

ISOMORPHISM OF COMMUTATIVE RHOTRIX RINGS

A. Mohammed¹, M. Shettima² and B. M. Ahmed³^{1,2}Department of Mathematics, Ahmadu Bello University, Zaria, Nigeria³Department of Mathematics, University of Ilorin, Ilorin, Nigeria¹Corresponding Author E-mail:abdulmaths@yahoo.com Phone: +2348065519683

ABSTRACT

Isomorphisms of certain commutative rhotrix rings that preserve the same algebraic structure under mapping are constructed. The results are presented for mathematical consideration.

Keywords: Rhotrix, ring, rhotrix ring, rhotrix isomorphism

INTRODUCTION

A rhotrix ring is a ring having rhotrix set as an underlying set. Two rhotrix rings are said to be isomorphic when there is an isomorphism mapping between them. An isomorphism is a bijective homomorphism. The idea of rhotrix ring started from the work of Mohammed (2007), including other algebraic structures such as rhotrix: semigroup, monoid, group and Boolean algebra.

The ideas serve as a form of extension to the initial algebra and analysis of rhotrices conceived by Ajibade (2003), which was also an extension of ideas on matrix-tertition and matrix-noitret created by Atanassov and Shannon (1998). A generalized definition of a rhotrix set was given by Mohammed (2011) as a mathematical rhomboid array of the form:

$$R_n(F) = \left\{ \begin{array}{ccccccc} & & & & r_1 & & \\ & & & & r_2 & r_3 & r_4 \\ & & & \dots & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \dots \\ r_{\left\{\frac{t+1}{2}\right\}-n\backslash 2} & \dots & \dots & \dots & r_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots & \dots & r_{\left\{\frac{t+1}{2}\right\}+n\backslash 2} \\ & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & & r_{t-3} & r_{t-2} & r_{t-1} & & & \\ & & & & r_t & & & & \end{array} \right\} : r_i \in F \quad (1)$$

where $1 \leq i \leq t$, $t = \frac{1}{2}(n^2 + 1)$, $n \in 2\mathbb{Z}^+ + 1$, $n\backslash 2$ is the integer division on division of n by 2 and $h(A_n) = r_{(t+1)/2}$ is called the heart of any rhotrix $A_n \in R_n(F)$. Thus, the weight of any rhotrix $A_n \in R_n(F)$ is given by $|A_n| = \frac{1}{2}(n^2 + 1)$. The rhotrix set $R_n(F)$ has an infinite cardinality, since F is any infinite ring of numbers. The operations of addition and multiplication defined in (Mohammed, 2011) are recorded and modified to compact form respectively as follows:

$$A_n + B_n = \langle a_i \rangle + \langle b_i \rangle = \langle a_i + b_i \rangle = \langle r_i \rangle$$

$$= \left\langle \begin{array}{ccccccc} & & & a_1 + b_1 & & & \\ & & a_2 + b_2 & a_3 + b_3 & a_4 + b_4 & & \\ - & - & - & - & - & - & - \\ - & - & - & a_{\frac{1}{4}(n^2+3)} + b_{\frac{1}{4}(n^2+3)} & - & - & - \\ - & - & - & - & - & - & - \\ & a_{\frac{1}{2}(n^2-5)} + b_{\frac{1}{2}(n^2-5)} & a_{\frac{1}{2}(n^2-3)} + b_{\frac{1}{2}(n^2-3)} & a_{\frac{1}{2}(n^2-1)} + b_{\frac{1}{2}(n^2-1)} & & & \\ & & a_{\frac{1}{2}(n^2+1)} + b_{\frac{1}{2}(n^2+1)} & & & & \end{array} \right\rangle \quad (2)$$

$$A_n \circ B_n = \langle a_i \rangle \circ \langle b_i \rangle = \langle c_i \rangle, \quad (3)$$

Where

$$c_i = \sum h(A)h(B), \text{ for } i = \left\{ \frac{t+1}{2} \right\} \\ a_i h(B) + b_i h(A), \text{ otherwise}$$

