

ON THE VARIANTS OF THE TAU METHODS FOR SOLUTIONS OF IVPs IN SYSTEMS OF FIRST ORDER DIFFERENTIAL EQUATIONS.

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ABSTRACT: This paper is concerned with two variants of a numerical integration scheme, namely the tau method for systems of first order ordinary differential equations. The error estimation of the two variants are obtained and numerical results confirm the accuracy and effectiveness of the scheme.

Key Words: Tau Method, Differential form, Integrated form, Error estimates, Perturbation term.

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1. INTRODUCTION

In 1938, Lanczos introduced tau method which provides an accurate approximate solution of linear differential equations with polynomial coefficients (see Lanczos(1956), Fox(1968) and Adeniyi(1991)). Since then, variants of tau methods have been developed. The scope of application of this method was improved upon by Ortiz(1969) when he proposed a recursive generation of canonical polynomials. Two of these variants are highlighted below.

1.1 DIFFERENTIAL FORM OF TAU METHOD

For the description of the differential form of the tau method, we consider the boundary value problem (BVP) in the m -th order linear DE:

$$Ly(x) = \sum_{r=0}^m P_r(x) y^{(r)}(x) = f(x), a \leq x \leq b \quad (1.1a)$$

$$Ly(x_k) = \sum_{r=0}^{m-1} a_{rk} y^{(r)}(x_k) = \alpha_k, k = 1(1)m \quad (1.1b)$$

$$\text{where: } y_n(x) = \sum_{r=0}^n a_r x^r, 0 < n < +\infty \quad (1.2)$$

of $y_n(x)$ which is the exact solution of perturbed problem:

$$Ly_n(x) = \sum_{r=0}^{m-1} P_r(x) y_n^{(r)}(x) = f(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}(x) \quad (1.3a)$$

$$L^* y(x_{rk}) = \sum_{r=0}^{m-1} a_{rk} y_n^{(r)}(x_{rk}) = \alpha_k, k = 1(a)m \quad (1.3b)$$

for $a \leq x \leq b$ and where $\tau_r, r = 1(1)m$, are parameter to be determined along with the a 's in (1.3); $T_r(x)$ is the r -th degree Chebyshev polynomial valid in the interval $[a, b]$, and s is the number of over-determination of equation (1.1a) defined below (see also Fox and Parker (1968), Adeniyi (1991)).

Definition 1.1 The number of over-determination s , τ the DE (1.1a) is defined as:

$$\# = \max\{|N_r - 1|, 0 \leq r \leq m\} \quad (1.5)$$

Definition 1.2 The problem (1.1) is said to be overdetermined if $\#$ given by (1.6) is greater than zero.

We determine $a_r, r = 1(1)n$ and $\tau_r, r = 1(1)m$, from the linear algebraic system

$$A\tau = b,$$

obtained by equating corresponding coefficients of powers of x from (1.3a) and then applying conditions (1.3b);

$$A = (a_{ij}), 1 \leq i, j \leq n + m + s + 1; b = (b_i)^T, 1 \leq i \leq n + m + s + 1;$$

$$\tau = (a_0, a_1, \dots, a_n, \tau_1, \dots, \tau_{m+s})^T.$$

Consequently, we obtain from (1.2) our desired approximant $Y_n(x)$.

1.2 THE INTEGRATED FORMULATION TO THE TAU METHOD

We still make use of BVP given in (1.1a) and (1.1b)

$$\text{Let } \iiint \dots i \int g(x) dx$$

i denote the indefinite integration i times applied to the function $g(x)$ and let

$$I_i = \iiint \dots m \int L(\cdot) dx \quad (1.7)$$

The integrated form of

$$Ly(x) := \sum_{r=0}^n p_r(x) y^{(r)}(x) = f(x) \quad (1.8)$$

is then

$$I_1(y_n) = \iiint \dots m \int f(x) dx + C_m(x) \quad (1.9)$$

where $C_m(x)$ denotes an arbitrary polynomial of degree $(m-1)$ arising from the constant of integration. The tau approximant, $y_n(x)$, thus satisfies the perturbed problem.

$$I_1(y_n(x)) = \iiint \dots m \int f(x) dx + C_m(x) + H_{m,n}(x) \quad (1.0a)$$

$$L^* y(x_k) := \sum_{r=0}^{m-1} a_r y_n^{(r)}(x_k) = \alpha_k, k = 1(i)m \quad (1.0b)$$

$$\text{where } H_{m,n}(x) = \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_n^*(x) \quad (1.11)$$

The integrated tau problem (1.10) often gives a more accurate approximation than its corresponding differential form (1.3) due to its higher order perturbation term (1.11) (Fox (1968)).

2. ERROR ESTIMATION OF THE TAU METHODS

2.1 ERROR ESTIMATION FOR THE DIFFERENTIAL FORM

If we define

$$e_n(x) = y(x) - y_n(x) \quad (2.1)$$

then it satisfies the problem

$$Le_n(x) = -H_n(x), L^* e_n(x_k) = 0 \quad (2.2)$$

where

$$H_n(x) = \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}^*(x), \quad s \geq 0 \quad (2.3)$$

A polynomial estimate $(e_n(x))_{n+1}$ of degree $n+1$ for $e_n(x)$ is given as:

$$(e_n(x))_{n+1} = \frac{\phi_n \mu_m(x) T_{n-m+1}^*(x)}{2^{2(n-m)+1}} \quad (2.4)$$

The role of $\mu_m(x)$ is to ensure that $(e_n(x))_{n+1}$ satisfies some or all the homogenous conditions of $e_n(x)$.

