

Analytical Solutions of Nonlinear Initial-Boundary Value Problems

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Abstract

This work presents solutions to initial-boundary value problems that are generally nonlinear exploring Taylor's theorem in a decomposition technic. Not until few decades back, real world phenomena in many fields of human endeavours were modelled into initial and boundary value problems. Due to identified deficiencies in doing that, initial-boundary value equations are used to achieve the aim. The problems are mostly nonlinear. Solution to such category of problems mostly defies the use of the established analytical methods. Most researchers employ numerical approaches that result in approximate solutions. The present approach uses Taylor's theorem alongside Adomian decomposition method. The nonlinear aspect is treated by decomposing it into polynomials, while the resulting unknown terms are evaluated through Taylor's expansion. The method is applied to eight problems in literature, all of which give exact solution. Mathematica 13.3 is employed for the simulations.

Keywords: Adomian polynomials, Boundary value problem, Initial value problem, Taylor's theorem for two variables.

Introduction

The importance of differential equations in the understanding of the physical world around us, especially as it affects modelling of phenomena that affect human and animal existence (Jimoh & Oyedeki, 2020).

Since the resulting differential equations are mostly nonlinear, solving them by the traditional methods (otherwise referred to as analytical methods) becomes very difficult and in most cases outrightly impossible. Thus, the need to employ other approaches such as numerical methods or better still semi-analytical methods. The latter is in many instances preferred since they give exact solution whenever such exists in closed form. When the problem does not have an exact solution, the truncated series solution that semi-analytical methods will give is always better than what numerical approach would give. Prominent among those methods are Adomian decomposition

method (ADM) Adomian (1994), variational iteration method (VIM) in (He, 1999), homotopy analysis method (HAM) introduced by (Liao, 1997), just to mention a few. The cornerstone of ADM lies in the decomposition of the nonlinear operator present in the equation into a series of specific polynomials, known as Adomian polynomials, which depend on the series components of the solution (Elsaid, 2012). The current study solves non-linear partial differential equations using Adomian decomposition method.

Boundary value problems (BVPs) as expressed by (Boyce, 2017; Agmon, 2010; Ladyzhenskaya, 2013; Gakhov, 2014) are the physical application that often led to another problem in which the value of the dependent variable or its derivatives is specified at two different points. Boundary conditions are described as the values or relationship prescribed for the function or its derivatives at the special boundary of the region. But when the conditions are specified at the same point, it is referred to as initial value problems (IVPs) (Sarbach & Tiglio, 2012; Synge, 2012; Keller, 2018).

(Ahmed & Dmitry, 2011) and few other scientists proved the convergence of Adomian decomposition method for various types of problems. The researchers made use of Majorants, which is a tool in Cauchy-Kovalevsky theorem for differential equations with analytic vector fields, others used Picard iterative approach to establish its convergence. ADM does not require any small or large parameter in its application, nor does it require any control parameter that guarantees convergence of its solution.

To simplify the derivation of Adomian polynomials for strongly nonlinear terms, (Yisa, 2019) presented solutions of ordinary differential equations with special nonlinearities. In that work, the author derived the Adomian polynomials for some selected difficult nonlinear terms and implemented them in the solutions of nonlinear initial value problems using ADM.

In the present work, ADM is used to solve initial-boundary value problems of one, two and three dimensions. The category of problems considered cannot be handled without incorporating Taylor's theorem involving two variables.

Literature Review

In this section, the definitions relevant to the discussion in what follows, statement of the problem considered, and some other useful things are presented.

The Definition of Taylor's Theorem for Two Variables

Let f be a function of two variables ξ and η . If all partial derivatives of order m are continuous in a closed region and if all the $(m + 1)$ partial derivatives exist in the open region, then

$$\begin{aligned} f(\xi_0 + h, \eta_0 + k) = & f(\xi_0, \eta_0) + \left[h \frac{\partial}{\partial \xi} + k \frac{\partial}{\partial \eta} \right] f(\xi_0, \eta_0) + \frac{1}{2!} \left[h \frac{\partial}{\partial \xi} + k \frac{\partial}{\partial \eta} \right]^2 f(\xi_0, \eta_0) + \dots \\ & + \frac{1}{m!} \left[h \frac{\partial}{\partial \xi} + k \frac{\partial}{\partial \eta} \right]^m f(\xi_0, \eta_0) + R_m, \end{aligned} \quad (1)$$

where

$$R_m = \frac{1}{(m+1)!} \left[h \frac{\partial}{\partial \xi} + k \frac{\partial}{\partial \eta} \right]^{m+1} f(\xi_0 + \theta h, \eta_0 + \theta k), \quad 0 < \theta < 1 \text{ (Wrede, 2010)}$$

