigma_1 + \gamma + 1)(\sigma_1 + \gamma + 2) \dots (\sigma_1 + \gamma + \sigma_0)}{(\sigma_1 + \gamma + k + 1)(\sigma_1 + \gamma + k + 2) \dots (\sigma_1 + \gamma + k + \sigma_0)} c_{\sigma_1, \gamma, k}^1 \\ &= c_k^* c_{\sigma_1, \gamma, k}^1. \end{aligned}$$

Hence we can represent $\phi_{\sigma_2, \gamma}^1$ as

$$\phi_{\sigma_2, \gamma}^1 = q * \phi_{\sigma_1, \gamma}^1$$

where $*$ means convolution (or Hadamard product) of analytic functions and $q(z) = 1 + \sum_{k=1}^{\infty} c_k^* z^k$. And it is easily verified that the sequence $\{c_k^*\}_{k=0}^{\infty}$, $c_0^* = 1$, is convex null. Thus it follows from Lemmas 1 and 2 that $\operatorname{Re} q(z) > 1/2$ so that $\phi_{\sigma_2, \gamma}^1$ takes values in the convex hull of $\phi_{\sigma_1, \gamma}^1$. Hence $\operatorname{Re} \phi_{\sigma_2, \gamma}^1 > \beta$.

Now for $j = 2$, it is trivially obvious from (2.3) by observing that

$$\phi_{\sigma_2, \gamma}^2(p(z)) = \phi_{\sigma_0, \gamma}^2(\phi_{\sigma_1, \gamma}^2(p(z))),$$

which shows that $\operatorname{Re} \phi_{\sigma_1, \gamma}^2 > \beta$ implies $\operatorname{Re} \phi_{\sigma_2, \gamma}^2 > \beta$ as required. The proof is thus complete. $\square$

The following theorems can be proved in the same way as in [1, 3].

Theorem 2.5. Let $\phi^j \in \Phi^j(\beta)$. Define

$$M_{\sigma,\gamma}^j(\beta, r) = 1 + 2(1 - \beta) \sum_{k=1}^{\infty} c_{\sigma,\gamma,k}^j r^k, \quad |z| = r$$

and

$$m_{\sigma,\gamma}^j(\beta, r) = 1 + 2(1 - \beta) \sum_{k=1}^{\infty} c_{\sigma,\gamma,k}^j (-r)^k, \quad |z| = r.$$

Then

$$m_{\sigma,\gamma}^j(\beta, r) \leq \operatorname{Re} \phi^j(z) \leq |\phi^j(z)| \leq M_{\sigma,\gamma}^j(\beta, r).$$

The extremal functions $\phi^j(L_{0,\beta}(z))$ and $\phi^j(L_{0,\beta}(-z))$ show that both bounds are best possible.

Corollary 2.6. $\phi^j \in \Phi^j(\beta)$ if and only if $\phi^j(p(z)) \prec \phi^j(L_{0,\beta}(z))$.

Remark 2.2. For $z \in E$, the following are equivalent:

- (i) $p \prec L_{0,\beta}(z)$,
- (ii) $p \in P(\beta)$,
- (iii) $\phi^j \in \Phi^j(\beta)$,
- (iv) $\phi^j(p(z)) \prec \phi^j(L_{0,\beta}(z))$.

Theorem 2.7. $\Phi^j(\beta)$ is a convex set.

3. PROPERTIES OF THE CLASSES $\mathcal{B}_{\sigma,\gamma}^j(\beta)$

Similar to Lemmas 4.2 and 2 of [1, 3] respectively we have

Lemma 3.1. Let $f \in A$. For real numbers σ and γ as defined above and $z \in E$, the following are equivalent:

- (i) $f \in \mathcal{B}_{\sigma,\gamma}^j(\beta)$,
- (ii) $\mathcal{L}_{\sigma,\gamma}^j f(z)/z \in P(\beta)$,
- (iii) $f(z)/z \in \Phi^j(\beta)$.

If $p \in P(\beta)$, then using the above lemma and the relation (1.3) we see that functions in $\mathcal{B}_{\sigma,\gamma}^j(\beta)$ have the following integral representations:

$$f(z) = \mathcal{J}_{\sigma,\gamma}^j[zp(z)] = z\phi_{\sigma,\gamma}^j(p(z)).$$

Using the above representation we find that

$$f_1(z) = z + (1 - \beta) \frac{\gamma + 1}{\sigma + \gamma + 1} z^2 + \cdots \in B_{\sigma,\gamma}^1(\beta)$$

and

$$f_2(z) = z + (1 - \beta) \left(\frac{\gamma + 1}{\gamma + 2} \right)^\sigma z^2 + \cdots \in B_{\sigma,\gamma}^2(\beta).$$