Now, let us consider the perturbed error equation

$$L(e_n(x))_{n+1} = -H_n(x) + H_M(x), \quad M \geq n+1 \quad (2.5)$$

where

$$H_M(x) = \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+2}^*(x) \quad (2.6)$$

The necessary $m+s-1$ equations to determine $\tau_{s+1}, \dots, \tau_{m+s}$ and Φ_n are given by equating corresponding coefficients of $x^{n+s+1}, \dots, x^{n+1-m}$ in (2.6). A forward elimination process is used. Substituting the expression obtained for Φ_n into (1.15) gives $(e_n(x))_{n+1}$ in the terms of known quantities. We then obtain for large n , the estimate

$$\varepsilon = \max_{0 \leq r \leq 1} |(e_n(x))_{n+1}| = \frac{|\Phi_n|}{2^{2(n-m)+1}} \quad (2.7)$$

for

$$|(e_n)| = \max_{0 \leq r \leq 1} |e_n(x)| \quad (2.8)$$

2.2 ERROR ESTIMATE OF THE INTEGRATED FORM

The interpreted error equation is considered here with the aim of improving the accuracy of the estimate (2.7). The error polynomial $e_n(x)$ given by (2.4) satisfies the perturbed problem.

$$I_L(e_n(x))_{n+1} = - \int \int_m \left(\sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}^*(x) \right) dx + C_m(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+s}^*(x) \quad (2.9).$$

By equating corresponding coefficients of like powers of x in (2.9), as described in section 2.1, will give the error estimate ε_1 .

3. SYSTEMS OF FIRST ORDER DIFFERENTIAL EQUATIONS

We consider here the differential and integral forms of the tau method for the class of non-overdetermined constant-coefficient homogeneous differential system.

$$AY' = 0; \quad (3.1a)$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \dots & -a_{m-2} & -a_{m-1} \end{pmatrix}$$

and

$$Y = (y_1, y_2, \dots, y_m)^T; Y(0) = \alpha = (\alpha_0, \alpha_1, \dots, \alpha_{m-1})^T \quad (3.1b).$$

we remark here that (3.1) is equivalent to an m -th order. Without loss of generality we shall assume that $a = 0$ and $b = 1$, since by the transformation

$$u = (x-a)/(b-a), \quad a \leq x \leq b,$$

we may readily transform (3.1) into the interval $[0, 1]$.

3.1 TAU APPROXIMANT BY THE DIFFERENTIAL FORM

The perturbed form of (3.1)

$$Y_n' = AY_n + \tau T_n(x) \quad (3.2)$$

where

$$Y_n = (y_{1,n}, y_{2,n}, \dots, y_{m,n})^T$$

is the approximate solution to be obtained, A is the matrix of coefficients earlier defined by (3.2) above, and the vector

$$\tau = (\tau_1, \tau_2, \dots, \tau_m)^T.$$

This is equivalent to matrix equation:

$$\begin{pmatrix} \sum_{r=0}^n b_r x^{r-1} \\ \sum_{r=0}^n c_r x^{r-1} \\ \sum_{r=0}^n d_r x^{r-1} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \dots & \dots & -a_{m-1} \end{pmatrix} \begin{pmatrix} \sum_{r=0}^n b_r x^r \\ \sum_{r=0}^n c_r x^r \\ \sum_{r=0}^n d_r x^r \\ \vdots \end{pmatrix} + \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \vdots \\ \tau_m \end{pmatrix} \begin{pmatrix} \sum_{r=0}^n C_r^{(n)} x^r \\ \sum_{r=0}^n C_r^{(n)} x^r \\ \sum_{r=0}^n C_r^{(n)} x^r \\ \vdots \end{pmatrix} \quad (3.3)$$

We equate corresponding coefficients of x in (3.3) to obtain a system of linear equations that is solved to get the values of the constants a_r , $r=0(1)n$.

3.2 ERROR ESTIMATION FOR DIFFERENTIAL FORM

Here we illustrate the error estimation for the differential form for the three problems considered in this paper.

Example 3.2.1

We first consider here the system:

$$\begin{bmatrix} \frac{d}{dx} - 1 & 0 & 0 \\ 0 & \frac{d}{dx} - 1 & 0 \\ -1 & 3 & \frac{d}{dx} - 1 \end{bmatrix} \begin{bmatrix} y \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} y(0) \\ v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 10 \end{bmatrix} \quad (i)$$