It was stated in (Mohammed, 2011) that for any two rhotrices A and I , if $A \circ I = I \circ A = A$ then I is called the identity rhotrix. Thus, the rhotrix

$$A = \left\langle \begin{array}{ccccccc} & & & a_1 & & & \\ & & a_2 & a_3 & a_4 & & \\ - & - & - & - & - & - & - \\ - & - & - & a_{\frac{1}{4}(n^2+3)} & - & - & - \\ - & - & - & - & - & - & - \\ & a_{\frac{1}{2}(n^2-5)} & a_{\frac{1}{2}(n^2-3)} & a_{\frac{1}{2}(n^2-1)} & & & \\ & & a_{\frac{1}{2}(n^2+1)} & & & & \end{array} \right\rangle \quad (4)$$

has

$$I = \left\langle \begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & 0 & 0 & & \\ - & - & - & - & - & - & - \\ 0 & - & 0 & 1 & 0 & - & 0 \\ - & - & - & - & - & - & - \\ & 0 & 0 & 0 & & & \\ & & 0 & & & & \end{array} \right\rangle \quad (5)$$

as its identity rhotrix. The inverse of the rhotrix A is the rhotrix P satisfying the property that

$A \circ P = P \circ A = I$, where P was given as

$$P = A^{-1} = -\frac{1}{(h(A))^2} \left\langle \begin{array}{ccccccc} & & & a_1 & & & \\ & & & a_2 & a_3 & a_4 & \\ - & - & - & - & - & - & - \\ - & - & - & -h(A) & - & - & - \\ - & - & - & - & - & - & - \\ & a_{\frac{1}{2}(n^2-5)} & a_{\frac{1}{2}(n^2-3)} & a_{\frac{1}{2}(n^2-1)} & & & \\ & & a_{\frac{1}{2}(n^2+1)} & & & & \end{array} \right\rangle \quad (6)$$

It was mentioned in Mohammed (2007) that the triple $(R_3(F), +, \circ)$ forms the commutative ring of all rhotrices of size 3 with entries from the ring of all real numbers. A hearty rhotrix set was defined in (Mohammed, 2007) as

$$R_3^*(F) = \left\{ \left\langle \begin{array}{ccc} 0 & & \\ 0 & c & 0 \\ 0 & & \end{array} \right\rangle : 0, c \in F \right\}$$

and it was mentioned that the triple $(R_3^*(F), +, \circ)$ forms 'the ring of all real hearty rhotrices of size 3', which is a subring of $(R_3(F), +, \circ)$. Furthermore, the set

$$R_3^*(Z) = \left\{ \left\langle \begin{array}{ccc} 0 & & \\ 0 & c & 0 \\ 0 & & \end{array} \right\rangle : 0, c \in Z \right\}$$

was called the set of all integer hearty rhotrices of the same size and it was established therein that the triple $(R_3^*(Z), +, \circ)$ forms 'the ring of all integer hearty rhotrices of size 3', which is a subring of $(R_3(F), +, \circ)$.

This later result motivated Tudunkaya and Makanjuola (2010) to establish that the triple $(R_3^*(Z_p), +, \circ)$ forms 'finite field of all residue integer hearty rhotrices of size 3', because the rhotrices contain their entries from finite field of integers modulo prime p , where

$$R_3^*(Z_p) = \left\{ \left\langle \begin{array}{ccc} 0 & & \\ 0 & c & 0 \\ 0 & & \end{array} \right\rangle : 0, c \in Z_p \right\}.$$

Following this, Usaini and Tudunkaya (2011) presented $(R_n^*(Z_p), +, \circ)$ as 'finite field of all residue integer hearty rhotrices of size n ' over finite field Z_p , where

$$R_n^*(Z_p) = \left\{ \begin{pmatrix} & & & 0 & & \\ & \dots & & \dots & \dots & \\ 0 & \dots & 0 & c & 0 & \dots & 0 \\ & & \dots & \dots & \dots & & \\ & & & 0 & & & \end{pmatrix} : 0, c \in Z_p \right\}.$$

Mohammed and Shettima (2017a) showed that the triple $(R_n(F), +, \circ)$ forms 'the ring of all real rhotrices of size n ' called commutative full rhotrix of numbers. Furthermore, Mohammed and Shettima (2017b) determined a number of subrings of $(R_3(F), +, \circ)$ in which the triple $(R_n^*(F), +, \circ)$ forms 'the ring of all real hearty rhotrices of size n ', a generalization of $(R_3^*(F), +, \circ)$ in (Mohammed, 2007). Ideals and quotient rings in commutative full rhotrix ring was presented by Mohammed and Shettima (2018a). Also, the idea of representing any ring (group) as hearty rhotrix ring (group) was discussed in Mohammed and Shettima (2018b). Following this, Mohammed and Shettima (2020) constructed the commutative full rhotrix ring of fraction.

While exploring the literature of rhotrix ring theory, it may be worthy of note to mention that a number of subrings of commutative full rhotrix ring over the usual ring of real numbers does exhibit the same algebraic structure when

mappings are considered between them. This reason serves as a motivation for this research.

