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NATURAL VIBRATION ANALYSIS OF A TAPERED ROTATING PRESTRESSED RAYLEIGH BEAM USING DIFFERENTIAL TRANSFORM METHOD

¹Gbadeyan, J. A. and ¹Olotu, O. T.

Department of Mathematics, University of Ilorin, PMB. 1515, Ilorin, Kwara State, Nigeria

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ABSTRACT

This study aims at carrying out the vibration analysis of a tapered rotating cantilever initially stressed Rayleigh beam. The governing partial differential equations of motion of a rotating tapered Rayleigh beam are reduced to ordinary differential equations using separation of variables. A semi-analytical approach known as Differential Transform Method (DTM) is then applied to the non-dimensional form of the governing equations of motion and a set of recursive algebraic equations are determined. Performing some direct algebraic operations on these derived equations and using some computer codes developed and implemented in MAPLE 18, the dimensionless natural frequencies and the associated mode shapes of the beam are obtained. In the illustrated example presented, the modulus of elasticity is assumed to vary along the beam's length while the mass density is kept constant. The effects of rotational speed, inverse of slenderness ratio and taper ratio on the dimensionless natural frequencies are also investigated. The results obtained are validated using six relevant existing studies and are found to compare favourably well with those in the open literature. It is found among others that (i) increasing the rotational speed and the inverse of slenderness ratio of the tapered rotating Rayleigh beam lead to an increase and a decrease respectively in its natural frequencies; and (ii) increasing the tapered ratio of the beam yields, generally, a decrease in its natural frequencies. Finally, it is shown that the convergence of the DTM for the governing ordinary differential equation holds.

Correspondence:
j.agbadeyan@yahoo.com; +2348035785943

1.0 INTRODUCTION

Beams are basic components in most of the structural systems conceived, designed and constructed in engineering. There are numerous engineering applications of rotating cantilever beams. Many engineering components that can be modeled as cantilever beams rotating about a fixed axis include turbine tapered blades, helicopter tapered blades, aircraft tapered propellers, and electrical concrete poles for which the determination of the natural frequencies and

mode shapes play an important role in the mechanical behaviour and structural reliability of these components. This is why it is of vital importance for the design engineers to evaluate the dynamic characteristics of beam structures effectively. Preliminary vibration analysis of beam structures is carried out to optimize the design, detect the early precursors to machine failure and avoid investments on repairs.

Beam-type structures find wide application in many branches of engineering. For this reason,

there have been distinguished research works on free and forced vibrations of uniform beams with different support conditions Gbadeyan and Hammed (2017), Gbadeyan *et al.* (2019).

Engineering problems are usually described by ordinary and partial differential equations. The vibration problems of tapered beams have been studied by some researchers. Most of these problems are difficult to solve analytically because of some complicated theoretical derivations that may be required. Alternatively, the numerical methods provide approximate solution rather than analytical solutions of the problems.

The behaviour of beam has been investigated by some researchers in the past using the differential transform method. Chen and Ho (1998) studied the analysis of general elastically end restrained non-uniform Euler-Bernoulli beam using differential transform method. Jackson and Oyadiji (2009) used the DTM to solve the free vibration problem of a rotating tapered beam based on the Bresse-Rayleigh beam theory. The accuracy of the Rayleigh beam theory with respect to Euler-Bernoulli hypotheses was clearly illustrated. Furthermore, in their work, both modulus of elasticity and mass density were kept constant.

Banerjee and Jackson (2013) studied the free vibration of rotating tapered Rayleigh beam using a dynamic stiffness method of solution. Auciello and Lippiello (2013) studied the vibration analysis of rotating non-uniform tapered Rayleigh beam using the 'Cell Discretization Method'. Sarkar *et al.* (2014) applied an inverse problem approach to obtain closed form solutions of the free vibration of non-uniform rotating Rayleigh cantilever beams. Li *et al.* (2015) investigated the flapwise bending vibration of a rotating tapered Rayleigh beam using Hamilton's principle and the integral equation method to

determine the natural frequencies. Chen *et al.* (2015) re-examined the free vibration of rotating tapered Timoshenko beams using the technique of variational iteration where the natural frequencies and mode shapes of the beams with linearly varying height as well as linearly varying height and width were investigated.

Oke (2017) examined the convergence of the DTM for ordinary differential equations. He established that if the solution of an ordinary differential equation can be written as Taylors' expansion, then the solution can be obtained using the DTM. Adair and Jaeger (2017) applied a systematic procedure developed for examining the dynamic behaviour of a rotating tapered Euler-Bernoulli beam with an elastically constrained root. In their work, the equation describing the motion of the beam was written as a first-order matrix differential equation and later solved using the power series method. Nourifar *et al.* (2018) employed both the differential transform method and the finite element method to investigate the free vibration analysis of non-uniform rotating Euler-Bernoulli beam with an exponentially decaying circular cross-section in which comparison were made between the two methods used. Ganguli and Bhat (2018) analyzed the spectral equivalence between certain non-uniform rotating Rayleigh beams and a given uniform Rayleigh beam, and also between non-uniform non-rotating Rayleigh beams and a given uniform rotating Rayleigh beam using the finite element method. Lee and Lee (2020) developed a method called Transfer Matrix Method (TMM) to analyze the effect of centrifugal force on the natural frequencies of a rotating double-tapered beam whose cross-sections have linearly reduced height and width.

In this work, the vibration problem of rotating tapered prestressed Rayleigh beam is studied. The transformation called differential transform is used to solve the problem. Thus, the purpose of

this paper is also to explore further the efficacy of the differential transform method in the solution of vibrational problems. However, unlike the work of Jackson and Oyadiji (2009), this present paper considered, for the purpose of numerical implementation, a model for which only the mass density per unit length remains constant while the modulus of elasticity is a function of coordinate x . The governing differential equation of the vibrating beam is a fourth order differential equation. Using the differential transform technique, the governing differential equation became a recursive algebraic equation and the boundary conditions became simple algebraic

equations which were suitable for symbolic computer implementation. Moreover, after some simple algebraic operations on these equations, any ith natural frequency, the closed formed series solution of any ith mode shape and the dynamic response can be obtained.

2.0 MATHEMATICAL FORMULATION OF THE PROBLEM

Considering a tapered Rayleigh beam which is rotating about a vertical axis with a uniform rotation speed Ω . The governing equation of motion of this tapered prestressed Rayleigh beam of finite length l is given by the fourth order partial differential equation [Li *et al.* (2015)]

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \left[E(x)I(x) \frac{\partial^2 W(x,t)}{\partial x^2} \right] + \rho(x)A(x) \frac{\partial^2 W(x,t)}{\partial t^2} - \frac{\partial}{\partial x} \left[N(x) \frac{\partial W(x,t)}{\partial x} \right] \\ - \frac{\partial}{\partial x} \left[\rho(x)I(x) \frac{\partial^3 W(x,t)}{\partial x \partial t^2} \right] + \Omega^2 \frac{\partial}{\partial x} \left[\rho(x)I(x) \frac{\partial W(x,t)}{\partial x} \right] = 0, \quad x \in (0, l) \end{aligned} \quad (1)$$

where $W(x,t)$ represents the dynamic response of the beam, $E(x)$ is Young's modulus, $I(x)$ is moment of inertia, $\rho(x)$ is the mass per unit volume of the beam, $A(x)$ is the variable cross-sectional area of the beam, x is the coordinate along the longitudinal axis, t is the time. $N(x)$ is the arbitrary axial tensile excitation force. If the beam is subjected to an axial end force N_0 , then $N(x)$ is a constant function where $N(x) = N_0$. However, if the beam is subjected to distributed axial force $g(x)$, then $N(x) = \int_x^l g(\eta) d\eta$.

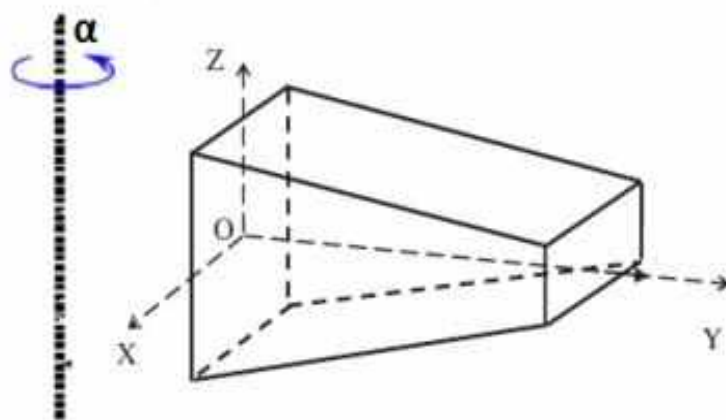


Figure 1: Model of a tapered beam system

The initial conditions are

$$W(x, 0) = W_0(x) \quad \text{and} \quad \frac{\partial W(x, 0)}{\partial t} = \dot{W}_0(x) \quad (2)$$

The boundary conditions are

At $x = 0$,

$$\frac{\partial W(x, t)}{\partial x} = 0 \quad (3)$$

$$W(x, t) = 0 \quad (4)$$

At $x = l$,

$$E(x)I(x) \frac{\partial^2 W(x, t)}{\partial x^2} = 0 \quad (5)$$

$$\frac{\partial}{\partial x} \left[E(x)I(x) \frac{\partial^2 W(x, t)}{\partial x^2} \right] - N(x) \frac{\partial W(x, t)}{\partial x} - \rho(x)I(x) \frac{\partial^3 W(x, t)}{\partial x \partial t^2} + \Omega^2 \rho(x)I(x) \frac{\partial W(x, t)}{\partial x} = 0 \quad (6)$$

