

TWO RADII FOR CERTAIN BAZILEVIČ MAPS

M. SIDIQ¹ AND K. O. BABALOLA²

ABSTRACT. The goal of this paper is the determination of the radii of two disks: (i) in which the solution of certain integral equation involving a function $f \in B_n^\lambda(\beta)$ is in $B_n^\lambda(\beta)$ and (ii) for which each function $f \in B_n^\lambda(\beta)$ is σ - n -spiral univalent.

1. INTRODUCTION

In this work we denote by A the class of functions of the form $f(z) = z + a_2z^2 + \dots$ which are regular in the unit disk $E = \{z : |z| < 1\}$. Also let P be the class of functions $p(z) = 1 + c_1z + \dots$, which are regular in E and have positive real part.

Bazilevič [4] proved that functions in E defined by the integral

$$f(z) = \left\{ \frac{\alpha}{1 + \beta^2} \int_0^z [p(t) - i\xi] t^{-\left(1 + \frac{i\alpha\xi}{1 + \xi^2}\right)} g(t) \left(\frac{\alpha}{1 + \xi^2}\right) dt \right\}^{\frac{1+i\xi}{\alpha}} \quad (1)$$

where $p \in P$ and $g(z) = z + b_2z^2 + \dots$ is starlike, are univalent. The parameters $\alpha > 0$ and ξ are real and all powers mean principal determinations only. The family of functions (1), known as Bazilevič maps, is denoted in this paper by $B(\alpha, \xi, g)$.

In [1] Babalola elaborated a new method for the study of Bazilevič maps as a whole. He defined a new class $B(\lambda, g)$ of functions satisfying

$$\frac{z(f(z)^\lambda)'}{\eta z^{i\mu} g^\eta(z)} \in P_\lambda. \quad (2)$$

where $\lambda = \eta + i\mu$ with $\eta > 0$ and μ being real numbers, and P_λ consists of functions $h(z) = p(z) + i\frac{\mu}{\eta} = 1 + i\frac{\mu}{\eta} + c_1z + \dots$, $p \in P$. The class $B(\alpha, \xi, g)$ of Bazilevič maps is the special case $\lambda = \alpha/(1 + i\xi)$ of $B(\lambda, g)$. Using the Salagean derivative D^n defined as $D^n f(z) = D(D^{n-1}f(z)) = z(D^{n-1}f(z))'$ with $D^0 f(z) = f(z)$ (see [8]), in aforementioned paper, Babalola considered a more general class, $B_n(\lambda, g)$ consisting in all functions f which satisfy

$$\frac{D^n f(z)^\lambda}{\eta \lambda^{n-1} z^{i\mu} g(z)^\eta} \in P_\lambda.$$

For $0 \leq \beta < 1$, in this paper are investigated two radius problems for the class $B_n^\lambda(\beta)$ which we define as follows:

Definition 1.1. A regular function $f \in A$ is said to belong to the class $B_n^\lambda(\beta)$ if and only if

$$\frac{D^n f(z)^\lambda}{\eta \lambda^{n-1} z^\lambda} \in P_\lambda(\beta)$$

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where D^n is a Salagean operator and all other parameters have their usual definitions, and the class $P_\lambda(\beta)$ is the class of regular functions of the form $h(z) = 1 + i\frac{\mu}{\eta} + (1 - \beta)c_1z + \dots$ in E with $\operatorname{Re} h(z) > \beta$.

The purpose of this paper is to determine the radii of two disks: (i) in which the solution of certain integral equation involving $f \in B_n^\lambda(\beta)$ is in $B_n^\lambda(\beta)$ and (ii) for which each $f \in B_n^\lambda(\beta)$ is σ - n -spiral univalent. Obviously, the class $B_n^\lambda(\beta)$ generalizes many other known classes. Hence by specifying its underlying parameters, our results agree with many existing ones and added many more to the study of univalent functions.