The error approximants $(e'_{1,n}(x))_{n+1}, (e'_{2,n}(x))_{n+1}, (e'_{3,n}(x))_{n+1}$ satisfy

$$\left. \begin{aligned}
 (e'_{1,n}(x))_{n+1} - (e_{2,n}(x))_{n+1} &= \tau_1 T_{n+1}(x) - \tau_1 T_n^*(x) \\
 (e'_{2,n}(x))_{n+1} - (e_{3,n}(x))_{n+1} &= \tau_2 T_{n+1}(x) - \tau_2 T_n^*(x) \\
 (e'_{3,n}(x))_{n+1} - 3(e_{3,n}(x))_{n+1} + 3(e_{2,n}(x))_{n+1} - (e_{1,n}(x))_{n+1} &= \tau_3 T_{n+1}(x) - \tau_3 T_n^*(x)
 \end{aligned} \right\} (ii)$$

$$\left. \begin{aligned}
 (e_{1,n}(x))_{n+1} &= \frac{\Phi_{1,n} x T_n^*(x)}{C_n^{(n)}}, (e_{2,n}(x))_{n+1} = \frac{\Phi_{2,n} x T_n^*(x)}{C_n^{(n)}}, (e_{3,n}(x))_{n+1} = \frac{\Phi_{3,n} x T_n^*(x)}{C_n^{(n)}} \\
 (e_{1,n}(x))_{n+1} &= \frac{\Phi_{1,n}}{C_n^{(n)}} \{C_n^{(n)} x^{n+1} + C_{n-1}^n x^n + \dots\} \\
 (e_{2,n}(x))_{n+1} &= \frac{\Phi_{2,n}}{C_n^{(n)}} \{C_n^{(n)} x^{n+1} + C_{n-1}^n x^n + \dots\} \\
 (e_{3,n}(x))_{n+1} &= \frac{\Phi_{3,n}}{C_n^{(n)}} \{C_n^{(n)} x^{n+1} + C_{n-1}^n x^n + \dots\}
 \end{aligned} \right\} (iii)$$

Putting (iii) in the system of equations (ii) we have

$$\begin{aligned}
 &\frac{\Phi_{1,n}}{C_n^{(n)}} \{(n+1)C_n^{(n)} x^n + \dots\} - \frac{\Phi_{2,n}}{C_n^{(n)}} \{C_n^{(n)} x^{n+1} + C_{n-1}^n x^n + \dots\} \\
 &\quad = \tau_1 \{C_{n+1}^{n+1} x^{n+1} + C_n^{(n+1)} x^n + \dots\} - \tau_1 \{C_n^{(n)} x^n + \dots\} \\
 &\frac{\Phi_{2,n}}{C_n^{(n)}} \{(n+1)C_n^{(n)} x^n + \dots\} - \frac{\Phi_{3,n}}{C_n^{(n)}} \{C_n^{(n)} x^{n+1} + C_{n-1}^n x^n + \dots\} \\
 &\quad = \tau_2 \{C_{n+1}^{n+1} x^{n+1} + C_n^{(n+1)} x^n + \dots\} - \tau_2 \{C_n^{(n)} x^n + \dots\} \\
 &\frac{\Phi_{3,n}}{C_n^{(n)}} \{(n+1)C_n^{(n)} x^n + \dots\} - \frac{3\Phi_{2,n}}{C_n^{(n)}} \{C_n^{(n)} x^{n+1} + C_{n-1}^n x^n + \dots\} - \frac{\Phi_{1,n}}{C_n^{(n)}} \{C_n^{(n)} x^{n+1} + C_{n-1}^n x^n + \dots\} \\
 &\quad = \tau_3 \{C_{n+1}^{n+1} x^{n+1} + C_n^{(n+1)} x^n + \dots\} - \tau_3 \{C_n^{(n)} x^n + \dots\} \\
 &\quad - \frac{\Phi_{2,n}}{C_n^{(n)}} \cdot C_n^{(n)} x^{n+1} + \frac{1}{C_n^{(n)}} \{(n+1)\Phi_{1,n} C_n^{(n)} - \Phi_{2,n} C_{n-1}^{(n)}\} x^n = \tau_1 C_{n+1}^{(n+1)} x^{n+1} + \{\tau_1 C_n^{(n+1)} - \tau_1 C_n^{(n+1)}\} x^n + \dots \\
 &\quad - \frac{\Phi_{3,n}}{C_n^{(n)}} \cdot C_n^{(n)} x^{n+1} + \frac{1}{C_n^{(n)}} \{(n+1)\Phi_{1,n} C_n^{(n)} - \Phi_{2,n} C_{n-1}^{(n)}\} x^n = \tau_2 C_{n+1}^{(n+1)} x^{n+1} + \{\tau_2 C_n^{(n+1)} - \tau_2 C_n^{(n+1)}\} x^n + \dots \\
 &\quad - \frac{1}{C_n^{(n)}} \{3C_n^{(n)} \Phi_{3,n} - 3C_n^{(n)} \Phi_{2,n} + C_n^{(n)} \Phi_{1,n}\} x^{n+1} + \frac{1}{C_n^{(n)}} \{(n+1)\Phi_{3,n} C_n^{(n)} - 3\Phi_{3,n} C_n^n + 3\Phi_{2,n} C_{n-1}^n - 3\Phi_{1,n} C_n^n\} x^n \\
 &\quad = \tau_3 C_{n+1}^{(n+1)} x^{n+1} + \{\tau_3 C_n^{(n+1)} - \tau_3 C_n^{(n+1)}\} x^n + \dots
 \end{aligned}$$