For free vibration, we assume that the response is of the form

$$W(x, t) = Y(x)(a_1 \cos \omega t + a_2 \sin \omega t) \quad (7)$$

Substituting equation (7) into equation (1) leads to

$$\begin{aligned} & \frac{d^2}{dx^2} \left[E(x)I(x) \frac{d^2 Y(x)}{dx^2} \right] - \rho(x)A(x)\omega^2 Y(x) - \frac{d}{dx} \left[N(x) \frac{dY(x)}{dx} \right] + \frac{d}{dx} \left[\rho(x)I(x)\omega^2 \frac{dY(x)}{dx} \right] \\ & + \Omega^2 \frac{d}{dx} \left[\rho(x)I(x) \frac{dY(x)}{dx} \right] = 0, \quad x \in (0, l) \end{aligned} \quad (8)$$

Also, using equation (7), equations (3) - (6) are expressed as follows:

At $x = 0$

$$\frac{dY(x)}{dx} = 0 \quad (9)$$

$$Y(x) = 0 \quad (10)$$

At $x = l$,

$$E(x)I(x) \frac{d^2 Y(x)}{dx^2} = 0 \quad (11)$$

$$\frac{d}{dx} \left[E(x)I(x) \frac{d^2 Y(x)}{dx^2} \right] - [N(x) - \rho(x)I(x)\omega^2 - \rho(x)I(x)\Omega^2] \frac{dY(x)}{dx} = 0 \quad (12)$$

The following non-dimensional parameters are used:

$$\left. \begin{aligned} \xi &= \frac{x}{l} \\ b(\xi) &= \frac{E(x)I(x)}{E(0)I(0)} \\ q(\xi) &= \frac{\rho(x)A(x)}{\rho(0)A(0)} \\ f(\xi) &= \frac{\rho(x)I(x)}{\rho(0)I(0)} \\ \gamma &= \frac{1}{l} \sqrt{\frac{I(0)}{A(0)}} \\ \Lambda^2 &= \frac{\rho(0)A(0)\omega^2 l^4}{E(0)I(0)} \end{aligned} \right\} \begin{aligned} y(\xi) &= \frac{Y(x)}{l} \\ n(\xi) &= -\frac{N(x)l^2}{E(0)I(0)} \\ \tau_R &= \left. \frac{b'(\xi)}{b(\xi)} \right|_{\xi=1} \\ \eta_R &= \frac{N(l)l^2}{E(l)I(l)} \\ \psi_R &= \left. \frac{f(\xi)}{b(\xi)} \right|_{\xi=1} \\ \alpha^2 &= \frac{\rho(0)A(0)\Omega^2 l^4}{E(0)I(0)} \end{aligned} \quad (13)$$

The governing differential equation (8) and the boundary conditions in equations (9) - (12) are written in the following non-dimensional forms as:

$$\begin{aligned} b(\xi) \frac{d^4 y(\xi)}{d\xi^4} + 2 \frac{db(\xi)}{d\xi} \cdot \frac{d^3 y(\xi)}{d\xi^3} + \frac{d^2 b(\xi)}{d\xi^2} \cdot \frac{d^2 y(\xi)}{d\xi^2} + \frac{d^2 y(\xi)}{d\xi^2} + \frac{dn(\xi)}{d\xi} \cdot \frac{dy(\xi)}{d\xi} \\ + \gamma^2 \left[f(\xi) \frac{d^2 y(\xi)}{d\xi^2} + \frac{df(\xi)}{d\xi} \cdot \frac{dy(\xi)}{d\xi} \right] (\Lambda^2 + \alpha^2) - \Lambda^2 q(\xi) y(\xi) = 0 \end{aligned} \quad (14)$$

At $\xi = 0$,

$$\frac{dy(\xi)}{d\xi} = 0 \quad (15)$$

$$y(\xi) = 0 \quad (16)$$

At $\xi = 1$,

$$\frac{d^2 y(\xi)}{d\xi^2} = 0 \quad (17)$$

$$\frac{d^3 y(\xi)}{d\xi^3} + \tau_R \frac{d^2 y(\xi)}{d\xi^2} - [\eta_R - \Lambda^2 \gamma^2 \psi_R - \alpha^2 \gamma^2 \psi_R] \frac{dy(\xi)}{d\xi} = 0 \quad (18)$$

3.0 METHOD OF SOLUTION

3.1 Differential Transform Method (DTM)

The DTM is a transformation technique based on the Taylor series expansion and is a useful tool to obtain analytical solutions of differential equations. This method was first proposed by Zhou (1986) for solving both linear and nonlinear initial value problems of electrical circuits. In this method, certain transformation rules are applied and the governing differential equations and the boundary conditions of the system are transformed into a set of algebraic equations in terms of the differential transforms of the original functions. The solution of these algebraic equations gives the desired solution of the

problem. It is different from high-order Taylor series expansions because Taylor series method requires symbolic computation of the necessary derivatives of the data functions and is expensive for large orders. The DTM provides an iterative procedure to obtain higher-order. This method gives an analytic solution in the form of a polynomial. Application of DTM leads to accurate results with fast convergence rate and small computational effort.

Basic definitions and operations of differential transform method are introduced as follows:

A function $x(t)$, analytical in domain D , can be represented by a power series around any arbitrary point in this domain. The differential transform of a function $x(t)$ is defined as:

$$\bar{X}(k) = \frac{1}{k!} \left[\frac{d^k x(t)}{dt^k} \right]_{t=0} \quad (19)$$

In equation (19), $x(t)$ is the original function and $\bar{X}(k)$ is the transformed function. The inverse differential transform of $\bar{X}(k)$ is defined as

$$x(t) = \sum_{k=0}^{\infty} \bar{X}(k) t^k \quad (20)$$

Combining equations (19) and (20), we have

$$x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left[\frac{d^k x(t)}{dt^k} \right]_{t=0} \quad (21)$$

which is the Taylor series of $x(t)$ at $t=0$. Equation (21) implies that the concept of differential transformation is derived from the Taylor series expansion. In practical applications, the function $x(t)$ is expressed by a finite series and equation (21) is written as

$$x(t) = \sum_{k=0}^m \bar{X}(k) t^k \quad (22)$$

which implies that

$$x(t) = \sum_{k=m+1}^{\infty} \bar{X}(k) t^k \quad (23)$$

is negligibly small.

In this study, the value of m depends on the convergence of the natural frequencies. In addition, we use the lower-case letters to represents the original functions and a bar over

the upper-case letters to stand for the transformed functions (T-functions). Table 1 contains some relevant basic operations for DTM.

Table 1: The Fundamental Operations for Differential Transform Method

Original Function	Transformed Function
$w(x) = u(x) \pm v(x)$	$\bar{W}(k) = \bar{U}(k) \pm \bar{V}(k)$
$w(x) = \lambda u(x)$	$\bar{W}(k) = \lambda \bar{U}(k)$, λ is a constant
$w(x) = x^r$	$\bar{W}(k) = \delta(k-r) = \begin{cases} 1 & \text{if } k = r \\ 0 & \text{if } k \neq r \end{cases}$
$w(x) = \frac{du(x)}{dx}$	$\bar{W}(k) = (k+1)\bar{U}(k+1)$
$w(x) = \frac{d^r u(x)}{dx^r}$	$\bar{W}(k) = (k+1)(k+2) \dots (k+r)\bar{U}(k+r)$
$w(x) = u(x)v(x)$	$\bar{W}(k) = \sum_{r=0}^k \bar{U}(r)\bar{V}(k-r) = \sum_{r=0}^k \bar{U}(k-r)\bar{V}(r)$
$w(x) = \frac{du(x)}{dx} \frac{dv(x)}{dx}$	$\bar{W}(k) = \sum_{r=0}^k (r+1)\bar{U}(r+1)(k-r+1)\bar{V}(k-r+1)$
$w(x) = u(x) \frac{dv(x)}{dx}$	$\bar{W}(k) = \sum_{r=0}^k \bar{U}(r)(k-r+1)\bar{V}(k-r+1)$
$w(x) = u(x) \frac{d^2 v(x)}{dx^2}$	$\bar{W}(k) = \sum_{r=0}^k \bar{U}(r)(k-r+1)(k-r+2)\bar{V}(k-r+2)$
$w(x) = u(x) \frac{dv(x)}{dx} \frac{dz(x)}{dx}$	$\bar{W}(k) = \sum_{r=0}^k \sum_{t=0}^{k-r} \bar{U}(r)(t+1)\bar{V}(t+1) \times (k-r-t+1)\bar{Z}(k-r-t+1)$
$w(x) = u(x) \frac{dv(x)}{dx} \frac{d^2 z(x)}{dx^2}$	$\bar{W}(k) = \sum_{r=0}^k \sum_{t=0}^{k-r} \bar{U}(r)(t+1)\bar{V}(t+1) \times (k-r-t+1)(k-r-t+2)\bar{Z}(k-r-t+2)$
$w(x) = u(x)v(x) \frac{dz(x)}{dx}$	$\bar{W}(k) = \sum_{r=0}^k \sum_{t=0}^{k-r} \bar{U}(r)\bar{V}(t) \times (k-r-t+1)\bar{Z}(k-r-t+1)$
$w(x) = u(x)v(x)z(x)$	$\bar{W}(k) = \sum_{r=0}^k \sum_{t=0}^{k-r} \bar{U}(r)\bar{V}(t)\bar{Z}(k-r-t)$

3.2 Application of Differential Transform Method

Defining $\lambda = \Lambda^2$, taking the differential transform of equation (14), the following recursive algebraic relation is obtained:

$$\begin{aligned}
 & \sum_{r=0}^k \bar{B}(k-r)(r+1)(r+2)(r+3)(r+4)\bar{Y}(r+4) + 2 \sum_{r=0}^k (k-r+1)\bar{B}(k-r+1) \\
 & (r+1)(r+2)(r+3)\bar{Y}(r+3) + \sum_{r=0}^k (k-r+1)(k-r+2)\bar{B}(k-r+2)(r+1)(r+2)\bar{Y}(r+2) \\
 & + \sum_{r=0}^k \bar{N}(k-r)(r+1)(r+2)\bar{Y}(r+2) + \sum_{r=0}^k (k-r+1)\bar{N}(k-r+1)(r+1)\bar{Y}(r+1) \\
 & + \gamma^2(\lambda + \alpha^2) \sum_{r=0}^k \bar{F}(k-r)(r+1)(r+2)\bar{Y}(r+2) + \gamma^2(\lambda + \alpha^2) \sum_{r=0}^k (k-r+1)\bar{F}(k-r+1)(r+1)\bar{Y}(r+1) \\
 & = \lambda \sum_{r=0}^k \bar{Q}(k-r)\bar{Y}(r)
 \end{aligned} \tag{24}$$

where $\bar{N}(k), \bar{Q}(k), \bar{Y}(k), \bar{F}(k)$ and $\bar{B}(k)$ are the T-functions of $n(\xi), q(\xi), y(\xi), f(\xi)$ and $b(\xi)$ respectively.