THE COMMUTATIVE RHOTRIX RING AND ITS SUBRINGS

As a necessary precursor for understanding the results discussed in the next section below, the reader is advised to study the subrings of the commutative rhotrix ring established by Mohammed and Shettima (2017b).

ISOMORPHIC RHOTRIX RINGS

The results in this section show that a number of rhotrix rings preserve the same algebraic structure under mappings.

Theorem 1

Let $UT_n(F)$ be the upper triangular rhotrix ring and let $DT_n(F)$ be the lower triangular rhotrix ring. Let $f : UT_n(F) \rightarrow DT_n(F)$ defined by $f(A_n^{ut}) = A_n^{dt}$,

where:

$$A_n^{ut} = \left\langle \begin{array}{ccccccc} & & & a_1 & & & \\ & & & a_2 & a_3 & a_4 & \\ & & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{\left\{\frac{t+1}{2}\right\}-n\setminus 2} & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} \\ & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & & \\ & & & & 0_t & & & & \end{array} \right\rangle \in UT_n(F)$$

and

$$A_n^{dt} = \left\langle \begin{array}{ccccccc} & & & 0_t & & & \\ & & & 0_{t-1} & 0_{t-2} & 0_{t-3} & \\ & & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{\left\{\frac{t+1}{2}\right\}-n\setminus 2} & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} \\ & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & & a_2 & a_3 & a_4 & & & \\ & & & & a_1 & & & & \end{array} \right\rangle \in DT_n(F).$$

Then

$$UT_n(F) \cong DT_n(F)$$

Proof

First, it will be shown that the mapping f is a homomorphism, Now, by hypothesis, $f : UT_n(F) \rightarrow DT_n(F)$ is defined by $f(A_n^{ut}) = A_n^{dt}$, for all $A_n^{ut} \in UT_n(F)$, $A_n^{dt} \in DT_n(F)$.

Therefore, for any $A_n^{ut}, B_n^{ut} \in UT_n(F)$, we have

$$(i) \quad f(A_n^{ut} + B_n^{ut})$$

$$\begin{aligned}
 &= f \left(\begin{array}{cccccccc} & & & & a_1 + b_1 & & & \\ & & & & a_2 + b_2 & a_3 + b_3 & a_4 + b_4 & \\ & & \dots & \dots & \dots & \dots & \dots & \\ & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & a_{\left\{\frac{t+1}{2}\right\}-n\setminus 2} + b_{\left\{\frac{t+1}{2}\right\}-n\setminus 2} & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} + b_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} + b_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} \\ & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & & & & 0_t & & & \end{array} \right) \\
 &= \left(\begin{array}{cccccccc} & & & & 0_t & & & \\ & & & 0_{t-1} & 0_{t-2} & 0_{t-3} & & \\ & \dots & \dots & \dots & \dots & \dots & \dots & \\ & \dots & \dots & \dots & \dots & \dots & \dots & \\ & a_{\left\{\frac{t+1}{2}\right\}-n\setminus 2} + b_{\left\{\frac{t+1}{2}\right\}-n\setminus 2} & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} + b_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} + b_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} \\ & \dots & \dots & \dots & \dots & \dots & \dots & \\ & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & & a_2 + b_2 & a_3 + b_3 & a_4 + b_4 & & \\ & & & & a_1 + b_1 & & & \end{array} \right) \\
 &= \left(\begin{array}{cccccccc} & & & & 0_t & & & \\ & & & 0_{t-1} & 0_{t-2} & 0_{t-3} & & \\ & \dots & \dots & \dots & \dots & \dots & \dots & \\ & \dots & \dots & \dots & \dots & \dots & \dots & \\ & a_{\left\{\frac{t+1}{2}\right\}-n\setminus 2} & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} \\ & \dots & \dots & \dots & \dots & \dots & \dots & \\ & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & & a_2 & a_3 & a_4 & & \\ & & & & a_1 & & & \end{array} \right)
 \end{aligned}$$

$$+ \left\langle \begin{array}{ccccccc} & & & 0_t & & & \\ & & 0_{t-1} & 0_{t-2} & 0_{t-3} & & \\ & \dots & \dots & \dots & \dots & \dots & \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{\left\{\frac{t+1}{2}\right\}-n\setminus 2} & \dots & \dots & b_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots & b_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} \\ & \dots & \dots & \dots & \dots & \dots & \\ & \dots & \dots & \dots & \dots & \dots & \\ & & b_2 & b_3 & b_4 & & \\ & & & b_1 & & & \end{array} \right\rangle$$