The paper is organized as follows: in Section 2, relevant lemmas are stated, and proved where necessary. The main results are stated and proved in Section 3.

2. PRELIMINARY LEMMAS

Lemma 2.1. [5] *Let $w = b_1z + b_2z^2 + \dots$ be a unit-bound function, then*

$$|w'(z)| \leq \frac{1 - |w(z)|^2}{1 - |z|^2}$$

strict inequality holds, unless $w(z) = e^{i\sigma}z$, for some real number σ .

Lemma 2.2. [9] *Let $p \in P$. Then*

$$|zp'(z)| \leq \frac{2r}{1 - r^2} \operatorname{Re} p(z).$$

Corollary 2.3. *Let $p \in P$. Then for any complex number λ ,*

$$\operatorname{Re} \frac{zp'(z)}{\lambda} \geq \frac{-2r}{|\lambda|(1 - r^2)} \operatorname{Re} p(z).$$

Lemma 2.4. [2] *For any complex number λ , if either of $D^{n+1}f^\lambda(z)/D^n f^\lambda(z)$ or $D^{n+1}f(z)/D^n f(z)$ takes a value independent of n , then*

$$\frac{D^{n+1}f^\lambda(z)}{D^n f^\lambda(z)} = \lambda \frac{D^{n+1}f(z)}{D^n f(z)}.$$

Lemma 2.5. *Let $h \in P_\lambda(\beta)$. Then*

$$\left| \frac{zh'(z)}{h(z)} \right| \leq \frac{2(1 - \beta)r}{1 - r^2}.$$

Proof. The proof follows as in Lemmas 2.1 and 2.2 in [1], noting that $h \in P_\lambda(\beta)$ can be written as

$$h(z) = \beta + (1 - \beta)p(z) + \frac{i\mu}{\eta}$$

for some $p \in P$. □

Corollary 2.6. *Let $h \in P_\lambda(\beta)$. Then*

$$\operatorname{Re} \frac{zh'(z)}{h(z)} \geq \frac{-2(1 - \beta)r}{1 - r^2}.$$

3. MAIN RESULT

Our main results are the following:

Theorem 3.1. *Let $f \in B_n^\lambda(\beta)$. Then the solution of the integral equation*

$$f^\lambda(z) = \frac{\lambda + c}{z^c} \int_0^z t^{c-1} F^\lambda(t) dt \quad (3),$$

given by

$$F(z) = \left(\frac{(z^c f^\lambda(z))'}{(\lambda + c)z^{c-1}} \right)^{\frac{1}{\lambda}} \quad (4),$$

is in $B_n^\lambda(\beta)$ provided $|z| < r_0(\lambda, c)$, where $r_0(\lambda, c)$ is defined as

$$r_0(\lambda, c) = \frac{\sqrt{1 + |\lambda + c|^2} - 1}{|\lambda + c|}.$$

The radius is the largest possible.

Proof. From (4) we have

$$(\lambda + c)F^\lambda(z) = z(f^\lambda(z))' + cf^\lambda(z)$$

so that

$$(\lambda + c) \frac{D^n F^\lambda(z)}{\eta \lambda^{n-1} z^\lambda} = c \frac{D^n f^\lambda(z)}{\eta \lambda^{n-1} z^\lambda} + \frac{D^{n+1} f^\lambda(z)}{\eta \lambda^{n-1} z^\lambda}. \quad (5)$$

Since $f \in B_n^\lambda(\beta)$, then there exist $p \in P$ such that

$$\frac{D^n f^\lambda(z)}{\eta \lambda^{n-1} z^\lambda} = \beta + (1 - \beta)p(z) + \frac{i\mu}{\eta}. \quad (6)$$