We solve this system by forward elimination for $\Phi_{1,n}$, $\Phi_{2,n}$ and $\Phi_{3,n}$ using established relation (Adeniyi [1]):

$$C_n^{(n)} = 2^{2n-1}, C_{n-1}^{(n)} = -\frac{1}{2} n C_n^{(n)}, \frac{C_n^{(n+1)}}{C_{n+1}^{(n+1)}} = -\frac{1}{2} (n+1)$$

to obtain

$$\Phi_{1,n} = \frac{-2^{2n-1}[(2n^2 + n + 0.5)\tau_1 + (n - 0.5)\tau_2 - 0.5(n+1)\tau_3]}{2n^3 + 3n^2 + 1.5n + 0.25};$$

$$\Phi_{2,n} = \frac{-2^{2n-1}[0.5\tau_1 + (2n^2 + n - 1)\tau_2 - (n+1)\tau_3]}{2n^3 + 3n^2 + 1.5n + 0.25};$$

$$\Phi_{3,n} = \frac{-2^{2n-1}[(n+1)\tau_1 - (3n+2.5)\tau_2 - 2(n+1)^2\tau_3]}{2n^3 + 3n^2 + 1.5n + 0.25};$$

$$\text{Thus, } \varepsilon_1 = \max_{0 \leq x \leq 1} |(e_{1,n}(x))_{n+1}|, \varepsilon_2 = \max_{0 \leq x \leq 1} |(e_{2,n}(x))_{n+1}| \text{ and } \varepsilon_3 = \max_{0 \leq x \leq 1} |(e_{3,n}(x))_{n+1}|.$$

$$\varepsilon_1 = \frac{(2n^2 + n + 0.5)\tau_1 + (n - 0.5)\tau_2 - 0.5(n+1)\tau_3}{2n^3 + 3n^2 + 1.5n + 0.25};$$

$$\varepsilon_2 = \frac{0.5\tau_1 + (2n^2 + n - 1)\tau_2 - (n+1)\tau_3}{2n^3 + 3n^2 + 1.5n + 0.25};$$

$$\varepsilon_3 = \frac{(n+1)\tau_1 - (3n+2.5)\tau_2 - 2(n+1)^2\tau_3}{2n^3 + 3n^2 + 1.5n + 0.25}.$$

Example 3.2.2

We shall be concerned here with the problem

$$\begin{pmatrix} \frac{d}{dx} - 1 & 0 & 0 & 0 \\ 0 & \frac{d}{dx} - 1 & 0 & 0 \\ 0 & 0 & \frac{d}{dx} - 1 & 0 \\ 9 & 0 & 10 & \frac{d}{dx} \end{pmatrix} \begin{pmatrix} y \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad \begin{pmatrix} y(0) \\ u(0) \\ v(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 32 \\ 0 \end{pmatrix}$$

Following the same procedure as in problem 3.1, we arrive at

$$\varepsilon_1 = \frac{[8(n+1)^2(16n^4 + 64n^3 + 144n + 56)\tau_1 + 4(n+1)(16n^4 + 64n^3 + 136n^2 + 144n + 56)\tau_2 + (32n^4 + 128n^3 + 192n^2 + 128n + 32)\tau_3 + 16(n+1)^3\tau_4]}{8(n+1)^3(16n^4 + 64n^3 + 136n^2 + 144n + 65)};$$

$$\varepsilon_2 = \frac{[72(n+1)^2\tau_1 + 4(n+1)(16n^4 + 64n^3 + 136n^2 + 144n + 56)\tau_2 - (32n^4 + 128n^3 + 192n^2 + 128n + 32)\tau_3 + 16(n+1)^3\tau_4]}{4(n+1)^2(16n^4 + 64n^3 + 136n^2 + 144n + 65)};$$

$$\varepsilon_3 = \frac{[72(n+1)^2\tau_1 + 36(n+1)\tau_2 - (32n^4 + 128n^3 + 192n^2 + 128n + 32)\tau_3 + 16(n+1)^3\tau_4]}{2(n+1)^2(16n^4 + 64n^3 + 136n^2 + 144n + 65)};$$

$$\varepsilon_4 = \frac{[72(n+1)^2\tau_1 + 36(n+1)\tau_2 - (40n^2 + 80n + 49)\tau_3 + 16(n+1)^3\tau_4]}{(16n^4 + 64n^3 + 136n^2 + 144n + 65)}.$$

Example 3.2.3

The last of the problems to be considered is

$$\begin{pmatrix} \frac{d}{dx} - 1 & 0 & 0 \\ 0 & \frac{d}{dx} - 1 & 0 \\ 1 & -1 & \frac{d}{dx} - 1 \end{pmatrix} \begin{bmatrix} y \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \begin{bmatrix} y(0) \\ v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

The resulting estimates obtained for this problem by the same procedures employed for the previous examples, are:

$$\begin{aligned} \varepsilon_1 &= \frac{(4n^2 + 6n + 1)r_1 + (2n + 1)r_2 + r_3}{4n^3 + 10n^2 + 7n + 1.5}; \\ \varepsilon_2 &= \frac{-r_1 + (4n^2 + 6n + 1)r_2 + 2(n + 1)r_3}{4n^3 + 10n^2 + 7n + 1.5}; \\ \varepsilon_3 &= \frac{-2(n + 1)r_1 + (2n + 1)r_2 + 4(n + 1)^2 r_3}{4n^3 + 10n^2 + 7n + 1.5}. \end{aligned}$$