$$= f(A_n^{ut}) + f(B_n^{ut})$$

$$(ii) f(A_n^{ut} \circ B_n^{ut})$$

$$= f \left(\left\langle \begin{array}{ccccccc} & & & a_1 h(B_n^{ut}) + b_1 h(A_n^{ut}) & & & \\ & & a_2 h(B_n^{ut}) + b_2 h(A_n^{ut}) & a_3 h(B_n^{ut}) + b_3 h(A_n^{ut}) & a_4 h(B_n^{ut}) + b_4 h(A_n^{ut}) & & \\ & \dots & \dots & \dots & \dots & \dots & \\ & \dots & \dots & \dots & \dots & \dots & \\ p & \dots & \dots & h(A_n^{ut}) h(B_n^{ut}) & \dots & \dots & q \\ & \dots & \dots & \dots & \dots & \dots & \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & & & 0_t & & & \end{array} \right\rangle \right)$$

$$= \left\langle \begin{array}{ccccccc} & & & 0_t & & & \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & \dots & \dots & \dots & \dots & \dots & \\ & \dots & \dots & \dots & \dots & \dots & \\ p & \dots & \dots & h(A_n^{dt}) h(B_n^{dt}) & \dots & \dots & q \\ & \dots & \dots & \dots & \dots & \dots & \\ & & a_2 h(B_n^{dt}) + b_2 h(A_n^{dt}) & a_3 h(B_n^{dt}) + b_3 h(A_n^{dt}) & a_4 h(B_n^{dt}) + b_4 h(A_n^{dt}) & & \\ & & & a_1 h(B_n^{dt}) + b_1 h(A_n^{dt}) & & & \end{array} \right\rangle$$

$$\begin{aligned}
 &= \left\langle \begin{array}{ccccccc} & & & 0_t & & & \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & \dots & \dots & \dots & \dots & \dots & \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{\left\{\frac{t+1}{2}\right\}-n\setminus 2} & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} \\ & \dots & \dots & \dots & \dots & \dots & \\ & \dots & \dots & \dots & \dots & \dots & \\ & & a_2 & a_3 & a_4 & & \\ & & & a_1 & & & \end{array} \right\rangle \\
 &\circ \left\langle \begin{array}{ccccccc} & & & 0_t & & & \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & \dots & \dots & \dots & \dots & \dots & \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{\left\{\frac{t+1}{2}\right\}-n\setminus 2} & \dots & \dots & b_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots & b_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} \\ & \dots & \dots & \dots & \dots & \dots & \\ & \dots & \dots & \dots & \dots & \dots & \\ & & b_2 & b_3 & b_4 & & \\ & & & b_1 & & & \end{array} \right\rangle
 \end{aligned}$$

$$= f(A_n^{ut}) \circ f(B_n^{ut}) \in DT_n(F),$$

where:

$$h(A_n^{ut}) = h(A_n^{dt}) = a_{\left\{\frac{t+1}{2}\right\}}$$

$$h(B_n^{ut}) = h(B_n^{dt}) = b_{\left\{\frac{t+1}{2}\right\}},$$

$$p = a_{\left\{\frac{t+1}{2}\right\}-n\setminus 2} b_{\left\{\frac{t+1}{2}\right\}} + b_{\left\{\frac{t+1}{2}\right\}-n\setminus 2} a_{\left\{\frac{t+1}{2}\right\}},$$

$$q = a_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} b_{\left\{\frac{t+1}{2}\right\}} + b_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} a_{\left\{\frac{t+1}{2}\right\}}.$$

Implying that f is a homomorphism.

Next, f is an injection, since $\ker f = \{O_n \in UT_n(F)\}$ and in addition, if $f(A_n^{ut}) = f(B_n^{ut})$ then it implies that $A_n^{ut} = B_n^{ut}$, since their image is the same down triangular rhotrices. Also, f is a surjection, since the image of

the mapping f , given by $im f = f(UT_n(F)) = DT_n(F)$. Thus, f is a bijective homomorphism. Hence, $UT_n(F) \cong DT_n(F)$.