Definition of Adomian polynomials

A less cumbersome formula for the derivation of polynomials corresponding to respective nonlinear terms was reported in Adomian (1994), as

$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \left[N \left(\sum_{k=0}^n \lambda^k u_k \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots \quad (2)$$

Considered the nonlinear n th-order initial-boundary value problem:

$$Lu(\xi, \eta) + Ru(\xi, \eta) + N(u(\xi, \eta)) = g(\xi, \eta) \quad (3)$$

with boundary conditions: $u(0, \eta) = \beta_0$, $u(L, \eta) = \beta_1$

and associated initial conditions: $u(\xi, 0) = \alpha_0$, $u_\eta(\xi, 0) = \alpha_1, \dots, u_{\eta \dots \eta}(\xi, 0) = \alpha_{n-1}$

Methodology

Generation of Adomian polynomials for some nonlinear terms

In this section, Adomian polynomials for nonlinear terms were generated for the first few polynomials A_0, A_1, A_2 .

Consider the nonlinear term $N(u) = (u_{\xi\xi})^2$

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$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \left[N \left(\sum_{k=0}^n \lambda^k u_k \right) \right]_{\lambda=0},$$

For $n = 0$:

$$A_0 = N(u_0) = (u_0)_{\xi\xi}^2$$

For $n = 1$:

$$A_1 = \frac{\partial}{\partial \lambda} \left[N \left(\sum_{k=0}^1 \lambda^k u_k \right) \right]_{\lambda=0},$$

$$A_1 = \frac{\partial}{\partial \lambda} [N(u_0 + \lambda u_1)]_{\lambda=0}$$

$$A_1 = \frac{\partial}{\partial \lambda} [(u_0)_{\xi\xi} + \lambda (u_1)_{\xi\xi}]^2_{\lambda=0}$$

$$A_1 = 2(u_1)_{\xi\xi} [(u_0)_{\xi\xi} + \lambda (u_1)_{\xi\xi}]_{\lambda=0}$$

$$A_1 = 2(u_0)_{\xi\xi} (u_1)_{\xi\xi}$$

For $n = 2$:

$$A_2 = \frac{1}{2!} \frac{\partial^2}{\partial \lambda^2} \left[N \left(\sum_{k=0}^2 \lambda^k u_k \right) \right]_{\lambda=0},$$

$$A_2 = \frac{1}{2!} \frac{\partial^2}{\partial \lambda^2} [N(u_0 + \lambda u_1 + \lambda^2 u_2)]_{\lambda=0}$$

$$A_2 = \frac{1}{2!} \frac{\partial^2}{\partial \lambda^2} [(u_0)_{\xi\xi} + \lambda (u_1)_{\xi\xi} + \lambda^2 (u_2)_{\xi\xi}]^2_{\lambda=0}$$

$$A_2 = \frac{\partial}{\partial \lambda} [((u_0)_{\xi\xi} + \lambda (u_1)_{\xi\xi} + \lambda^2 (u_2)_{\xi\xi})((u_1)_{\xi\xi} + 2\lambda (u_2)_{\xi\xi})]_{\lambda=0}$$

$$[A_2 = [((u_1)_{\xi\xi} + 2\lambda (u_2)_{\xi\xi})^2 + ((u_0)_{\xi\xi} + \lambda (u_1)_{\xi\xi} + \lambda^2 (u_2)_{\xi\xi})(2(u_2)_{\xi\xi})]_{\lambda=0}$$

$$A_2 = (u_1)_{\xi\xi}^2 + 2(u_0)_{\xi\xi} (u_2)_{\xi\xi}$$

Consider the nonlinear term $N(u) = \sin(u_\xi)$

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$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \left[N \left(\sum_{k=0}^n \lambda^k u_k \right) \right]_{\lambda=0},$$

For $n = 0$:

$$A_0 = N(u_0) = \sin(u_0)_\xi$$

For $n = 1$:

$$A_1 = \frac{\partial}{\partial \lambda} \left[N \left(\sum_{k=0}^1 \lambda^k u_k \right) \right]_{\lambda=0},$$

$$A_1 = \frac{\partial}{\partial \lambda} [N(u_0 + \lambda u_1)]_{\lambda=0}$$

$$A_1 = \frac{\partial}{\partial \lambda} [\sin(u_0 + \lambda u_1)_\xi]_{\lambda=0}$$

$$A_1 = (u_1)_\xi \cos(u_0 + \lambda u_1)_\xi \big|_{\lambda=0}$$

$$A_1 = (u_1)_\xi \cos(u_0)_\xi$$

For $n = 2$:

$$A_2 = \frac{1}{2!} \frac{\partial^2}{\partial \lambda^2} \left[N \left(\sum_{k=0}^2 \lambda^k u_k \right) \right]_{\lambda=0},$$