Thus

$$D^{n+1} f^\lambda(z) = \eta \lambda^n z^\lambda \left[\beta + (1 - \beta)p(z) + \frac{i\mu}{\eta} \right] + (1 - \beta) \eta \lambda^{n-1} z^{\lambda+1} p'(z)$$

and

$$\frac{D^{n+1} f^\lambda(z)}{\eta \lambda^{n-1} z^\lambda} = \lambda \left[\beta + (1 - \beta)p(z) + \frac{i\mu}{\eta} \right] + (1 - \beta) z p'(z). \quad (7)$$

Substituting (6) and (7) in (5) and simplifying, we obtain

$$(\lambda + c) \frac{D^n F^\lambda(z)}{\eta \lambda^{n-1} z^\lambda} = (\lambda + c) \left[\beta + (1 - \beta)p(z) + \frac{i\mu}{\eta} \right] + (1 - \beta) z p'(z)$$

which yields

$$\frac{D^n F^\lambda(z)}{\eta \lambda^{n-1} z^\lambda} = \beta + (1 - \beta) \left[p(z) + \frac{z p'(z)}{\lambda + c} \right] + \frac{i\mu}{\eta}$$

Therefore

$$\operatorname{Re} \frac{D^n F^\lambda(z)}{\eta \lambda^{n-1} z^\lambda} = \beta + \operatorname{Re} \left((1 - \beta) \left[p(z) + \frac{z p'(z)}{\lambda + c} \right] \right).$$

Applying Corollary 2.3, we have

$$\begin{aligned} \operatorname{Re} \frac{D^n F^\lambda(z)}{\eta \lambda^{n-1} z^\lambda} - \beta &\geq (1 - \beta) \left(1 - \frac{2r}{|\lambda + c|(1 - r^2)} \right) \operatorname{Re} p(z) \\ &= (1 - \beta) \left(\frac{|\lambda + c|(1 - r^2) - 2r}{|\lambda + c|(1 - r^2)} \right) \operatorname{Re} p(z) > 0 \end{aligned}$$

provided $|z| = r < r_0(\lambda, c)$, where $r_0(\lambda, c)$ is the smallest positive root of $|\lambda + c|r^2 + 2r - |\lambda + c| = 0$. That is

$$|z| = r < r_0(\lambda, c) = \frac{\sqrt{1 + |\lambda + c|^2} - 1}{|\lambda + c|}.$$

Hence

$$\operatorname{Re} \frac{D^n F(z)^\lambda}{\eta \lambda^{n-1} z^\lambda} > \beta$$

in the subdisk $|z| < r_0(\lambda, c)$. □

If $\xi = 0$ then $\lambda = \eta = \alpha$, then Theorem 3.1 yields:

Corollary 3.2. *Let $f \in B_n^\alpha(\beta)$. Then the solution of the integral equation*

$$f^\alpha(z) = \frac{\alpha + c}{z^c} \int_0^z t^{c-1} F^\alpha(t) dt,$$

given by

$$F(z) = \left(\frac{(z^c f^\alpha(z))'}{(\alpha + c)z^{c-1}} \right)^{\frac{1}{\alpha}},$$

is in $B_n^\alpha(\beta)$ provided $|z| < r_0(\alpha, c)$, where $r_0(\alpha, c)$ is given by,

$$r_0(\alpha, c) = \frac{\sqrt{1 + (\alpha + c)^2} - 1}{\alpha + c}.$$

The result is sharp.

Putting $\beta = 0$ and $\lambda = \frac{\alpha}{1+i\xi}$ in Theorem 3.1, we obtain:

Corollary 3.3. *Let $f \in B_n(\alpha, \xi)$. Then the solution of the integral equation*

$$f^{\frac{\alpha}{1+i\xi^2}}(z) = \frac{\alpha + c(1+i\xi)}{(1+i\xi)z^c} \int_0^z t^{c-1} F^{\frac{\alpha}{1+i\xi}}(t) dt,$$

given by

$$F(z) = \left(\frac{(1+i\xi)(z^c f^{\frac{\alpha}{1+i\xi}}(z))'}{[\alpha + c(1+i\xi)]z^{c-1}} \right)^{\frac{1+i\xi}{\alpha}}$$

is in $B_n(\alpha, \xi)$, provided $|z| < r_0(\alpha, \xi, c)$, where

$$r_0(\alpha, \xi, c) = \frac{\sqrt{1 + \xi^2(1 + c^2) + (\alpha + c)^2} - \sqrt{1 + \xi^2}}{\sqrt{(\alpha + c)^2 + \xi^2 c^2}}.$$

The result is sharp.