Using equation (19), the boundary conditions in equations (15) and (16) become

At $\xi = 0$,

$$\bar{Y}(1) = 0 \quad (25)$$

and

$$\bar{Y}(0) = 0 \quad (26)$$

Using equation (20), the boundary conditions in equations (17) and (18) become

At $\xi = 1$,

$$\sum_{k=0}^m k(k-1)\bar{Y}(k) = 0 \quad (27)$$

and

$$\sum_{k=0}^m k(k-1)(k-2)\bar{Y}(k) + \tau_R \sum_{k=0}^m k(k-1)\bar{Y}(k) - [\eta_R - \gamma^2(\lambda + \alpha^2)\psi_R] \sum_{k=0}^m k\bar{Y}(k) = 0 \quad (28)$$

Equations (25) - (28) are algebraic equations which were implemented in MAPLE 18.

Let

$$\bar{Y}(2) = s \quad (29)$$

$$\bar{Y}(3) = z \quad (30)$$

where s and z are unknown parameters.

Putting $k = 0$ into equation (24) to obtain

$$\bar{Y}(4) = - \left[\left(\frac{\bar{B}(2)}{6\bar{B}(0)} + \frac{\bar{N}(0)}{12\bar{B}(0)} + \frac{\gamma^2(\lambda + \alpha^2)\bar{F}(0)}{12\bar{B}(0)} \right) s + \frac{\bar{B}(1)}{2\bar{B}(0)} z \right] \quad (31)$$

Similarly, putting $k = 1$ into equation (24) to obtain

$$\begin{aligned} \bar{Y}(5) = \frac{1}{120} \left\{ \left(\frac{12\bar{B}(1)\bar{B}(2)}{\bar{B}^2(0)} + \frac{6\bar{B}(1)\bar{N}(0)}{\bar{B}^2(0)} + \frac{6\gamma^2(\lambda + \alpha^2)\bar{B}(1)\bar{F}(0)}{\bar{B}^2(0)} - \frac{12\bar{B}(3)}{\bar{B}(0)} - \frac{4\bar{N}(1)}{\bar{B}(0)} \right. \right. \\ \left. \left. \frac{4\gamma^2(\lambda + \alpha^2)\bar{F}(1)}{\bar{B}(0)} \right) s + \left(\frac{36\bar{B}^2(1)}{\bar{B}^2(0)} - \frac{36\bar{B}(2)}{\bar{B}(0)} - \frac{6\bar{N}(0)}{\bar{B}(0)} - \frac{6\gamma^2(\lambda + \alpha^2)\bar{F}(0)}{\bar{B}(0)} \right) z \right\} \quad (32) \end{aligned}$$

Following the same recurrence step, the m th term $\bar{Y}(m)$ can be calculated, with m being decided by the convergence of natural frequency as described later. Substituting $\bar{Y}(0)$ to $\bar{Y}(m)$ into equations (27) and (28) to give

$$a_{j1}^{(m)}(\lambda)s + a_{j2}^{(m)}(\lambda)z = 0, \quad j = 1, 2, \quad (33)$$

where $a_{ji}^{(m)}(\lambda)$, $a_{j2}^{(m)}(\lambda)$ are polynomials of λ corresponding to m .

By equation (33), the frequency equation is given as follows:

$$\begin{vmatrix} a_{11}^{(m)}(\lambda) & a_{12}^{(m)}(\lambda) \\ a_{21}^{(m)}(\lambda) & a_{22}^{(m)}(\lambda) \end{vmatrix} = 0 \quad (34)$$

Solving equation (34), the dimensionless frequencies are calculated with $\lambda = \Lambda^2$, we get $\Lambda = \Lambda_i^{(m)}$, $i = 1, 2$, where $\Lambda_i^{(m)}$ is the i th estimated dimensionless natural frequency corresponding to m , with m being decided by the following criterion:

$$|\Lambda_i^{(m)} - \Lambda_i^{(m-1)}| \leq \delta \quad (35)$$

where $\Lambda_i^{(m-1)}$ is the i th estimated dimensionless natural frequency corresponding to $m-1$ and δ is the error tolerance parameter. In this study, $\delta = 0.0001$. From equation (35), two cases were discussed.

CASE 1: If equation (35) is satisfied, then $\Lambda_i^{(m)}$ is the i th estimated dimensionless natural frequency Λ_i .

Substituting Λ_i into equation (33), gives

$$z = c_1 s \quad \text{where } c_1 \text{ is a constant.} \quad (36)$$

Substituting Λ_i and equation (36) into $\bar{Y}(0)$ to $\bar{Y}(m)$ using equation (22), leads to

$$y_i(\xi) = \sum_{k=0}^m \xi^k \bar{Y}^*(k) \quad (37)$$

where $\bar{Y}^*(k)$ is $\bar{Y}(k)$ whose Λ and z are substituted by Λ_i and $c_1 s$, $y_i(\xi)$ is the i th mode shape corresponding to dimensionless natural frequency Λ_i . Moreover, by $\xi = \frac{x}{l}$ leads to

$$Y_i(x) = y_i\left(\frac{x}{l}\right) \quad (38)$$

CASE 2: If equation (35) is not satisfied, the following steps should be repeated until i th dimensionless natural frequency and i th mode shape are found.

Step 1: Substituting $m+1$ for m .

Step 2: The same procedure as shown in equations (34) - (38). By mode shape functions, the free vibration response is expressed as follows:

$$\Phi(x, t) = \sum_{i=1}^{\infty} Y_i(x) (a_1 \cos \omega t + a_2 \sin \omega t) \quad (39)$$

where a_1 and a_2 are determined by initial conditions in equation (2).

4.0 NUMERICAL IMPLEMENTATION

To demonstrate the method introduced in this study, the free vibrations of a rotating cantilever beam is studied. In this section, the rotating Rayleigh beam is fixed at $x = 0$ and its material properties are assumed to be

$$\left. \begin{aligned} E(x)I(x) &= E(0)\left(1+\beta\frac{x}{l}\right)I(0)\left(1-c\frac{x}{l}\right)^{n+2} \\ \rho(x)A(x) &= \rho(0)A(0)\left(1-c\frac{x}{l}\right)^n \end{aligned} \right\} \quad (40)$$

where $E(0)$, $I(0)$, $\rho(0)$ and $A(0)$ are Young's modulus, moment of inertia, mass per unit volume and cross-sectional area at $x = 0$ respectively. A general case of varying cross-section was considered. That is, the cross-sectional area and its moment of inertia are assumed to respectively take the following expressions.

$$\left. \begin{aligned} A(x) &= A(0)\left(1-c\frac{x}{l}\right)^n \\ I(x) &= I(0)\left(1-c\frac{x}{l}\right)^{n+2} \end{aligned} \right\} \quad (41)$$

where c is the taper ratio which satisfies $0 \leq c < 1$ and n takes the value 1 for the first type and 2 for the second type of taper beam (n is a constant reflecting the profile of various cross-sections).

The centrifugal force $N(x)$ at a distance of x from the center of rotation (origin) is given by

$$N(x) = \int_x^l \rho(x)A(x)\Omega^2 x dx \quad (42)$$

where Ω is the rotation speed.

The cases $n = 1$ and $n = 2$ were considered in equations (40) and (41).