$$A_2 = \frac{1}{2!} \frac{\partial^2}{\partial \lambda^2} [N(u_0 + \lambda u_1 + \lambda^2 u_2)]_{\lambda=0}$$

$$A_2 = \frac{1}{2!} \frac{\partial^2}{\partial \lambda^2} [\sin((u_0)_\xi + \lambda(u_1)_\xi + \lambda^2(u_2)_\xi)]_{\lambda=0}$$

$$A_2 = \frac{1}{2!} \frac{\partial}{\partial \lambda} [\cos((u_0)_\xi + \lambda(u_1)_\xi + \lambda^2(u_2)_\xi)((u_1)_\xi + 2\lambda(u_2)_\xi)]_{\lambda=0}$$

$$A_2 = \frac{1}{2} [-\sin((u_0)_\xi + \lambda(u_1)_\xi + \lambda^2(u_2)_\xi)((u_1)_\xi + 2\lambda(u_2)_\xi)^2 + \cos((u_0)_\xi + \lambda(u_1)_\xi + \lambda^2(u_2)_\xi) 2(u_2)_\xi]_{\lambda=0}$$

$$A_2 = \cos(u_0)_\xi (u_2)_\xi - \frac{1}{2} \sin(u_0)_\xi (u_1)_\xi^2$$

A Review of Adomian Decomposition Method

Consider equation (3)

$$u_0(\xi, \eta) = \psi_0(\xi, \eta) + \beta(\xi, \eta) \quad (4)$$

where $\psi_0(\xi, \eta)$ is derived from Taylor's series using the boundary conditions as follows:

$$\psi_0(\xi, \eta) = u(0, \eta) + \xi u_\xi(0, \eta) + \frac{\xi^2}{2!} u_{\xi\xi}(0, \eta) + \dots + \frac{\xi^{n-1}}{(n-1)!} u^{(n-1)}(0, \eta) \quad (5)$$

Applying the boundary conditions, to have

$$\psi_0(\xi, \eta) = \beta_0 + \xi h(\eta) + \frac{\xi^2}{2!} h'(\eta) + \dots + \frac{\xi^{n-1}}{(n-1)!} h^{(n-2)}(\eta) \quad (6)$$

where $h(\eta)$ denotes $u_\xi(0, t)$, $h'(\eta) = u_{\xi\xi}(0, \eta) \dots$ and

$\beta(\xi, \eta)$ is obtained from application of inverse of the linear differential operator L to $g(\xi, \eta)$ in (3). That is,

$$\beta(\xi, \eta) = L^{-1} g(\xi, \eta) \quad (7)$$

Therefore, adding (6) and (7) yields the initial approximation $u_0(\xi, \eta)$,

$$u_0(\xi, \eta) = \beta_0 + \xi h(\eta) + \frac{\xi^2}{2!} h'(\eta) + \dots + \frac{\xi^{n-1}}{(n-1)!} h^{(n-2)}(\eta) + L^{-1}g(\xi, \eta) \quad (8)$$

The recurrence relation

$$u_{n+1}(\xi, \eta) = - \int_0^\xi \int_0^{\xi_{n-1}} \dots \int_0^{\xi_2} \int_0^{\xi_1} M u_n(\varepsilon, \eta) d\varepsilon d\xi_1 \dots d\xi_{n-1} \\ - \int_0^\xi \int_0^{\xi_{n-1}} \dots \int_0^{\xi_2} \int_0^{\xi_1} A_n(\varepsilon, \eta) d\varepsilon d\xi_1 \dots d\xi_{n-1}, \quad n = 0, 1, 2, \dots \quad (9)$$

where A_n is the Adomian polynomial of the nonlinear term given. Other members, that is, $u_1(\xi, \eta)$, $u_2(\xi, \eta)$... are obtained from the recurrence relation.

The final solution is obtained as,

$$u(\xi, \eta) = u_0(\xi, \eta) + u_1(\xi, \eta) + \dots \quad (10)$$

The initial conditions are then implemented in (6), after which the coefficients of like powers of ξ are compared. Through that process, the Taylor expansion produces the unknown term. That is then substituted back into (8) to get the exact solution.(Yisa, 2019).

Numerical Experiments

In this section, numerical examples are presented to demonstrate the method explained in section 3.