Theorem 2

Let $LT_n(F)$ be the left triangular rhotrix ring and let $RT_n(F)$ be the right triangular rhotrix ring. Let $f : LT_n(F) \rightarrow RT_n(F)$ defined by $f(A_n^{lt}) = A_n^{rt}$, where:

$$A_n^{lt} = \left\langle \begin{array}{ccccccc} & & & & a_1 & & \\ & & & & a_2 & a_3 & 0_4 \\ & & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & a_{t-3} & a_{t-2} & 0_{t-1} \\ & & & & & a_t & \end{array} \right\rangle \in LT_n(F)$$

and

$$A_n^{rt} = \left\langle \begin{array}{ccccccc} & & & & a_1 & & \\ & & & & 0_4 & a_3 & a_2 \\ & & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & 0_{t-1} & a_{t-2} & a_{t-3} \\ & & & & & a_t & \end{array} \right\rangle \in RT_n(F).$$

Then

$$LT_n(F) \cong RT_n(F)$$

Proof

First, it will be shown that the mapping f is a homomorphism. Now, by hypothesis, $f : LT_n(F) \rightarrow RT_n(F)$ is defined by $f(A_n^{lt}) = A_n^{rt}$, for all $A_n^{lt} \in LT_n(F)$, $A_n^{rt} \in RT_n(F)$. Therefore, for any $A_n^{lt}, B_n^{lt} \in LT_n(F)$, we have by computation that:

$$(ii) \quad f(A_n^{lt} + B_n^{lt}) = f(A_n^{lt}) + f(B_n^{lt})$$

$$(iii) \quad f(A_n^{lt} \circ B_n^{lt}) = f(A_n^{lt}) \circ f(B_n^{lt})$$

Implying that f is a homomorphism.

Next, f is an injection, since $\ker f = \{O_n \in LT_n(F)\}$ and in addition, if $f(A_n^{lt}) = f(B_n^{lt})$ then it implies that $A_n^{lt} = B_n^{lt}$, since their image is the same down triangular rhotrices. Also, f is a surjection, since the image of the mapping f , given by $\text{im } f = f(LT_n(F)) = RT_n(F)$. Thus, f is a bijective homomorphism. Hence, $LT_n(F) \cong RT_n(F)$.

Theorem 3

Let $ULT_n(F)$ be the left triangular rhotrix ring and let $URT_n(F)$ be the upper right triangular rhotrix ring. Let $f : ULT_n(F) \rightarrow URT_n(F)$ defined by $f(A_n^{ult}) = A_n^{urt}$,

where:

$$A_n^{ult} = \left\langle \begin{array}{ccccccc} & & & & a_1 & & \\ & & & & a_2 & a_3 & 0_4 \\ & & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & 0_{t-3} & 0_{t-2} & 0_{t-1} \\ & & & & & 0_t & \end{array} \right\rangle \in ULT_n(F)$$

and

$$A_n^{urt} = \left\langle \begin{array}{ccccccc} & & & & a_1 & & \\ & & & & 0_4 & a_3 & a_2 \\ & & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & 0_{t-3} & 0_{t-2} & 0_{t-1} \\ & & & & & 0_t & \end{array} \right\rangle \in URT_n(F).$$

Then

$$ULT_n(F) \cong URT_n(F)$$

Proof

First, it will be shown that the mapping f is a homomorphism, Now, by hypothesis, $f : ULT_n(F) \rightarrow URT_n(F)$ is defined by $f(A_n^{ult}) = A_n^{urt}$, for all $A_n^{ult} \in ULT_n(F)$, $A_n^{urt} \in URT_n(F)$. Therefore, for any $A_n^{ult}, B_n^{ult} \in ULT_n(F)$, we have by computation that:

$$(iv) \quad f(A_n^{ult} + B_n^{ult}) = f(A_n^{ult}) + f(B_n^{ult})$$

$$(v) \quad f(A_n^{ult} \circ B_n^{ult}) = f(A_n^{ult}) \circ f(B_n^{ult})$$

Implying that f is a homomorphism.

Next, f is an injection, since $\ker f = \{O_n \in ULT_n(F)\}$ and in addition, if $f(A_n^{ult}) = f(B_n^{ult})$ then it implies that $A_n^{ult} = B_n^{ult}$, since their image is the same upper right triangular rhotrices. Also, f is a surjection, since the image of the mapping f , given by $im f = f(ULT_n(F)) = URT_n(F)$. Thus, f is a bijective homomorphism.

Hence, $ULT_n(F) \cong URT_n(F)$.