For example, taking $n = 1$, we have

$$N(x) = \int_x^l \rho(\xi)A(\xi)\Omega^2 \xi d\xi \quad (43)$$

$$\begin{aligned} &= \int_x^l \rho(0)A(0)\left(1-c\frac{\xi}{l}\right)\Omega^2 \xi d\xi \\ &= \rho(0)A(0)\Omega^2 l^2 \left[\left(\frac{1}{2} - \frac{c}{3} - \frac{\xi^2}{2} + \frac{c\xi^3}{3} \right) \right] \end{aligned} \quad (44)$$

The radius of gyration $r(x)$ is given by

$$r^2(x) = \frac{I(x)}{A(x)} = \frac{I(0)(1-c\xi)^3}{A(0)(1-c\xi)} = \frac{I(0)}{A(0)}(1-c\xi)^2 \quad (45)$$

$$r^2(0) = \frac{I(0)}{A(0)} = \frac{I(0)(1-c \times 0)}{A(0)(1-c \times 0)} = \frac{I(0)}{A(0)} \quad (46)$$

$$b(\xi) = \frac{E(x)I(x)}{E(0)I(0)} = \frac{E(0)I(0)\left(1+\beta\frac{x}{l}\right)\left(1-c\frac{x}{l}\right)^3}{E(0)I(0)} = (1+\beta\xi)(1-c\xi)^3 \quad (47)$$

$$n(\xi) = -\frac{N(x)l^2}{E(0)I(0)} = -\alpha^2 \left(\frac{1}{2} - \frac{c}{3} + \frac{c\xi^2}{2} + \frac{c\xi^3}{3} \right) \quad (48)$$

$$q(\xi) = \frac{\rho(x)A(x)}{\rho(0)A(0)} = \frac{\rho(0)A(0)(1-c\xi)}{\rho(0)A(0)} = (1-\xi c) \quad (49)$$

$$f(\xi) = \frac{\rho(x)I(x)}{\rho(0)I(0)} = \frac{\rho(x)A(x)}{\rho(0)A(0)} \frac{r^2(x)}{r^2(0)} = (1-\xi c)^3 \quad (50)$$

For free vibration, substitute equations (47) - (50) into equation (14) to give

$$\begin{aligned} & \left[1 + (\beta - 3c)\xi + 3(c^2 - \beta c)\xi^2 + (3\beta c^2 - c^3)\xi^3 - \beta c^3\xi^4 \right] \frac{d^4 y(\xi)}{d\xi^4} + 2 \left[(\beta - 3c) + 6(c^2 - \beta c)\xi \right. \\ & \left. + 3(3\beta c^2 - c^3)\xi^2 - 4\beta c^3\xi^3 \right] \frac{d^3 y(\xi)}{d\xi^3} + 6 \left[(c^2 - \beta c) + (3\beta c^2 - c^3)\xi - 2\beta c^3\xi^2 \right] \frac{d^2 y(\xi)}{d\xi^2} \\ & - \left(\frac{1}{2} - \frac{c}{3} - \frac{\xi^2}{2} + \frac{c\xi^3}{3} \right) \alpha^2 \frac{d^2 y(\xi)}{d\xi^2} + (\xi - c\xi^2) \alpha^2 \frac{dy(\xi)}{d\xi} + \gamma^2 (\Lambda^2 + \alpha^2) (1 - 3c\xi + 3c^2\xi^2 - c^3\xi^3) \frac{d^2 y(\xi)}{d\xi^2} \\ & + 3\gamma^2 (\Lambda^2 + \alpha^2) (-c + 2c^2\xi - c^3\xi^2) \frac{dy(\xi)}{d\xi} - \Lambda^2 (1 - c\xi) y(\xi) = 0 \end{aligned} \quad (51)$$

Taking the differential transform of equation (51) to obtain

$$\begin{aligned} & \sum_{r=0}^k \left[\delta(k-r) + (\beta - 3c)\delta(k-r-1) + (3c^2 - 3\beta c)\delta(k-r-2) + (3\beta c^2 - c^3)\delta(k-r-3) \right. \\ & \left. - \beta c^3\delta(k-r-4) \right] (r+1)(r+2)(r+3)(r+4) \bar{Y}(r+4) + 2 \sum_{r=0}^k \left[(\beta - 3c)\delta(k-r) \right. \\ & \left. + 6(c^2 - \beta c)\delta(k-r-1) + 3(3\beta c^2 - c^3)\delta(k-r-2) - 4\beta c^3\delta(k-r-3) \right] (r+1)(r+2)(r+3) \bar{Y}(r+3) \\ & + 6 \sum_{r=0}^k \left[(c^2 - \beta c)\delta(k-r) + (3\beta c^2 - c^3)\delta(k-r-1) - 2\beta c^3\delta(k-r-2) \right] (r+1)(r+2) \bar{Y}(r+2) \\ & - \sum_{r=0}^k \left[\frac{1}{2}\delta(k-r) - \frac{c}{3}\delta(k-r) - \frac{1}{2}\delta(k-r-2) + \frac{c}{3}\delta(k-r-3) \right] \alpha^2 (r+1)(r+2) \bar{Y}(r+2) \\ & + \sum_{r=0}^k \left[\delta(k-r-1) - c\delta(k-r-2) \right] \alpha^2 (r+1) \bar{Y}(r+1) \\ & + \gamma^2 (\Lambda^2 + \alpha^2) \sum_{r=0}^k \left[\delta(k-r) - 3c\delta(k-r-1) + 3c^2\delta(k-r-2) - c^3\delta(k-r-3) \right] (r+1)(r+2) \bar{Y}(r+2) \\ & + 3\gamma^2 (\Lambda^2 + \alpha^2) \sum_{r=0}^k \left[-c\delta(k-r) + 2c^2\delta(k-r-1) - c^3\delta(k-r-2) \right] (r+1) \bar{Y}(r+1) \\ & = \sum_{r=0}^k \lambda \left[\delta(k-r) - c\delta(k-r-1) \right] \bar{Y}(r) \end{aligned} \quad (52)$$

In equation (52), the following terms were defined:

$$\left. \begin{aligned} \bar{B}(k) &= \delta(k) + (\beta - 3c)\delta(k-1) + (3c^2 - 3\beta c)\delta(k-2) \\ &\quad + (3\beta c^2 - c^3)\delta(k-3) - \beta c^3\delta(k-4) \\ \bar{N}(k) &= -\left[\frac{1}{2}\delta(k) - \frac{c}{3}\delta(k) - \frac{1}{2}\delta(k-2) + \frac{c}{3}\delta(k-3)\right]\alpha^2 \\ \bar{Q}(k) &= \delta(k) - c\delta(k-1) \\ \bar{F}(k) &= \delta(k) - 3c\delta(k-1) + 3c^2\delta(k-2) - c^3\delta(k-3) \end{aligned} \right\} \quad (53)$$

Substituting these values of $\bar{Y}(0)$, $\bar{Y}(1)$, $\bar{Y}(2)$, $\bar{Y}(3)$, and $k=0$ into equation (24), gives

$$\bar{Y}(4) = \frac{1}{24} \left[\left(\frac{50}{3} - 2\lambda\gamma^2 - 50\gamma^2 \right) s + 12z \right] \quad (54)$$

Substituting the values of $\bar{Y}(0)$, $\bar{Y}(1)$, $\bar{Y}(2)$, $\bar{Y}(3)$, $\bar{Y}(4)$ and $k=1$ into equation (24), gives

$$\bar{Y}(5) = \frac{1}{120} [47s + (-6\lambda\gamma^2 - 150\gamma^2 + 86)z] \quad (55)$$

Substituting the values of $\bar{Y}(0)$, $\bar{Y}(1)$, $\bar{Y}(2)$, $\bar{Y}(3)$, $\bar{Y}(4)$, $\bar{Y}(5)$ and $k=2$ into equation (24), yields

$$\begin{aligned} \bar{Y}(6) &= \frac{1}{360} \left[\left(\frac{1619}{18} + \lambda - \frac{127}{6}\lambda\gamma^2 - \frac{3175\gamma^2}{6} + 50\lambda\gamma^4 + 625\gamma^4 + \lambda^2\gamma^4 \right) s \right. \\ &\quad \left. + \left(204 - \frac{9}{2}\lambda\gamma^2 - \frac{225}{2}\gamma^2 \right) z \right] \end{aligned} \quad (56)$$

Following the same recurrence step, the m th term $\bar{Y}(m)$ can be calculated. Computer codes developed using MAPLE 18 were used to calculate both the natural frequency and corresponding mode shape for $\gamma = 0.01$, $\beta = 0.5$, $\alpha = 5$ and $c = 0.5$ using equation (52). The first three dimensionless natural frequencies are $\lambda_1 = 6.8324$ at $m = 29$, $\lambda_2 = 23.4058$ at $m = 32$ and $\lambda_3 = 55.7606$ at $m = 34$. Now to calculate the first mode shape, substitute λ_1 into $\bar{Y}(0)$ to $\bar{Y}(29)$ and using equation (22), the series function of the first mode shape is obtained as

$$\begin{aligned} y_1(\xi)_{29} &= s (\xi^2 - 0.7004\xi^3 + 0.3437\xi^4 - 0.1100\xi^5 - 0.0177\xi^6 + 0.0329\xi^7 - 0.0215\xi^8 \\ &\quad + 0.0076\xi^9 - 0.0010\xi^{10} + 0.0001\xi^{11} + 0.0006\xi^{12} + 0.0001\xi^{13} + 0.0002\xi^{14} + 0.0001\xi^{15} \\ &\quad + 4.5639 \times 10^{-5}\xi^{16} + 1.5790 \times 10^{-5}\xi^{17} + 6.2364 \times 10^{-6}\xi^{18} + 3.1801 \times 10^{-7}\xi^{19} - 1.3961 \times 10^{-6}\xi^{20} \\ &\quad - 1.7951 \times 10^{-6}\xi^{21} - 1.4890 \times 10^{-6}\xi^{22} - 1.0772 \times 10^{-6}\xi^{23} - 6.9804 \times 10^{-7}\xi^{24} - 4.2351 \times 10^{-7}\xi^{25} \\ &\quad - 2.4028 \times 10^{-7}\xi^{26} - 1.2898 \times 10^{-7}\xi^{27} - 6.4857 \times 10^{-8}\xi^{28} - 3.0335 \times 10^{-8}\xi^{29}) \end{aligned} \quad (57)$$