If $\xi = 0$ and $n = 1$ in Corollary 3.3, we have:

Corollary 3.4. *Let $f \in B(\alpha)$. Then the solution of the integral equation*

$$f(z)^\alpha = \frac{\alpha + c}{z^c} \int_0^z t^{c-1} F(t)^\alpha dt$$

is given as

$$F(z) = \left(\frac{(z^c f(z)^\alpha)'}{(\alpha + c)z^{c-1}} \right)^{\frac{1}{\alpha}}$$

is in $B(\alpha)$, provided $|z| < r_0(\alpha, c)$ where

$$r_0(\alpha, c) = \frac{\sqrt{1 + (\alpha + c)^2} - 1}{\alpha + c}.$$

The result is sharp.

The above corollary coincides with a result of Singh [9, Theorem 4'].

If $c = 1$ in Corollary 3.4, we obtain:

Corollary 3.5. *Let $f \in B(\alpha)$. Then the solution of the integral equation*

$$f^\alpha(z) = \frac{\alpha + 1}{z^c} \int_0^z F^\alpha(t) dt$$

given as

$$F(z) = \left(\frac{(zf^\alpha(z))'}{\alpha + 1} \right)^{\frac{1}{\alpha}}$$

is in $B(\alpha)$, provided $|z| < r_0(\alpha)$, where

$$r_0(\alpha) = \frac{\sqrt{2 + 2\alpha + \alpha^2} - 1}{\alpha + 1}.$$

The result is sharp.

This is the result of Bernardi [3, Theorem 4].

If $\alpha = 1$ in Corollary 3.5 it yields

Corollary 3.6. *Let $f \in R$. Then the solution of the integral equation*

$$f(z) = \frac{2}{z} \int_0^z F(t) dt$$

given as

$$F(z) = \frac{(zf(z))'}{2}$$

is in R , provided $|z| < \frac{1}{2}(\sqrt{5} - 1)$. The result is sharp.

This is the result of Livingston [6, Theorem 4].

Theorem 3.7. *Let $f \in B_n^\lambda(\beta)$. Then f is σ - n -spiral univalent in the disk $|z| < r_0(\eta, \mu, \beta)$ where $r_0(\eta, \mu, \beta)$ is given by*

$$r < r_0(\eta, \mu, \beta) = \frac{1}{\eta - \beta|\lambda|} \left(\sqrt{(1 - \beta)^2 + (\eta - \beta|\lambda|)^2} - (1 - \beta) \right),$$

where $|\lambda| = \sqrt{\eta^2 + \mu^2}$ and $\sigma = \tan^{-1} \frac{\mu}{\eta}$. The radius is the largest possible.

Proof. Since $f \in B_n^\lambda(\beta)$, it implies that there exist $h \in P_\lambda(\beta)$ such that

$$\frac{D^n f^\lambda(z)}{\eta \lambda^{n-1} z^\lambda} = h(z).$$

Then

$$D^{n+1} f^\lambda(z) = \eta \lambda^n z^\lambda h(z) + \eta \lambda^{n-1} z^{\lambda+1} h'(z),$$

hence

$$\frac{D^{n+1} f^\lambda(z)}{D^n f(z)^\lambda} = \lambda + \frac{zh'(z)}{h(z)}. \quad (8)$$