Theorem 4

Let $URT_n(F)$ be the upper right triangular rhotrix ring and let $DRT_n(F)$ be the down right triangular rhotrix ring. Let $f : URT_n(F) \rightarrow DRT_n(F)$ defined by $f(A_n^{urt}) = A_n^{drt}$,

where:

$$A_n^{urt} = \left\langle \begin{array}{ccccccc} & & & & a_1 & & \\ & & & & 0_2 & a_3 & a_4 \\ & & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ 0_{\left\{\frac{t+1}{2}\right\}-n\setminus 2} & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} \\ & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \dots \\ & & & 0_{t-3} & 0_{t-2} & 0_{t-1} & \\ & & & & 0_t & & \end{array} \right\rangle \in URT_n(F)$$

and

$$A_n^{drt} = \left(\begin{array}{ccccccc} & & & & 0_t & & \\ & & & & 0_{t-3} & 0_{t-2} & 0_{t-1} \\ & & & \dots & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \dots \\ 0_{\left\{\frac{t+1}{2}\right\}-n\setminus 2} & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} \\ & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \dots & \\ & & & 0_2 & a_3 & a_4 & & \\ & & & & a_1 & & & \end{array} \right) \in DRT_n(F)$$

Then

$$URT_n(F) \cong DRT_n(F)$$

Proof

First, it will be shown that the mapping f is a homomorphism, Now, by hypothesis, $f : URT_n(F) \rightarrow DRT_n(F)$ is defined by $f(A_n^{urt}) = A_n^{drt}$, for all $A_n^{urt} \in URT_n(F)$, $A_n^{drt} \in DRT_n(F)$.

Therefore, for any $A_n^{urt}, B_n^{urt} \in URT_n(F)$, we have by computation that:

$$(vi) \quad f(A_n^{urt} + B_n^{urt}) = f(A_n^{urt}) + f(B_n^{urt})$$

$$(vii) \quad f(A_n^{urt} \circ B_n^{urt}) = f(A_n^{urt}) \circ f(B_n^{urt})$$

Implying that f is a homomorphism.

Next, f is an injection, since $\ker f = \{O_n \in URT_n(F)\}$ and in addition, if $f(A_n^{urt}) = f(B_n^{urt})$ then it implies that $A_n^{urt} = B_n^{urt}$, since their image is the same down right triangular rhotrices. Also, f is a surjection, since the image of the mapping f , given by $im f = f(URT_n(F)) = DRT_n(F)$. Thus, f is a bijective homomorphism.

Hence, $URT_n(F) \cong DRT_n(F)$.

Theorem 5

Let $DRT_n(F)$ be the down right triangular rhotrix ring and let $DLT_n(F)$ be the down left triangular rhotrix ring.

Let $f : DRT_n(F) \rightarrow DLT_n(F)$ defined by $f(A_n^{drt}) = A_n^{dlt}$,

where:

$$A_n^{drt} = \left\langle \begin{array}{ccccccc} & & & 0_1 & & & \\ & & & 0_2 & 0_3 & 0_4 & \\ & & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots \\ 0_{\left\{\frac{t+1}{2}\right\}-n\setminus 2} & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & 0_{t-3} & a_{t-2} & a_{t-1} & \\ & & & & a_t & & \end{array} \right\rangle \in DRT_n(F)$$

and

$$A_n^{dlt} = \left\langle \begin{array}{ccccccc} & & & 0_1 & & & \\ & & & 0_2 & 0_3 & 0_4 & \\ & & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots \\ a_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & a_{t-1} & a_{t-2} & 0_{t-3} & \\ & & & & a_t & & \end{array} \right\rangle \in DLT_n(F).$$

Then

$$DRT_n(F) \cong DLT_n(F)$$

Proof

First, it will be shown that the mapping f is a homomorphism, Now, by hypothesis, $f : DRT_n(F) \rightarrow DLT_n(F)$

is defined by $f(A_n^{drt}) = A_n^{dlt}$, for all $A_n^{drt} \in DRT_n(F)$, $A_n^{dlt} \in DLT_n(F)$.

Therefore, for any $A_n^{drt}, B_n^{drt} \in DRT_n(F)$, we have by computation that:

$$(viii) \quad f(A_n^{ut} + B_n^{ut}) = f(A_n^{drt}) + f(B_n^{drt})$$

$$(ix) \quad f(A_n^{drt} \circ B_n^{drt}) = f(A_n^{drt}) \circ f(B_n^{drt})$$

Implying that f is a homomorphism.

Theorem 6

$$A_n^{dlt} = \left\langle \begin{array}{ccccccc} & & & & 0_1 & & \\ & & & & 0_2 & 0_3 & 0_4 \\ & & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{\left\{\frac{t+1}{2}\right\}-n\setminus 2} & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots & 0_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \\ & & & a_{t-3} & a_{t-2} & 0_{t-1} & \\ & & & & a_t & & \end{array} \right\rangle \in DLT_n(F)$$

and

$$A_n^{ult} = \left\langle \begin{array}{ccccccc} & & & a_t & & & \\ & & & a_{t-2} & 0_{t-1} & & \\ & & a_{t-3} & & & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & 0_2 & 0_3 & 0_4 & & \\ & & & 0_1 & & & \end{array} a_{\left\{\frac{t+1}{2}\right\}-n\setminus 2} \quad 0_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} \right\rangle \in ULT_n(F).$$

$$DLT_n(F) \cong ULT_n(F)$$

Proof

First, it will be shown that the mapping f is a homomorphism, Now, by hypothesis, $f : DLT_n(F) \rightarrow ULT_n(F)$ is defined by $f(A_n^{dlt}) = A_n^{ult}$, for all $A_n^{dlt} \in DLT_n(F)$, $A_n^{ult} \in ULT_n(F)$.