3.3 TAU APPROXIMANT BY THE INTEGRAL FORM

From the perturbed equation

$$\begin{aligned} \underline{Y}_n^1 &= A\underline{Y}_n + \underline{\tau}T_n(x), \\ \int_0^x \underline{Y}_n^1(u) du &= \int_0^x A\underline{Y}_n du + \underline{\tau}T_{n+m}(x), \quad (M = 1); \\ \underline{Y}_n(u)|_0^x &= \int_0^x A\underline{Y}_n du + \underline{\tau}T_{n+1}(x); \\ \underline{Y}_n(x) &= \int_0^x A\underline{Y}_n du + \underline{\tau}T_{n+1}(x) + \underline{Y}_n(0); \end{aligned}$$

we have

$$\begin{aligned} \int_0^x \underline{Y}_n^1(u) du &= \int_0^x A\underline{Y}_n du + \underline{\tau}T_{n+m}(x), \quad (M = 1) \\ \underline{Y}_n(u) \int_0^x &= \int_0^x A\underline{Y}_n du + \underline{\tau}T_{n+1}(x) \\ \underline{Y}_n(x) &= \int_0^x A\underline{Y}_n du + \underline{\tau}T_{n+1}(x) + \underline{Y}_n(0) \end{aligned}$$

This is written as

$$\begin{bmatrix} \sum_{r=0}^n b_r x^r \\ \sum_{r=0}^n b_r x^r \\ \sum_{r=0}^n d_r x^r \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \dots & -a_{m-1} & 0 \end{bmatrix} \begin{bmatrix} \sum_{r=0}^n \frac{b_r}{r+1} x^r \\ \sum_{r=0}^n \frac{C_r}{r+1} x^r \\ \sum_{r=0}^n \frac{d_r}{r+1} x^r \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} + \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \vdots \\ \vdots \\ \tau_m \end{bmatrix} \begin{bmatrix} T_{n+1}(x) \\ T_{n+1}(x) \\ T_{n+1}(x) \\ \vdots \\ \vdots \\ T_{n+1}(x) \end{bmatrix} + \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_m \end{bmatrix}$$

The values of the unknowns $b_r, C_r, d_r, \dots, (r = 0(1)n)$ and the free tau parameters, $\tau_r, r = 0(1)m$, are obtained from the initial values and by solving the system of equations resulting from equating corresponding coefficients of powers of x .

3.4. ERROR ESTIMATION OF THE INTEGRATED FORM

From 1.7, we obtain the error estimate of the problems in the examples of sub-section 3.2 for the tau integrated form.

Example 3.4.1

We consider again the third order IVP

$$\begin{bmatrix} \frac{d}{dx} - 1 & 0 & 0 \\ 0 & \frac{d}{dx} - 1 & 0 \\ -1 & 3 & \frac{d}{dx} - 3 \end{bmatrix} \begin{bmatrix} y \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} y(0) \\ v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 10 \end{bmatrix}$$

The error approximants $(e_{1,n}(x))_{n+1}, (e_{2,n}(x))_{n+1}, (e_{3,n}(x))_{n+1}$ satisfy

$$(e_{1,n}^1(x))_{n+1} - (e_{2,n}(x))_{n+1} = \tau_1 T_{n+1}^*(x) - \tau_1 T_n^*(x)$$

$$(e_{2,n}^1(x))_{n+1} - (e_{3,n}(x))_{n+1} = \tau_2 T_{n+1}^*(x) - \tau_2 T_n^*(x)$$

$$(e_{3,n}^1(x))_{n+1} - 3(e_{3,n}(x))_{n+1} + (e_{1,n}(x))_{n+1} - 3(e_{2,n}(x))_{n+1} = \tau_3 T_{n+1}^*(x) - \tau_3 T_n^*(x)$$

$$(e_{1,n}(x))_{n+1} = \frac{\Phi_{1,n} T_n^*(x)}{2^{2n-1}} = \frac{\Phi_{1,n}}{2^{2n-1}} \left\{ C_n^{(n)} x^{n+1} + C_{n-1}^{(n)} x^n + \dots \right\}$$

$$\int_0^x e_{1,n}(u) du = \frac{\Phi_{1,n}}{C_n^{(n)}} \left\{ \frac{C_n^{(n)}}{n+2} x^{n+2} + \frac{C_{n-1}^{(n)}}{n+1} x^{n+1} + \dots \right\}$$

$$\int_0^x e_{2,n}(u) du = \frac{\Phi_{2,n}}{C_n^{(n)}} \left\{ \frac{C_n^{(n)}}{n+2} x^{n+2} + \frac{C_{n-1}^{(n)}}{n+1} x^{n+1} + \dots \right\}$$

$$\int_0^x e_{3,n}(u) du = \frac{\Phi_{3,n}}{C_n^{(n)}} \left\{ \frac{C_n^{(n)}}{n+2} x^{n+2} + \frac{C_{n-1}^{(n)}}{n+1} x^{n+1} + \dots \right\}$$