Applying Lemma 2.4, then (8) yields

$$\lambda \frac{D^{n+1} f(z)}{D^n f(z)} = \lambda + \frac{zh'(z)}{h(z)}.$$

Taking $\sigma = \tan^{-1} \frac{\mu}{\eta}$ we have now

$$\begin{aligned} \operatorname{Re} e^{i\sigma} \frac{D^{n+1}f(z)}{D^n f(z)} &= \operatorname{Re} \left(\frac{\lambda}{|\lambda|} + \frac{zh'(z)}{|\lambda|h(z)} \right) \\ &= \frac{\eta}{|\lambda|} + \operatorname{Re} \frac{zh'(z)}{|\lambda|h(z)}. \end{aligned}$$

Using Corollary 2.6, we get

$$\begin{aligned} \operatorname{Re} e^{i\sigma} \frac{D^{n+1}f(z)}{D^n f(z)} &\geq \frac{\eta}{|\lambda|} - \frac{-2(1-\beta)r}{(1-r^2)|\lambda|} \\ &= \frac{\eta(1-r^2) - 2(1-\beta)r}{(1-r^2)|\lambda|} > \beta \end{aligned}$$

provided

$$(\eta - \beta|\lambda|)r^2 + 2(1-\beta)r - (\eta - \beta|\lambda|) < 0,$$

that is

$$r < r_0(\eta, \mu, \beta) = \frac{1}{\eta - \beta|\lambda|} \left(\sqrt{(1-\beta)^2 + (\eta - \beta|\lambda|)^2} - (1-\beta) \right).$$

Thus f is σ - n -spiral univalent in the disk $|z| < r_0(\eta, \mu, \beta)$, where $r_0(\eta, \mu, \beta)$ is smallest positive root of the equation $(\eta - \beta|\lambda|)r^2 + 2(1-\beta)r - (\eta - \beta|\lambda|) = 0$. $\square$

If $\beta = 0$ in Theorem 3.7, we have:

Corollary 3.8. *Let $f \in B_n(\lambda)$. Then f is σ - n -spiral univalent in the disk $|z| < r_0(\eta)$, where $r_0(\eta)$ is given by*

$$r_0(\eta) = \frac{1}{\eta} \left(\sqrt{1 + \eta^2} - 1 \right),$$

where $\sigma = \tan^{-1} \frac{\mu}{\eta}$.

This result agrees Theorem 3.7 of Babalola [2].

For $\beta = 0$, $n = 1$ and $\lambda = \frac{\alpha}{1+i\xi}$, we obtain:

Corollary 3.9. *Let $f \in B(\alpha, \xi)$. Then f is σ -spiral univalent in the disk $|z| < r_0(\alpha, \xi)$, where $r_0(\alpha, \xi)$ is given by*

$$r_0(\alpha, \xi) = \frac{1}{\alpha} \left(\sqrt{(1+\xi^2)^2 + \alpha^2} - 1 - \xi^2 \right),$$

where $\sigma = \tan^{-1}(-\xi)$.

If $\xi = 0$ in Corollary 3.9, we have:

Corollary 3.10. *Let $f \in B(\alpha)$. Then f is 0-1-spiral (convex) univalent in the disk $|z| < r_0(\alpha)$, where $r_0(\alpha)$ is given by*

$$r_0(\alpha) = \frac{1}{\alpha} \left(\sqrt{1 + \alpha^2} - 1 \right).$$

Putting $\alpha = 1$ in Corollary 3.10, we get:

Corollary 3.11. *Let $f \in B(1)$. Then f is 0-1-spiral (convex) univalent in the disk $|z| < \sqrt{2} - 1$.*

This also agrees Corollary 3.5 of Babalola [1], which gave the radius of convexity of functions of bounded turning as $\sqrt{2} - 1$ as earlier obtained by MacGregor [7].

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^{1,2} DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILORIN, ILORIN, NIGERIA
E-mail address: ¹ kobabalola@gmail.com, ² mashoodsidiq@yahoo.com