Similarly, the second mode shape is given by

$$\begin{aligned} y_2(\xi)_{32} &= s (\xi^2 - 1.3392\xi^3 + 0.02001\xi^4 - 0.5642\xi^5 + 1.010\xi^6 - 0.3226\xi^7 + 0.1293\xi^8 - 0.0837\xi^9 \\ &\quad + 0.0219\xi^{10} - 0.0156\xi^{11} - 0.0004\xi^{12} - 0.0034\xi^{13} - 0.0012\xi^{14} - 0.0009\xi^{15} - 0.0004\xi^{16} \end{aligned}$$

$$\begin{aligned}
 & + \frac{d^2}{d\xi^2} \left(\sum_{r=0}^{\infty} \bar{B}(r) \xi^r \right) \cdot \frac{d^2}{d\xi^2} \left(\sum_{r=0}^{\infty} \bar{Y}(r) \xi^r \right) + \sum_{r=0}^{\infty} \bar{N}(r) \xi^r \cdot \frac{d^2}{d\xi^2} \left(\sum_{r=0}^{\infty} \bar{Y}(r) \xi^r \right) \\
 & + \frac{d}{d\xi} \left(\sum_{r=0}^{\infty} \bar{N}(r) \xi^r \right) \cdot \frac{d}{d\xi} \left(\sum_{r=0}^{\infty} \bar{Y}(r) \xi^r \right) + \gamma^2 (\Lambda^2 + \alpha^2) \left[\sum_{r=0}^{\infty} \bar{F}(r) \xi^r \cdot \frac{d^2}{d\xi^2} \left(\sum_{r=0}^{\infty} \bar{Y}(r) \xi^r \right) \right] \\
 & + \gamma^2 (\Lambda^2 + \alpha^2) \left[\frac{d}{d\xi} \left(\sum_{r=0}^{\infty} \bar{F}(r) \xi^r \right) \cdot \frac{d}{d\xi} \left(\sum_{r=0}^{\infty} \bar{Y}(r) \xi^r \right) \right] = \Lambda^2 \left[\sum_{r=0}^{\infty} \bar{Q}(r) \xi^r \cdot \sum_{r=0}^{\infty} \bar{Y}(r) \xi^r \right] \quad (61)
 \end{aligned}$$

Simplifying equation (61), we obtain

$$\begin{aligned}
 & \sum_{r=0}^{\infty} \bar{B}(r) \xi^r \cdot \sum_{r=4}^{\infty} r(r-1)(r-2)(r-3) \bar{Y}(r) \xi^{r-4} + 2 \left[\sum_{r=1}^{\infty} r \bar{B}(r) \xi^{r-1} \cdot \sum_{r=3}^{\infty} r(r-1)(r-2) \bar{Y}(r) \xi^{r-3} \right] \\
 & + \sum_{r=2}^{\infty} r(r-1) \bar{B}(r) \xi^{r-2} \cdot \sum_{r=2}^{\infty} r(r-1) \bar{Y}(r) \xi^{r-2} + \sum_{r=0}^{\infty} \bar{N}(r) \xi^r \cdot \sum_{r=2}^{\infty} r(r-1) \bar{Y}(r) \xi^{r-2} \\
 & + \sum_{r=1}^{\infty} r \bar{N}(r) \xi^{r-1} \cdot \sum_{r=1}^{\infty} r \bar{Y}(r) \xi^{r-1} + \gamma^2 (\Lambda^2 + \alpha^2) \left[\sum_{r=0}^{\infty} \bar{F}(r) \xi^r \cdot \sum_{r=2}^{\infty} r(r-1) \bar{Y}(r) \xi^{r-2} \right] \\
 & + \gamma^2 (\Lambda^2 + \alpha^2) \left[\sum_{r=1}^{\infty} r \bar{F}(r) \xi^{r-1} \cdot \sum_{r=1}^{\infty} r \bar{Y}(r) \xi^{r-1} \right] = \Lambda^2 \sum_{r=0}^{\infty} \bar{Q}(r) \xi^r \sum_{r=0}^{\infty} \bar{Y}(r) \xi^r \quad (62)
 \end{aligned}$$

Applying the rule of multiplication of power series to equation (62), we have

$$\begin{aligned}
 & \sum_{r=0}^{\infty} \sum_{r=0}^k \bar{B}(k-r)(r+1)(r+2)(r+3)(r+4) \bar{Y}(r+4) \xi^r + 2 \sum_{r=0}^{\infty} \sum_{r=0}^k (k-r+1) \bar{B}(k-r+1) \\
 & (r+1)(r+2)(r+3) \bar{Y}(r+3) \xi^r + \sum_{r=0}^{\infty} \sum_{r=0}^k (k-r+1)(k-r+2) \bar{B}(k-r+2)(r+1)(r+2) \cdot \\
 & \bar{Y}(r+2) \xi^r + \sum_{r=0}^{\infty} \sum_{r=0}^k \bar{N}(k-r)(r+1)(r+2) \bar{Y}(r+2) \xi^r + \sum_{r=0}^{\infty} \sum_{r=0}^k (k-r+1) \bar{N}(k-r+1) \cdot \\
 & (r+1) \bar{Y}(r+1) \xi^r + \gamma^2 (\Lambda^2 + \alpha^2) \sum_{r=0}^{\infty} \sum_{r=0}^k \bar{F}(k-r)(r+1)(r+2) \bar{Y}(r+2) \xi^r + \gamma^2 (\Lambda^2 + \alpha^2) \\
 & \sum_{r=0}^{\infty} \sum_{r=0}^k (k-r+1) \bar{F}(k-r+1)(r+1) \bar{Y}(r+1) \xi^r = \Lambda^2 \sum_{r=0}^{\infty} \sum_{r=0}^k \bar{Q}(k-r) \bar{Y}(r) \xi^r \quad (63)
 \end{aligned}$$

Equation (63) is written as

$$\begin{aligned}
 & \sum_{r=0}^{\infty} \left[\sum_{r=0}^k \bar{B}(k-r)(r+1)(r+2)(r+3)(r+4) \bar{Y}(r+4) + 2 \sum_{r=0}^k (k-r+1) \bar{B}(k-r+1) \right. \\
 & (r+1)(r+2)(r+3) \bar{Y}(r+3) + \sum_{r=0}^k (k-r+1)(k-r+2) \bar{B}(k-r+2)(r+1)(r+2) \bar{Y}(r+2) \\
 & + \sum_{r=0}^k \bar{N}(k-r)(r+1)(r+2) \bar{Y}(r+2) + \sum_{r=0}^k (k-r+1) \bar{N}(k-r+1)(r+1) \bar{Y}(r+1) \\
 & \left. + \gamma^2 (\Lambda^2 + \alpha^2) \sum_{r=0}^k \bar{F}(k-r)(r+1)(r+2) \bar{Y}(r+2) + \gamma^2 (\Lambda^2 + \alpha^2) \right]
 \end{aligned}$$

$$\begin{aligned}
& -0.0002\xi^{17} - 0.0001\xi^{18} - 2.2702 \times 10^{-5}\xi^{19} + 3.4571 \times 10^{-6}\xi^{20} + 1.0422 \times 10^{-5}\xi^{21} \\
& + 1.0888 \times 10^{-5}\xi^{22} + 8.5078 \times 10^{-6}\xi^{23} + 5.9420 \times 10^{-6}\xi^{24} + 3.7832 \times 10^{-6}\xi^{25} + 2.2715 \times 10^{-6}\xi^{26} \\
& + 1.2848 \times 10^{-6}\xi^{27} + 6.9127 \times 10^{-7}\xi^{28} + 3.5089 \times 10^{-7}\xi^{29} + 1.6720 \times 10^{-7}\xi^{30} + 7.2838 \times 10^{-8}\xi^{31} \\
& + 2.7570 \times 10^{-8}\xi^{32}
\end{aligned} \quad (58)$$

Also, the third mode shape is given by

$$\begin{aligned}
y_3(\xi)_{34} = & s(\xi^2 - 2.1417\xi^3 - 0.4025\xi^4 - 1.1097\xi^5 + 7.6631\xi^6 - 4.0806\xi^7 - 0.6596\xi^8 - 2.5494\xi^9 \\
& + 2.7428\xi^{10} - 0.5011\xi^{11} + 0.3923\xi^{12} - 0.4002\xi^{13} + 0.1494\xi^{14} - 0.0728\xi^{15} + 0.0401\xi^{16} - 0.0172\xi^{17} \\
& + 0.0085\xi^{18} - 0.0036\xi^{19} + 0.0018\xi^{20} - 0.0007\xi^{21} + 0.0004\xi^{22} - 0.0001\xi^{23} + 0.0001\xi^{24} \\
& - 2.1676 \times 10^{-5}\xi^{25} + 1.7370 \times 10^{-5}\xi^{26} - 3.9091 \times 10^{-6}\xi^{27} + 3.5904 \times 10^{-6}\xi^{28} - 8.1407 \times 10^{-7}\xi^{29} \\
& + 6.7463 \times 10^{-7}\xi^{30} - 2.1984 \times 10^{-7}\xi^{31} + 9.8623 \times 10^{-8}\xi^{32} - 7.2967 \times 10^{-8}\xi^{33} + 3.2682 \times 10^{-9}\xi^{34})
\end{aligned} \quad (59)$$

By going through similar recurrence process, other higher order dimensionless natural frequencies (λ_4 , λ_5 and λ_6) and their corresponding mode shape functions can be obtained.

5.0 PRESENTATION OF RESULTS

5.1 Convergence study

Successful application of numerical methods in engineering problems could be guaranteed only after their convergence is studied and their limitations are well understood. To the authors' best knowledge, DTM has been used for analysis of either uniform beams or tapered beams where it shows a great convergence Refs. [5,7,12,13,18,20]. In other words, DTM has been used for those types of differential equations whose coefficients are either constant or show a decreasing trend along the beam axis. It was observed that when the variable coefficients in the differential equation show strong increasing trends, DTM would provide satisfactory results

when a large number of terms are used. This article also shows below that the DTM used to solve the dimensionless governing ordinary differential equation (14) in this work converges.