Therefore, for any $A_n^{dlt}, B_n^{dlt} \in DLT_n(F)$, we have by computation that:

$$(x) \quad f(A_n^{dlt} + B_n^{dlt}) = f(A_n^{dlt}) + f(B_n^{dlt})$$

$$(xi) \quad f(A_n^{dlt} \circ B_n^{dlt}) = f(A_n^{dlt}) \circ f(B_n^{dlt})$$

Implying that f is a homomorphism.

Next, f is an injection, since $\ker f = \{O_n \in DLT_n(F)\}$ and in addition, if $f(A_n^{dlt}) = f(B_n^{dlt})$ then it implies that $A_n^{dlt} = B_n^{dlt}$, since their image is the same upper left triangular rhotrices. Also, f is a surjection, since the image of the mapping f , given by $im f = f(DLT_n(F)) = ULT_n(F)$. Thus, f is a bijective homomorphism.

Hence, $DLT_n(F) \cong ULT_n(F)$.

Theorem 7

Let $ULT_n(F)$ be the upper left triangular rhotrix ring and let $DRT_n(F)$ be the down right triangular rhotrix ring.

Let $f : ULT_n(F) \rightarrow DRT_n(F)$ defined by $f(A_n^{ult}) = A_n^{drt}$,

where:

$$A_n^{ult} = \left\langle \begin{array}{ccccccc} & & & & a_1 & & \\ & & & & a_2 & a_3 & 0_4 \\ & & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & 0_{t-3} & 0_{t-2} & 0_{t-1} \\ & & & & & 0_t & \end{array} \right\rangle \in ULT_n(F)$$

and

$$A_n^{drt} = \left(\begin{array}{ccccccc} & & & & 0_1 & & \\ & & & & 0_2 & 0_3 & 0_4 \\ & & & \dots & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \dots \\ 0_{\left\{\frac{t+1}{2}\right\}-n\setminus 2} & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} \\ & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots & \dots & \\ & & & 0_{t-3} & a_{t-2} & a_{t-1} & & \\ & & & & a_t & & & \end{array} \right) \in DRT_n(F).$$

Then

$$ULT_n(F) \cong DRT_n(F)$$

Proof

First, it will be shown that the mapping f is a homomorphism, Now, by hypothesis, $f : ULT_n(F) \rightarrow DRT_n(F)$ is defined by $f(A_n^{ult}) = A_n^{drt}$, for all $A_n^{ult} \in ULT_n(F)$, $A_n^{drt} \in DRT_n(F)$.

Therefore, for any $A_n^{ult}, B_n^{ult} \in ULT_n(F)$, we have by computation that:

$$(xii) \quad f(A_n^{ult} + B_n^{ult}) = f(A_n^{ult}) + f(B_n^{ult})$$

$$(xiii) \quad f(A_n^{ult} \circ B_n^{ult}) = f(A_n^{ult}) \circ f(B_n^{ult})$$

Implying that f is a homomorphism.

Next, f is an injection, since $\ker f = \{O_n \in ULT_n(F)\}$ and in addition, if $f(A_n^{ult}) = f(B_n^{ult})$ then it implies that $A_n^{ult} = B_n^{ult}$, since their image is the same down right triangular rhotrices. Also, f is a surjection, since the image of the mapping f , given by $im f = f(ULT_n(F)) = DRT_n(F)$. Thus, f is a bijective homomorphism. Hence, $ULT_n(F) \cong DRT_n(F)$.