$$\int_0^x e_{1,n}^1(u) du = \frac{\Phi_{1,n}}{C_n^{(n)}} \left\{ C_n^{(n)} x^{n+1} + C_{n-1}^{(n)} x^n + \dots \right\}$$

$$\int_0^x e_{2,n}^1(u) du = \frac{\Phi_{2,n}}{C_n^{(n)}} \left\{ C_n^{(n)} x^{n+1} + C_{n-1}^{(n)} x^n + \dots \right\}$$

$$\int_0^x e_{3,n}^1(u) du = \frac{\Phi_{3,n}}{C_n^{(n)}} \left\{ C_n^{(n)} x^{n+1} + C_{n-1}^{(n)} x^n + \dots \right\}$$

$$\int_0^x e_{1,n}^1(u)_{n+1} du - \int_0^x e_{2,n}^1(u)_{n+1} du = \bar{\tau}_1 \left\{ C_{n+2}^{(n+1)} + C_{n-1}^{(n+2)} + \dots \right\} - \tau_1 \int_0^x \sum_{r=0}^n C_r^{(n)} u^r du$$

$$\int_0^x e_{2,n}^1(u)_{n+1} du - \int_0^x e_{3,n}^1(u)_{n+1} du = \bar{\tau}_2 \left\{ C_{n+2}^{(n+1)} + C_{n-1}^{(n+2)} + \dots \right\} - \tau_2 \int_0^x \sum_{r=0}^n C_r^{(n)} u^r du$$

$$\begin{aligned} \int_0^x e_{3,n}^1(u)_{n+1} du + 3 \int_0^x e_{2,n}(u)_{n+1} du - \int_0^x e_{1,n}(u)_{n+1} du + 3 \int_0^x e_{3,n}(u)_{n+1} du \\ = \bar{\tau}_3 \left\{ C_{n+2}^{(n+1)} + C_{n-1}^{(n+2)} + \dots \right\} - \tau_3 \int_0^x \sum_{r=0}^n C_r^{(n)} u^r du \end{aligned}$$

Expanding the above system of equations and comparing the coefficients of various powers of x leads to

$$\varepsilon_1 = \frac{(2n^2 + n + 0.5)\tau_1 + (n - 0.5)\tau_2 - 0.5(n+1)\tau_3}{2n^3 + 3n^2 + 1.5n + 0.25};$$

$$\varepsilon_2 = \frac{0.5\tau_1 + (2n^2 + n - 1)\tau_2 - (n+1)\tau_3}{2n^3 + 3n^2 + 1.5n + 0.25};$$

$$\varepsilon_3 = \frac{(n+1)\tau_1 + (3n + 2.5)\tau_2 - 2(n+1)^2\tau_3}{2n^3 + 3n^2 + 1.5n + 0.25}.$$

Example 3.4.2

We consider here again problem

$$\begin{bmatrix} \frac{d}{dx}-1 & 0 & 0 & 0 \\ 0 & \frac{d}{dx}-1 & 0 & 0 \\ 0 & 0 & \frac{d}{dx}-1 & 0 \\ 9 & 0 & 10 & \frac{d}{dx} \end{bmatrix} \begin{bmatrix} y \\ u \\ u \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} y(0) \\ u(0) \\ v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 32 \\ 0 \end{bmatrix}$$

The error approximants $(e_{1,n}(x))_{n+1}$, $(e_{2,n}(x))_{n+1}$ and $(e_{3,n}(x))_{n+1}$ and satisfy

$$(e_{1,n}^1(x))_{n+1} - (e_{1,n}(x))_{n+1} = \bar{\tau}_1 T_{n+1}^*(x) - \tau_1 T_n^*(x)$$

$$(e_{2,n}^1(x))_{n+1} - (e_{2,n}(x))_{n+1} = \bar{\tau}_2 T_{n+1}^*(x) - \tau_2 T_n^*(x)$$

$$(e_{3,n}^1(x))_{n+1} - (e_{3,n}(x))_{n+1} = \bar{\tau}_3 T_{n+1}^*(x) - \tau_3 T_n^*(x)$$

$$(e_{4,n}^1(x))_{n+1} - 10(e_{3,n}(x))_{n+1} + 9(e_{1,n}(x))_{n+1} = \bar{\tau}_4 T_{n+1}^*(x) - \tau_4 T_n^*(x)$$