Theorem 5.1

Consider a variable coefficient 4th order homogeneous differential equation given in equation (14), then the differential transform method converges to the exact solution

Proof. Taking the differential transform of equation (14), this gives equation (24). Substituting the variable coefficients in equation (60) into the original problem given in equation (14)

$$\left. \begin{aligned} y(\xi) &= \sum_{r=0}^{\infty} \bar{Y}(r)\xi^r, \quad n(\xi) = \sum_{r=0}^{\infty} \bar{N}(r)\xi^r, \quad q(\xi) = \sum_{r=0}^{\infty} \bar{Q}(r)\xi^r \\ f(\xi) &= \sum_{r=0}^{\infty} \bar{F}(r)\xi^r, \quad b(\xi) = \sum_{r=0}^{\infty} \bar{B}(r)\xi^r \end{aligned} \right\} \quad (60)$$

[where $y(\xi), n(\xi), q(\xi), f(\xi)$ and $b(\xi)$ are the variable coefficients while $\bar{Y}(r), \bar{N}(r), \bar{Q}(r), \bar{F}(r)$ and $\bar{B}(r)$ are the T-functions respectively] to obtain

$$\sum_{r=0}^{\infty} \bar{B}(r)\xi^r \cdot \frac{d^4}{d\xi^4} \left(\sum_{r=0}^{\infty} \bar{Y}(r)\xi^r \right) + 2 \left[\frac{d}{d\xi} \left(\sum_{r=0}^{\infty} \bar{B}(r)\xi^r \right) \cdot \frac{d^3}{d\xi^3} \left(\sum_{r=0}^{\infty} \bar{Y}(r)\xi^r \right) \right]$$

$$\sum_{r=0}^k (k-r+1)\bar{F}(k-r+1)(r+1)\bar{Y}(r+1) \Big] \xi^r = \sum_{r=0}^{\infty} \left[\Lambda^2 \sum_{r=0}^k \bar{Q}(k-r)\bar{Y}(r) \right] \xi^r \quad (64)$$

Comparing coefficients, we have

$$\begin{aligned} & \sum_{r=0}^k \bar{B}(k-r)(r+1)(r+2)(r+3)(r+4)\bar{Y}(r+4) + 2 \sum_{r=0}^k (k-r+1)\bar{B}(k-r+1) \\ & (r+1)(r+2)(r+3)\bar{Y}(r+3) + \sum_{r=0}^k (k-r+1)(k-r+2)\bar{B}(k-r+2)(r+1)(r+2)\bar{Y}(r+2) \\ & + \sum_{r=0}^k \bar{N}(k-r)(r+1)(r+2)\bar{Y}(r+2) + \sum_{r=0}^k (k-r+1)\bar{N}(k-r+1)(r+1)\bar{Y}(r+1) + \gamma^2(\lambda + \alpha^2) \\ & \sum_{r=0}^k \bar{F}(k-r)(r+1)(r+2)\bar{Y}(r+2) + \gamma^2(\lambda + \alpha^2) \sum_{r=0}^k (k-r+1)\bar{F}(k-r+1)(r+1)\bar{Y}(r+1) \\ & = \lambda \sum_{r=0}^k \bar{Q}(k-r)\bar{Y}(r) \end{aligned} \quad (65)$$

Since equation (24) is equivalent to equation (65), it therefore follows that the DTM converges.

Table 2: Convergence of the first three dimensionless natural frequencies of rotating cantilever tapered Rayleigh beam for different number of terms (m) at $n=1$

m	λ_1	λ_2	λ_3
6	8.5582	-	-
7	6.6163	32.9551	-
8	7.9418	33.7375	-
9	7.1049	25.4972	-
10	7.1354	27.6595	45.8166
11	7.0531	24.9483	55.3646
12	6.9384	24.9515	45.0520
13	6.9192	24.1972	-
14	6.8694	23.9313	50.4313
15	6.8516	23.6487	104.012
16	6.8345	23.4959	53.2578
17	6.8271	23.3933	61.6515
18	6.8233	23.3457	54.8818
19	6.8229	23.3279	57.0638
20	6.8240	23.3308	55.5414
21	6.8258	23.3425	56.1648
22	6.8277	23.3574	55.7239
23	6.8292	23.3713	55.8880
24	6.8304	23.3827	55.7575
25	6.8312	23.3912	55.7979
26	6.8318	23.3971	55.7599
27	6.8321	23.4009	55.7698
28	6.8323	23.4033	55.7593
29	6.8324	23.4046	55.7620
30	6.8324	23.4053	55.7594
31	6.8325	23.4057	55.7604
32	6.8325	23.4058	55.7599
33	6.8325	23.4059	55.7603
34	6.8325	23.4059	55.7603

Table 3: Convergence of the fourth, fifth and sixth dimensionless natural frequencies of tapered rotating cantilever Rayleigh beam for different number of terms (m) at $n=1$

m	λ_4	λ_5	λ_6
25	100.5571	-	-
26	108.0450	126.7527	285.3475
27	102.6597	-	-
28	104.9251	-	-
29	103.5132	350.8895	393.2581
30	104.2029	-	394.6025
31	103.7940	-	-
32	103.9937	159.8187	-
33	103.8754	420.4198	-
34	103.9313	164.4034	526.4662
35	103.8980	170.7830	203.1676
36	103.9131	166.5889	-
37	103.9037	168.5465	219.9921
38	103.9075	167.4031	-
39	103.9056	167.9808	232.0535
40	103.9065	167.6575	618.8319
41	103.9059	167.8246	240.2032
42	103.9063	167.7332	258.9152
43	103.9058	167.7857	244.5003
44	103.9057	167.7651	249.1916
45	103.9058	167.7633	246.5275
46	103.9059	167.7632	247.8399
47	103.9059	167.7648	247.0173
48	103.9059	167.7649	247.5386
49	103.9058	167.7650	247.1806
50	103.9058	167.7651	247.2594
60	103.9059	167.7638	247.3832
65	103.9060	167.7579	247.5689
70	103.9061	167.7578	247.5734
79	103.9060	167.7648	247.4063
80	103.9060	167.7648	247.4056
81	103.9061	167.7617	247.4992
82	103.9061	167.7617	247.4992

Table 4: Effects of the rotational speed(α), inverse of slenderness ratio(γ) and taper ratio(c) on the dimensionless natural frequencies of a rotating cantilever tapered Rayleigh beam for $n = 1$.

γ	α	λ_1			λ_2			λ_3		
		$c = 0.25$	$c = 0.50$	$c = 0.75$	$c = 0.25$	$c = 0.50$	$c = 0.75$	$c = 0.25$	$c = 0.50$	$c = 0.75$
0.100	0	3.7220	3.9666	4.3879	19.6558	18.6173	17.1343	47.6335	44.2633	39.1292
0.050		3.7908	4.0157	4.4235	21.4681	19.7267	17.6983	56.4114	50.0421	42.1462
0.033		3.8039	4.0250	4.4301	21.8617	19.9543	17.8088	58.6817	51.3864	42.7842
0.025		3.8085	4.0282	4.4325	22.0045	20.0358	17.8481	59.5446	51.8830	43.0143
0.020		3.8106	4.0297	4.4336	22.0716	20.0738	17.8663	59.9572	52.1178	43.1221
0.013		3.8128	4.0312	4.4346	22.1383	20.1116	17.8844	60.3732	52.3527	43.2293
0.010		3.8135	4.0318	4.4350	22.1619	20.1249	17.8907	60.5209	52.4357	43.2670
BE		3.8145	4.0324	4.4355	22.1922	20.1420	17.8989	60.7123	52.5430	43.3157
0.100	5	6.4043	6.6727	7.0672	22.3180	21.5428	20.3021	50.0447	46.9533	42.1717
0.050		6.5821	6.7933	7.1509	24.5325	22.9154	21.0118	59.3997	53.1820	45.4744
0.033		6.6155	6.8159	7.1666	25.0108	23.1958	21.1508	61.8230	54.6304	46.1723
0.025		6.6271	6.8239	7.1721	25.1841	23.2962	21.2001	62.7442	55.1654	46.4239
0.020		6.6326	6.8275	7.1747	25.2654	23.3430	21.2230	63.1845	55.4182	46.5418
0.013		6.6379	6.8312	7.1772	25.3463	23.3895	21.2456	63.6285	55.6713	46.6590
0.010		6.6398	6.8324	7.1781	25.3747	23.4058	21.2536	63.7860	55.7607	46.7003
BE		6.6422	6.8341	7.1792	25.4115	23.4268	21.2638	63.9903	55.8762	46.7534
0.100	10	10.9335	11.2525	11.6509	28.7788	28.5460	27.6761	56.5355	54.1400	50.1734
0.050		11.2492	11.4676	11.8093	31.9830	30.5292	28.7051	67.4801	61.5887	54.2186
0.033		11.3074	11.5077	11.8389	32.6651	30.9304	28.9062	70.3285	63.3188	55.0718
0.025		11.3277	11.5217	11.8493	32.9111	31.0736	28.9774	71.4109	63.9573	55.3793
0.020		11.3371	11.5282	11.8541	33.0264	31.1404	29.0105	71.9282	64.2590	55.5233
0.013		11.3464	11.5347	11.8589	33.1410	31.2067	29.0433	72.4495	64.5609	55.6665
0.010		11.3496	11.5369	11.8605	33.1813	31.2299	29.0548	72.6345	64.6674	55.7169
BE		11.3538	11.5398	11.8626	33.2333	31.2599	29.0696	72.8744	64.8052	55.7818

Table 5: Effects of the rotational speed(α), inverse of slenderness ratio(γ) and taper ratio(c) on the dimensionless natural frequencies of a rotating cantilever tapered Rayleigh beam for $n = 2$