Theorem 3.2. If $\sigma_2 > \sigma_1 > 0$, then

$$B_{\sigma_2,\gamma}^j(\beta) \subset B_{\sigma_1,\gamma}^j(\beta).$$

Proof. Let $f \in B_{\sigma_2, \gamma}^j(\beta)$. Then by Lemma 3, $f(z)/z \in \Phi_{\sigma_2}^j(\beta)$. Then using Theorem 2, $f(z)/z \in \Phi_{\sigma_1}^j(\beta)$. By Lemma 3 again, $f \in B_{\sigma_1, \gamma}^j(\beta)$. $\square$

Theorem 3.3. $B_{1, \gamma}^j(\beta)$ consists only of univalent functions in E .

Proof. It follows from the equivalence (1.4). $\square$

The following corollary is a consequence of Theorems 5 and 6.

Corollary 3.4. For $\sigma \geq 1$, $\mathcal{B}_{\sigma, \gamma}^j(\beta)$ is a subclass of univalent functions in E .

The next result is also a consequence of Lemma 3 and Remark 1(ii).

Theorem 3.5. Let $f \in \mathcal{B}_{\sigma, \gamma}^j(\beta)$. Then

$$|a_k| \leq 2(1 - \beta)c_{\sigma, \gamma, k}^j, \quad k \geq 2.$$

Equality is attained for

$$f(z) = z + 2(1 - \beta) \sum_{k=2}^{\infty} c_{\sigma, \gamma, k}^j z^k. \quad (3.1)$$

If we take $\phi^j(z) = f(z)/z$ in Theorem 3, we have the following

Theorem 3.6. Let $f \in \mathcal{B}_{\sigma, \gamma}^j(\beta)$. Then

$$rm_{\sigma, \gamma}^j(\beta, r) \leq |f(z)| \leq rM_{\sigma, \gamma}^j(\beta, r).$$

Upper bound equality is realized for the function given by (3.1) while equality in the lower bound is attained for the function

$$f(z) = z + 2(1 - \beta) \sum_{k=2}^{\infty} (-1)^{k-1} c_{\sigma, \gamma, k}^j z^k. \quad (3.2)$$

Theorem 3.7. Each function $f(z)$ in the class $\mathcal{B}_{\sigma, \gamma}^j(\beta)$ maps the unit disk onto a domain which covers the disk $|w| < m_{\sigma, \gamma}^j(\beta, 1)$. The result is sharp.

Proof. From Theorem 8, we have $|f(z)| \geq rm_{\sigma, \gamma}^j(\beta, r)$. This implies that the range of every function $f(z)$ in the class $\mathcal{B}_{\sigma, \gamma}^j(\beta)$ covers the disk

$$|w| < m_{\sigma, \gamma}^j(\beta, 1) = \inf_{r \rightarrow 1} m_{\sigma, \gamma}^j(\beta, r).$$

The function given by (3.2) shows that the result is sharp. $\square$

Theorem 3.8. Let $f \in B_{\sigma, \gamma}^1(\beta)$ and $\sigma + \gamma > 0$, then

$$m_{\sigma, \gamma}^1(\beta, r) \leq \left| \frac{(\sigma + \gamma - 1)f(z)/z + f'(z)}{\sigma + \gamma} \right| \leq M_{\sigma, \gamma}^1(\beta, r).$$

Similarly, if $f \in B_{\sigma, \gamma}^2(\beta)$ then

$$m_{\sigma, \gamma}^2(\beta, r) \leq \left| \frac{\gamma f(z)/z + f'(z)}{\gamma + 1} \right| \leq M_{\sigma, \gamma}^2(\beta, r).$$

Upper bound equality is realized for the function $f(z)$ given by (3.1) while equality in the lower bound is attained for the function $f(z)$ defined by (3.2).

Proof. Since $f \in \mathcal{B}_{\sigma,\gamma}^j(\beta)$, by Lemma 2, there exists $\phi^j \in \Phi^j(\beta)$ such that $f(z) = z\phi^j$, so that

$$f'(z) = z(\phi^j)' + \phi^j.$$

Using relations (2.4) and (2.5) we get

$$(\sigma + \gamma - 1)\frac{f(z)}{z} + f'(z) = (\sigma + \gamma)\phi_{\sigma-1,\gamma}^3,$$

and

$$\gamma\frac{f(z)}{z} + f'(z) = (\gamma + 1)\phi_{\sigma-1,\gamma}^4.$$

Hence by Theorem 3 we have the inequalities. □

Lastly, using Theorem 4, it is easy to prove

Theorem 3.9. $\mathcal{B}_{\sigma,\gamma}^j(\beta)$ is a convex family of analytic and univalent functions.

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