Following the same procedure as in sector 3.2, we have

$$\begin{aligned} & [-8(n+1)^2(16n^4 + 64n^3 + 136n^2 + 144n + 56)\tau_1 - 4(n+1)(16n^4 + 64n^3 + 136n^2 + 144n + 56)\tau_2 \\ & - (32n^4 + 128n^3 + 192n^2 + 128n + 32)\tau_3 + 16(n+1)^3\tau_4] \\ \varepsilon_1 = & \frac{}{8(n+1)^3(16n^4 + 64n^3 + 136n^2 + 144n + 65)} \end{aligned}$$

$$\begin{aligned} & [72(n+1)^2\tau_1 - (32n^4 + 128n^3)(16n^4 + 64n^3 + 136n^2 + 144n + 56)\tau_2 \\ & + (32n^4 + 128n^3 + 192n^2 + 128n + 32)\tau_3 + 16(n+1)^3\tau_4] \\ \varepsilon_2 = & \frac{}{4(n+1)^2(16n^4 + 64n^3 + 136n^2 + 144n + 65)} \end{aligned}$$

$$\varepsilon_3 = \frac{[72(n+1)^2\tau_1 + 36(n+1)\tau_2 - (32n^4 + 128n^3 + 192n^2 + 128n + 32)\tau_3 - 16(n+1)^3\tau_4]}{2(n+1)(16n^4 + 64n^3 + 136n^2 + 144n + 65)}$$

$$\varepsilon_4 = \frac{72(n+1)^2\tau_1 + 36(n+1)\tau_2 + 2(40n^2 + 80n + 49)\tau_3 - 16(n+1)^2\tau_4}{16n^4 + 64n^3 + 136n^2 + 144n + 65}$$

Example 34.3

We revisit here the problem

$$\begin{bmatrix} \frac{d}{dx}-1 & 0 & 0 \\ 0 & \frac{d}{dx}-1 & 0 \\ 1 & -1 & \frac{d}{dx}-1 \end{bmatrix} \begin{bmatrix} y \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} y(0) \\ v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

The error approximants $(e_{1,n}(x))_{n+1}$, $(e_{2,n}(x))_{n+1}$ and $(e_{3,n}(x))_{n+1}$ satisfy

$$(e_{1,n}^1(x))_{n+1} - (e_{2,n}(x))_{n+1} = \tau_1 T_{n+1}^*(x) - \tau_1 T_n^*(x)$$

$$(e_{1,n}^1(x))_{n+1} - (e_{2,n}(x))_{n+1} = \tau_2 T_{n+1}^*(x) - \tau_2 T_n^*(x)$$

$$(e_{3,n}^1(x))_{n+1} - (e_{3,n}(x))_{n+1} - (e_{2,n}(x))_{n+1} + (e_{1,n}(x))_{n+1} = \tau_3 T_{n+1}^*(x) - \tau_3 T_n^*(x)$$

The results obtained for this problem are:

$$\varepsilon_1 = \frac{(4n^2 + 6n + 1)\tau_1 + (2n + 1)\tau_2 + \tau_3}{4n^3 + 10n^2 + 7n + 1.5};$$

$$\varepsilon_2 = \frac{-\tau_1 + (4n^2 + 6n + 2)\tau_2 + 2(n + 1)\tau_3}{4n^3 + 10n^2 + 7n + 1.5};$$

$$\varepsilon_3 = \frac{-2(n + 1)\tau_1 + (2n + 1)\tau_2 + 4(n + 1)^2 \tau_3}{4n^3 + 10n^2 + 7n + 1.5};$$

4. NUMERICAL EXPERIMENTS

The three systems of initial value problems presented below are used to experiment the entire discussions in this paper. The numerical results obtained are presented in Tables 4.1 – 4.4.

Problem 4.1

Consider the system

$$\begin{bmatrix} \frac{d}{dx} - 1 & 0 & 0 \\ 0 & \frac{d}{dx} - 1 & 0 \\ -1 & 3 & \frac{d}{dx} - 3 \end{bmatrix} \begin{bmatrix} y \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} y(0) \\ v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 10 \end{bmatrix};$$

Problem 4.2

Consider the system

$$\begin{bmatrix} \frac{d}{dx} - 1 & 0 & 0 & 0 \\ 0 & \frac{d}{dx} - 1 & 0 & 0 \\ 0 & 0 & \frac{d}{dx} - 1 & 0 \\ 9 & 0 & 10 & \frac{d}{dx} \end{bmatrix} \begin{bmatrix} y \\ u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} y(0) \\ y(0) \\ v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3.2 \\ 0 \end{bmatrix};$$

Problem 4.3

Consider the system

$$\begin{bmatrix} \frac{d}{dx} - 1 & 0 & 0 \\ 0 & \frac{d}{dx} - 1 & 0 \\ 1 & -1 & \frac{d}{dx} - 1 \end{bmatrix} \begin{bmatrix} y \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} y(0) \\ v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix};$$

The analytic solutions of the three problems are shown below.

Table 4.1: Analytic Solutions of Problems 4.1 – 4.3

Problem 4.1	Problem 4.2	Problem 4.3
$Y(x) = 2e^x(1+2x^2)$	$Y(x) = 4\cos x - 4\cos 3x$	$Y(x) = (2-x)e^x$
$V(x) = 2e^x(1+4x+2x^2)$	$U(x) = 4\sin x + 12\sin 3x$ $V(x) = 4\cos x + 36\cos 3x$	$V(x) = (1-x)e^x$
$W(x) = 2e^x(5+8x+2x^2)$	$W(x) = 4\sin x - 108\sin 3x$	$W(x) = -xe^x$