γ	α	λ_1			λ_2			λ_3		
		$c = 0.25$	$c = 0.50$	$c = 0.75$	$c = 0.25$	$c = 0.50$	$c = 0.75$	$c = 0.25$	$c = 0.50$	$c = 0.75$
0.100	0	4.0506	4.7982	6.1082	20.1983	19.8439	19.4807	48.0951	45.4146	41.5078
0.050		4.1282	4.8655	6.1773	22.0564	21.0432	20.1856	56.9768	51.3918	44.8375
0.033		4.1430	4.8782	6.1903	22.4595	21.2895	20.3247	59.2702	52.7819	45.5451
0.025		4.1483	4.8827	6.1949	22.6057	21.3777	20.3740	60.1416	53.2955	45.8006
0.020		4.1507	4.8848	6.1971	22.6743	21.4189	20.3969	60.5582	53.5382	45.9203
0.013		4.1531	4.8869	6.1992	22.7427	21.4598	20.4197	60.9781	53.7812	46.0395
0.010		4.1539	4.8876	6.1999	22.7668	21.4742	20.4277	61.1271	53.8670	46.0814
BE		4.1550	4.8885	6.2008	22.7978	21.4926	20.4379	61.3204	53.9780	46.1355
0.100	5	6.6142	7.2345	8.3136	22.6750	22.3391	21.9133	50.3107	47.6697	43.8024
0.050		6.8013	7.3772	8.4410	24.9083	23.7692	22.7467	59.7323	54.0364	47.3654
0.033		6.8365	7.4040	8.4650	25.3902	24.0619	22.9109	62.1678	55.5161	48.1222
0.025		6.8488	7.4135	8.4734	25.5648	24.1666	22.9691	63.0931	56.0626	48.3952
0.020		6.8546	7.4178	8.4773	25.6466	24.2155	22.9962	63.5353	56.3210	48.5232
0.013		6.8602	7.4222	8.4812	25.7281	24.2640	23.0231	63.9811	56.5795	48.6506
0.010		6.8622	7.4237	8.4825	25.7568	24.2810	23.0325	64.1393	56.6708	48.6954
BE		6.8647	7.4256	8.4843	25.7938	24.3030	23.0446	64.3444	56.7888	48.7532
0.100	10	11.0832	11.6638	12.5872	28.7911	28.5339	27.9406	56.3346	53.8231	50.0478
0.050		11.4142	11.9168	12.8207	31.9546	30.5119	29.0759	67.2526	61.2591	54.2361
0.033		11.4754	11.9642	12.8646	32.6276	30.9132	29.2991	70.0844	62.9839	55.1241
0.025		11.4968	11.9808	12.8800	32.8704	31.0566	29.3783	71.1597	63.6204	55.4446
0.020		11.5067	11.9885	12.8871	32.9841	31.1234	29.4151	71.6734	63.9211	55.5947
0.013		11.5164	11.9961	12.8942	33.0972	31.1898	29.4516	72.1912	64.2220	55.7441
0.010		11.5199	11.9988	12.8967	33.1369	31.2131	29.4644	72.3749	64.3283	55.7967
BE		11.5243	12.0022	12.8998	33.1882	31.2431	29.4808	72.6130	64.4656	55.8644

Table 6: Comparison of results with published results for a rotating cantilever tapered Rayleigh beam

for $c = 0.5$, $\gamma = \frac{1}{30}$, $E(x) = E(0)$ and $\rho(x) = \rho(0)$ for $n = 1$.

α	λ_i	Ref. [3]	Ref. [13]	Ref. [15]	Present
0	λ_1	3.8177	3.8211	3.8180	3.8135
	λ_2	18.1601	18.2246	18.1689	18.1449
	λ_3	46.2759	46.5758	46.3266	46.2650
	λ_4	86.9716	87.7974	87.1371	87.0239
	λ_5	139.0892	140.8193	139.4778	139.3135
5	λ_1	6.7332	6.7357	6.72860	6.7168
	λ_2	21.7123	21.7911	21.71756	21.6858
	λ_3	49.8385	50.1877	49.90902	49.8411
	λ_4	90.4926	91.4413	90.7426	90.6243
	λ_5	142.4541	144.4463	143.0589	142.8912
10	λ_1	11.4838	11.4856	11.4748	11.4536
	λ_2	29.8974	30.0232	29.9097	29.8606
	λ_3	59.1543	59.6737	59.3123	59.2270
	λ_4	100.1831	101.5423	100.7309	100.5976
	λ_5	151.9492	154.7865	153.2657	153.0851

Table 7: Comparison of results with published results for a rotating cantilever tapered Rayleigh beam

for $c = 0.5$, $\gamma = \frac{1}{100}$, $E(x) = E(0)$ and $\rho(x) = \rho(0)$.

α	λ_i	$n = 1$			$n = 2$		
		Ref.[4]	Ref.[18]	Present	Ref.[4]	Ref.[18]	Present
1	λ_1	3.9860	3.9866	3.9856	4.7632	4.822	4.7632
	λ_2	18.460	18.474	18.458	19.665	19.718	19.665
	λ_3	47.330	47.417	47.324	48.617	48.747	48.617
2	λ_1	4.4361	4.4368	4.4355	5.1555	5.279	5.1555
	λ_2	18.922	18.937	18.920	20.058	20.126	20.058
	λ_3	47.784	47.872	47.778	48.999	49.135	48.999
3	λ_1	5.0918	5.0927	5.0910	5.7446	5.896	5.7446
	λ_2	19.669	19.684	19.667	20.696	20.805	20.696
	λ_3	48.529	48.619	48.523	49.630	49.779	49.630
4	λ_1	5.8776	5.8780	5.8767	6.4712	6.644	6.4712
	λ_2	20.669	20.685	20.667	21.558	21.734	21.558
	λ_3	49.554	49.646	49.547	50.499	50.680	50.499

Table 8: Comparison of convergence of the dimensionless natural frequencies of a rotating cantilever tapered Rayleigh beam to an Euler-Bernoulli beam for $E(x)=E(0)$ and $\rho(x)=\rho(0)$ and $c=0.5$

γ	α	λ_1		λ_2		λ_3	
		Present	Ref.[4]	Present	Ref.[4]	Present	Ref.[4]
0.100	0	3.7337	3.7727	16.9208	17.097	40.0559	40.412
0.050		3.8007	3.8108	17.9362	17.988	45.1004	45.229
0.033		3.8135	3.8180	18.1449	18.169	46.2650	46.327
0.025		3.8180	3.8205	18.2197	18.233	46.6945	46.730
0.020		3.8201	3.8217	18.2547	18.263	46.8974	46.921
0.013		3.8221	3.8229	18.2893	18.293	47.1005	47.111
0.010		3.8229	3.8233	18.3016	18.304	47.1722	47.178
0.002		3.8237	—	18.3166	—	47.2611	—
0.001		3.8238	—	18.3171	—	47.2639	—
0.0001		3.8238	—	18.3173	—	47.2648	—
0.00001		3.8238	—	18.3173	—	47.2648	—
BE(0)		3.8238	3.8238	18.3173	18.317	47.2648	47.265
0.100	5	6.5096	6.6118	20.1203	20.356	43.0511	43.444
0.050		6.6838	6.7102	21.4196	21.489	48.5680	48.710
0.033		6.7168	6.7286	21.6858	21.718	49.8411	49.909
0.025		6.7284	6.7351	21.7811	21.799	50.3105	50.350
0.020		6.7338	6.7381	21.8256	21.837	50.5323	50.558
0.013		6.7391	6.7410	21.8698	21.875	50.7542	50.766
0.010		6.7410	6.7421	21.8853	21.888	50.8326	50.839
0.002		6.7433	—	21.9045	—	50.9297	—
0.001		6.7434	—	21.9051	—	50.9328	—
0.0001		6.7434	—	21.9053	—	50.9338	—
0.00001		6.7434	—	21.9053	—	50.9338	—
BE(0)		6.7434	6.7434	21.9053	21.905	50.9338	50.934

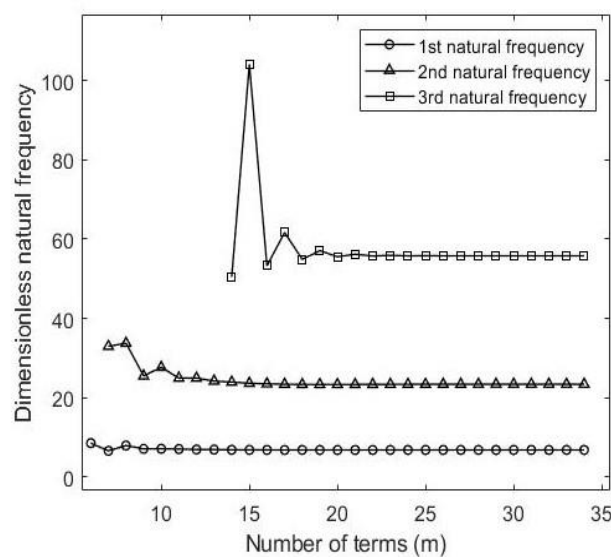


Figure 2: The convergence of the first, second and third dimensionless natural frequencies ($\lambda_1 = 6.8324$, $\lambda_2 = 23.4058$, $\lambda_3 = 55.7606$).

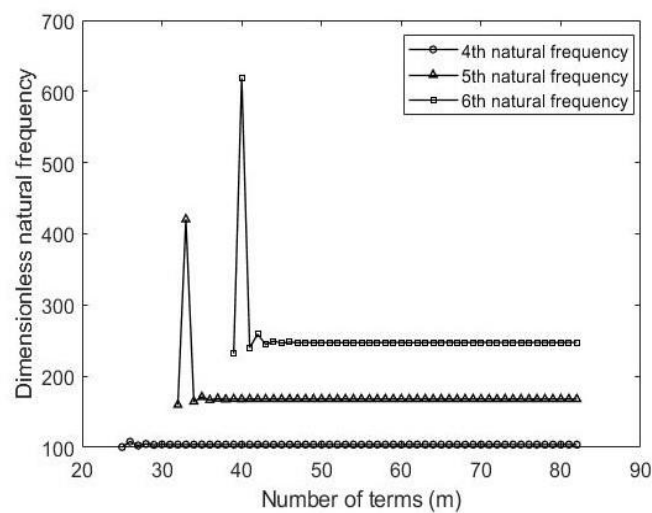


Figure 3: The convergence of the fourth, fifth and sixth dimensionless natural frequencies ($\lambda_4 = 103.9057$, $\lambda_5 = 167.7632$, $\lambda_6 = 247.4992$).