Theorem 8

Let $URT_n(F)$ be the upper right triangular rhotrix ring and let $DLT_n(F)$ be the down left triangular rhotrix ring. Let $f : URT_n(F) \rightarrow DLT_n(F)$ defined by $f(A_n^{urt}) = A_n^{dlt}$, where:

$$A_n^{urt} = \left(\begin{array}{ccccccc} & & & a_1 & & & \\ & & & 0_2 & a_3 & a_4 & \\ & & \dots & \dots & \dots & \dots & \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} \\ & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & \dots & \dots & \dots & \dots & & \\ & & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & & & & 0_t & & & \end{array} \right) \in URT_n(F)$$

and

$$A_n^{dlt} = \left(\begin{array}{ccccccc} & & & 0_1 & & & \\ & & & 0_2 & 0_3 & 0_4 & \\ & & \dots & \dots & \dots & \dots & \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ & \dots & \dots & \dots & a_{\left\{\frac{t+1}{2}\right\}} & \dots & \dots & 0_{\left\{\frac{t+1}{2}\right\}+n\setminus 2} \\ & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & \dots & \dots & \dots & \dots & & \\ & & & 0_{t-3} & a_{t-2} & 0_{t-1} & & \\ & & & & a_t & & & \end{array} \right) \in DLT_n(F).$$

Then

$$URT_n(F) \cong DLT_n(F)$$

Proof

First, it will be shown that the mapping f is a homomorphism, Now, by hypothesis, $f : URT_n(F) \rightarrow DLT_n(F)$ is defined by $f(A_n^{urt}) = A_n^{dlt}$, for all $A_n^{urt} \in URT_n(F)$, $A_n^{dlt} \in DLT_n(F)$.

Therefore, for any $A_n^{urt}, B_n^{urt} \in URT_n(F)$, we have by computation that:

$$(xiv) \quad f(A_n^{urt} + B_n^{urt}) = f(A_n^{urt}) + f(B_n^{urt})$$

$$(xv) \quad f(A_n^{urt} \circ B_n^{urt}) = f(A_n^{urt}) \circ f(B_n^{urt})$$

Implying that f is a homomorphism.

Next, f is an injection, since $\ker f = \{O_n \in URT_n(F)\}$ and in addition, if $f(A_n^{urt}) = f(B_n^{urt})$ then it implies that $A_n^{urt} = B_n^{urt}$, since their image is the same down left triangular rhotrices. Also, f is a surjection, since the image of the mapping f , given by $\text{im } f = f(URT_n(F)) = DLT_n(F)$. Thus, f is a bijective homomorphism. Hence, $URT_n(F) \cong DLT_n(F)$.

REFERENCES

- Ajibade, A. O. (2003). The Concept of Rhotrix in Mathematical Enrichment, *International Journal of Mathematical Education in Science and Technology*, 34, 175-79.
- Atanassov, K.T. and Shannon, A.G.(1998). Matrix-tertions and Matrix-noitrets: Exercises in Enrichment, *International Journal of Mathematical Education in Science and Technology*, 29, 898 – 903.
- Mohammed, A., (2007). "Enrichment exercises through extension to rhotrices", *International Journal of Mathematical Education in Science and Technology*, 38, 131 – 136.
- Mohammed, A. (2011). Theoretical Development and Applications of Rhotrices, Ph.D. Thesis, Ahmadu Bello University, Zaria (Unpublished).
- Mohammed, A. and Shettima, M. (2017a). "Axiomatic Developments for Commutative Rhotrix Spaces", *Katsina Journal of Natural and Applied Sciences*, Published by Umaru Musa Yaradua University, 6 (1), 1-10.
- Mohammed, A. and Shettima, M. (2017b). "On Subrings of Commutative Ring of All Real Rhotrices of Size Three", *Journal of the Nigerian Association of Mathematical Physics*, Published by Nigerian Association of Mathematical Physics, 43, 19-30.
- Mohammed, A. and Shettima, M. (2018a). "Ideals and Quotient Rings in Commutative Full Rhotrix Rings", *Katsina Journal of Natural and Applied Sciences*, Published by Umaru Musa Yaradua University, 7 (2), 126-142.
- Mohammed, A. and Shettima, M. (2018b). "Representation of an Arbitrary Ring (or Group) as Hearty Rhotrix Ring (or Group): Two Analogous Cases to Cayle's Theorem", *Katsina Journal of Natural and Applied Sciences*, Published by Umaru Musa Yaradua University, 7 (2), 110-118.
- Mohammed, A. and Shettima, M. (2020). "The Commutative Full Rhotrix Ring of Fractions", *Katsina Journal of Natural and Applied Sciences*, Published by Umaru Musa Yaradua University, 9 (1), 9-22.
- Tudunkaya, S. M. and Mekanjuola, S. O., (2010). "Rhotrices and the construction of finite fields", *Bulletin of Pure and Applied Sciences*, 29E, 225-229.
- Usaini, S. and Tudunkaya, S. M., (2011). "Note on rhotrices and the construction of finite fields", *Bulletin of Pure and Applied Sciences*, 30E, 53-59.