Table 4.2: Error and Error Estimates for Problem 4.1

		Differential Form		Integrated form	
		Error	Error Estimates	Error	Error Estimates
n = 8	Y	3.14×10^{-8}	4.68×10^{-8}	5.05×10^{-8}	1.30×10^{-9}
	V	3.86×10^{-8}	5.16×10^{-8}	6.11×10^{-8}	1.43×10^{-9}
	W	4.86×10^{-8}	8.91×10^{-8}	7.29×10^{-8}	2.47×10^{-9}
n = 9	Y	9.39×10^{-8}	1.44×10^{-9}	1.48×10^{-9}	3.59×10^{-11}
	V	1.13×10^{-9}	1.58×10^{-9}	1.77×10^{-9}	3.94×10^{-11}
	W	1.34×10^{-9}	2.08×10^{-9}	2.08×10^{-9}	6.44×10^{-9}
n = 10	Y	2.55×10^{-11}	3.94×10^{-11}	4.17×10^{-11}	8.95×10^{-9}
	V	3.03×10^{-11}	4.30×10^{-11}	4.85×10^{-11}	9.76×10^{-12}
	W	3.54×10^{-11}	6.72×10^{-11}	5.63×10^{-11}	1.52×10^{-12}
n = 11	Y	6.13×10^{-13}	9.74×10^{-13}	9.87×10^{-13}	2.03×10^{-14}
	V	7.18×10^{-13}	1.06×10^{-12}	1.14×10^{-12}	2.20×10^{-14}
	W	8.33×10^{-13}	1.59×10^{-12}	1.32×10^{-12}	3.31×10^{-14}

Table 4.3: Error and Error Estimates for Problem 4.2

		Differential Form		Integrated Form	
		Error	Error Estimates	Error	Error Estimates
n = 8	Y	3.71×10^{-6}	6.00×10^{-6}	3.08×10^{-6}	1.68×10^{-7}
	U	3.60×10^{-6}	1.70×10^{-6}	5.03×10^{-6}	4.79×10^{-8}
	V	3.34×10^{-5}	5.40×10^{-5}	2.78×10^{-5}	1.51×10^{-6}
	W	3.24×10^{-5}	3.79×10^{-5}	4.52×10^{-5}	1.06×10^{-6}
n = 9	Y	7.64×10^{-8}	3.57×10^{-8}	1.35×10^{-7}	9.00×10^{-10}
	U	8.02×10^{-7}	1.37×10^{-6}	6.91×10^{-7}	3.45×10^{-8}
	V	6.88×10^{-7}	3.22×10^{-7}	1.21×10^{-6}	8.10×10^{-9}
	W	7.21×10^{-6}	1.20×10^{-5}	6.22×10^{-6}	3.02×10^{-7}
n = 10	Y	1.76×10^{-8}	3.14×10^{-8}	1.60×10^{-8}	7.19×10^{-10}
	U	1.32×10^{-8}	6.11×10^{-9}	2.77×10^{-8}	1.40×10^{-10}
	V	1.58×10^{-7}	2.83×10^{-7}	1.44×10^{-7}	6.47×10^{-9}
	W	1.18×10^{-7}	1.74×10^{-7}	2.49×10^{-7}	3.98×10^{-9}
n = 11	Y	2.71×10^{-10}	1.06×10^{-10}	5.60×10^{-10}	2.22×10^{-12}
	U	3.39×10^{-9}	5.95×10^{-9}	2.98×10^{-9}	1.25×10^{-10}
	V	2.43×10^{-9}	9.56×10^{-10}	5.04×10^{-9}	2.00×10^{-11}
	W				

Table 4.4: Error and Error Estimates for Problem 4. 4

		Differential Form		Integrated Form	
		Error	Error Estimates	Error	Error Estimates
n = 8	Y	7.13×10^{-10}	1.12×10^{-9}	1.08×10^{-9}	3.11×10^{-11}
	V	8.07×10^{-10}	1.27×10^{-9}	1.21×10^{-9}	3.52×10^{-11}
	W	9.01×10^{-10}	1.42×10^{-9}	1.34×10^{-9}	3.94×10^{-11}
n = 9	Y	1.95×10^{-11}	3.15×10^{-11}	2.93×10^{-11}	7.87×10^{-13}
	V	2.18×10^{-11}	3.52×10^{-11}	3.25×10^{-11}	8.79×10^{-13}
	W	2.41×10^{-11}	3.89×10^{-11}	3.56×10^{-11}	9.71×10^{-13}
n = 10	Y	4.93×10^{-13}	7.95×10^{-13}	7.57×10^{-13}	1.80×10^{-14}
	V	5.44×10^{-13}	8.78×10^{-13}	8.31×10^{-13}	1.99×10^{-14}
	W	5.96×10^{-13}	9.61×10^{-13}	9.04×10^{-13}	2.18×10^{-14}
n = 11	Y	1.10×10^{-14}	1.82×10^{-14}	1.65×10^{-14}	4.15×10^{-16}
	V	1.21×10^{-14}	1.99×10^{-14}	1.82×10^{-14}	4.54×10^{-16}
	W	1.31×10^{-14}	2.16×10^{-14}	1.97×10^{-14}	4.94×10^{-16}

5. CONCLUSION

Two variants of the tau method and their corresponding error estimates for first order systems of IVPs in ordinary differential equations have been presented. For the two variants, numerical results showed that the order of the tau approximant is closely estimated, thus, showing the effectiveness of the method.

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