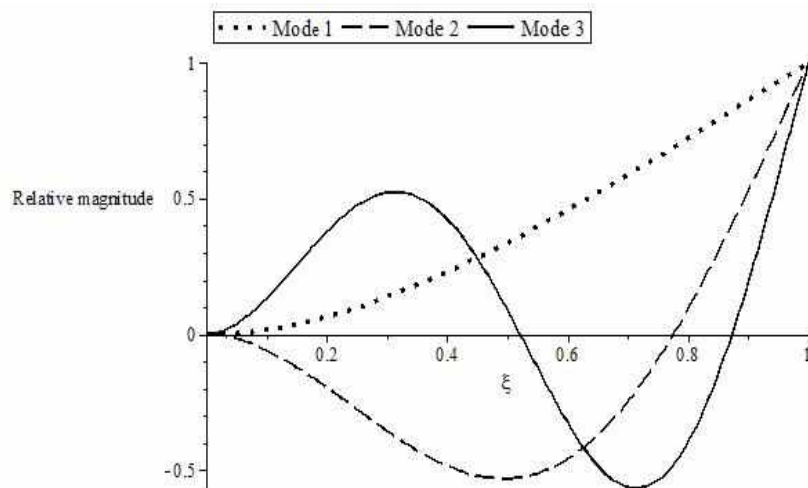


Figure 4: The first, second and third mode shapes of a rotating tapered Rayleigh beam.

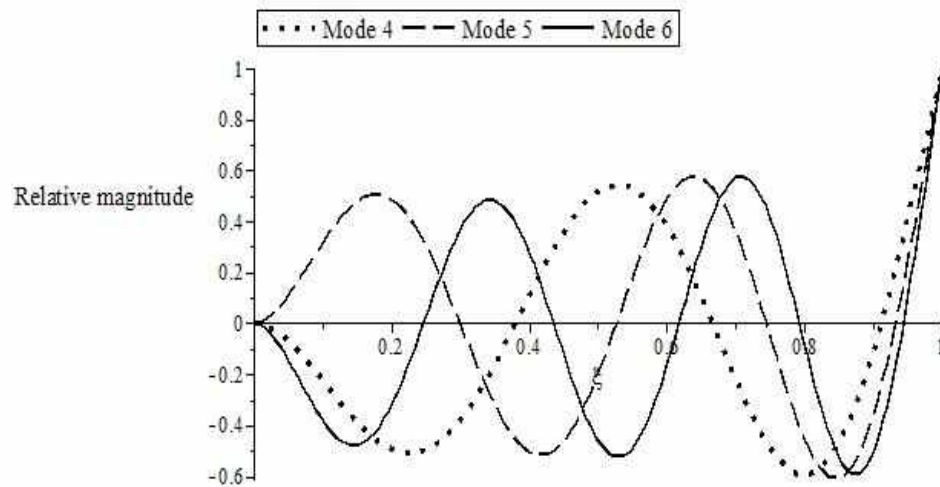


Figure 5: The fourth, fifth and sixth mode shapes of a rotating tapered Rayleigh beam.

6.0 DISCUSSION OF RESULTS

We remarked in this section that the parameters (i.e. rotational speed (α), inverse of slenderness ratio (γ) and tapered ratio (c)) are independent variables while lambda (Λ or λ), $b(\xi)$, $f(\xi)$ and $q(\xi)$ are the dependent variables. In other words, lambda (Λ or λ), $b(\xi)$, $f(\xi)$ and $q(\xi)$ are functions of α , γ and c (see equations 13 and 14). Therefore, varying each of α , γ and c will lead to different values of the dimensionless natural frequencies (Λ or λ). Also, in the numerical analysis, the values of α , γ and c are chosen following Refs.[4, 13 and 15], see page 14, Tables 4, 5, 6, 7 and 8.

Figures 2 and 3 (obtained using Tables 2 and 3) show the trend in the convergences of the first six dimensionless natural frequencies of a rotating prestressed tapered cantilever Rayleigh beam. It was noted that as the number of terms (m) increases, the first six dimensionless natural frequencies λ_1 to λ_6 converged to 6.8324, 23.4058, 55.7606, 103.9057, 167.7632 and 247.4992 (see also Tables 2 and 3) one by one without missing any frequency for $\delta = 0.0001$. The corresponding number of terms (m) are 29, 32, 34, 44, 46, and 82 respectively (see also Tables 2 and 3). It was observed that the convergence speed increased with decreasing frequency order, i.e. the first frequency λ_1 needed 29 terms to reach exact solution, second frequency λ_2 used 32 terms while the third frequency λ_3 took 34 terms to reach the convergence point as shown in Table 2 and 3 as well as Figures 2 and 3.

Figures 4 and 5 show the dynamical behaviour of a rotating prestressed tapered cantilever Rayleigh beam at different frequencies, thereby obtaining various mode shapes. In particular the first three mode shapes, for example, are obtained using equations (57)-(59) respectively. The remaining mode shapes were obtained using their corresponding mode shape functions. It is noted that the first mode shape intercepted the x -axis at a point, the second mode shape at two points while the third mode does this at three points. The same trend was applicable to other mode shapes.

For $n=1$ and 2, Tables 4 and 5 show the results for the first three dimensionless natural frequencies λ for various values of inverse of the slenderness ratio γ , for three different values of the tapered ratio c and rotational speed α . It was observed that the rotational speed has an increasing effect on the dimensionless natural frequencies; while as the inverse of slenderness ratio increases, there was a decrease in the dimensionless natural frequencies. The natural frequencies except the fundamental ones decrease as the taper ratio increases. However, for the fundamental frequencies, the taper ratio leads to a slight increase, which can be observed from those listed in Tables 4 and 5. This means that the fundamental frequencies have a reverse trend of change as compared to other higher-order frequencies. The consideration of inverse of the slenderness ratio gives rise to a reduction of the natural frequencies, while the increase of rotational speed leads to an increase of the dimensionless natural frequencies. Comparing the results of Table 5 with Table 4, it can be found that the inverse of the slenderness ratio on dimensionless natural frequencies is slightly more significant for $n = 2$.

Tables 6 and 7 are presented for validation purposes. The results obtained from the present work are compared with those obtained using other theories reported in the literature. In Table 6, comparison of results with published results for a rotating cantilever tapered Rayleigh beam for $c = 0.5$, $\gamma = \frac{1}{30}$, $E(x) = E(0)$ and $\rho(x) = \rho(0)$ for $n=1$ are shown. We found out that the present results have a good agreement with those presented in Refs. [3, 13 and 15]. In Table 7, comparison of results with published results for a rotating cantilever tapered Rayleigh beam for $c = 0.5$, $\gamma = \frac{1}{100}$, $E(x) = E(0)$ and

$\rho(x) = \rho(0)$ for four different rotational speed at $n=1$ and $n=2$ are shown. Three dimensionless natural frequencies are calculated for a rotating tapered Rayleigh beam and found that our numerical results agreed with those in Refs. [4, 8 and 18].

BE represents Euler-Bernoulli beam. Note that the equation of motion of a tapered rotating prestressed Rayleigh cantilever beam in equation (14) reduces to that of Euler-Bernoulli beam when inverse of the slenderness ratio (γ) is set to 0. Solving this reduced equation, the frequencies of Euler-Bernoulli can be obtained. However, going through the numerical analysis, we noted that as the inverse of the slenderness ratio (γ) approaches zero, then the Rayleigh beam frequencies are found to converge to the Euler-Bernoulli beam frequencies as shown in Table 8. Furthermore, for validation purpose, we compared our frequency values for tapered rotating prestressed Rayleigh cantilever beam with those in Ref. [4] as γ approaches zero and found to be in good agreement. The results obtained from this method [DTM] shown in Tables 6 and 7 compare favourably well with the existing results obtained using other methods. The main findings are that: increasing the rotational speed and the inverse of slenderness ratio of the rotating tapered Rayleigh beam led to an increase and a decrease respectively in its natural frequencies; and increasing the tapered ratio of the beam yielded, in general, a decrease in its natural frequencies.

7.0 CONCLUSION

By the method proposed [DTM] in this study, the closed form series solutions of the free vibration problem of a rotating cantilever tapered Rayleigh beam was obtained. The essential steps of the DTM application consist of three main steps. The steps are: transforming the non-dimensional form of the governing equations of motion into algebraic equations, solving the transformed equations and then applying a process of inverse transform to obtain any desired natural frequency, series solution of any i th mode shape and the dynamic response. All the algebraic calculations are executed speedily by using MAPLE 18. A comparison between the present study and earlier ones for which different solution techniques were used was carried out and the results are found consistent. The convergence of the DTM for the non-dimensional form of the ordinary differential equation is established.

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NOMENCLATURE

$E(x)$: Modulus of elasticity of the beam

$I(x)$: Moment of inertia of the beam

x : The coordinate along the longitudinal axis

t : Time

ξ : Dimensionless length

ω : Natural frequency

$y(\xi)$: Mode shape function

l : Length of beam

Ω : Rotational speed

$E(x)I(x)$: Flexural rigidity

$N(x)$: Centrifugal force

β : Constant of proportionality

c : Taper ratio

$\rho(x)$: Mass per unit volume of the beam

α : Dimensionless rotational speed parameter

$W(x, t)$: Dynamic response of the beam

λ_i : Dimensionless natural frequency at i th mode

γ : Rotatory inertia parameter

$A(x)$: Variable cross-sectional area of the beam

$r(x)$: Radius of gyration