PROBLEM 1(Mohyud-Din, 2010)

Solve the two-dimensional initial-boundary value problem

$$u_{\eta\eta} = \frac{1}{2}\tau^2 u_{\xi\xi} + \frac{1}{2}\xi^2 u_{\tau\tau}, \quad 0 < \xi, \tau < 1, \eta > 0$$

with boundary conditions

$$u(0, \tau, \eta) = \tau^2 e^{-\eta}, \quad u(1, \tau, \eta) = (1 + \tau^2)e^{-\eta}$$

$$u(\xi, 0, \eta) = \xi^2 e^{-\eta}, \quad u(\xi, 1, \eta) = (1 + \xi^2)e^{-\eta}$$

and initial conditions

$$u(\xi, \tau, 0) = \xi^2 + \tau^2, \quad u_\eta(\xi, \tau, 0) = -(\xi^2 + \tau^2)$$

The exact solution for this problem is

$$u(\xi, \tau, \eta) = (\xi^2 + \tau^2)e^{-\eta}.$$

Solution

Solving in ξ -direction

$$u_{\xi\xi} = \frac{2}{\tau^2} u_{\eta\eta} - \frac{\xi^2}{\tau^2} u_{\tau\tau}$$

The initial approximation $u_0(\xi, \tau, \eta)$ is obtained as

$$u_0(\xi, \tau, \eta) = \psi_0(\xi, \tau, \eta) + \beta(\xi, \tau, \eta)$$

$$\psi_0(\xi, \tau, \eta) = u(0, \tau, \eta) + \xi u_\xi(0, \tau, \eta) = \tau^2 e^{-\eta} + \xi h(\eta),$$

where $h(\eta) = u_\xi(0, \tau, \eta)$

$$\beta(\xi, \tau, \eta) = 0$$

$$u_0(\xi, \tau, \eta) = \tau^2 e^{-\eta} + \xi h(\eta)$$

And the recurrence relation is derived as

$$u_{n+1}(\xi, \tau, \eta) = \int_0^\xi \int_0^s \left(\frac{2}{\tau^2} (u_n)_{\eta\eta} - \frac{\varepsilon^2}{\tau^2} (u_n)_{\tau\tau} \right) d\varepsilon ds, \quad n = 0, 1, 2, 3 \dots$$

When $n = 0$:

$$u_1(\xi, \tau, \eta) = \int_0^\xi \int_0^s \left(\frac{2}{\tau^2} (u_0)_{\eta\eta} - \frac{\varepsilon^2}{\tau^2} (u_0)_{\tau\tau} \right) d\varepsilon ds$$

$$u_1(\xi, \tau, \eta) = \int_0^\xi \int_0^s \left(\frac{2}{\tau^2} (\tau^2 e^{-\eta} + \varepsilon h(\eta))_{\eta\eta} - \frac{\varepsilon^2}{\tau^2} (\tau^2 e^{-\eta} + \varepsilon h(\eta))_{\tau\tau} \right) d\varepsilon ds$$

$$u_1(\xi, \tau, \eta) = \int_0^\xi \int_0^s \left(2e^{-\eta} + \frac{2\varepsilon}{\tau^2} h''(\eta) - \frac{2\varepsilon^2}{\tau^2} e^{-\eta} \right) d\varepsilon ds$$

$$u_1(\xi, \tau, \eta) = \xi^2 e^{-\eta} + \frac{\xi^3}{3\tau^2} h''(\eta) - \frac{\xi^4}{6\tau^2} e^{-\eta}$$

When $n = 1$:

$$u_2(\xi, \tau, \eta) = \int_0^\xi \int_0^s \left(\frac{2}{\tau^2} (u_1)_{\eta\eta} - \frac{\varepsilon^2}{\tau^2} (u_1)_{\tau\tau} \right) d\varepsilon ds$$

$$\begin{aligned} &= \int_0^\xi \int_0^s \left(\frac{2}{\tau^2} \left(\varepsilon^2 e^{-\eta} + \frac{\varepsilon^3}{3\tau^2} h''(\eta) - \frac{\varepsilon^4}{6\tau^2} e^{-\eta} \right)_{\eta\eta} \right. \\ &\quad \left. - \frac{\varepsilon^2}{\tau^2} \left(\varepsilon^2 e^{-\eta} + \frac{\varepsilon^3}{3\tau^2} h''(\eta) - \frac{\varepsilon^4}{6\tau^2} e^{-\eta} \right)_{\tau\tau} \right) d\varepsilon ds \end{aligned}$$

$$= \int_0^\xi \int_0^s \left(\frac{2\varepsilon^2}{\tau^2} e^{-\eta} + \frac{2\varepsilon^3}{3\tau^4} h''''(\eta) - \frac{2\varepsilon^4}{6\tau^4} e^{-\eta} + \frac{6\varepsilon^5}{3\tau^6} h''(\eta) + \frac{6\varepsilon^6}{6\tau^6} e^{-\eta} \right) d\varepsilon ds$$

$$u_2(\xi, \tau, \eta) = \frac{\xi^4}{6\tau^2} e^{-\eta} + \frac{2\xi^5}{60\tau^4} h''''(\eta) - \frac{2\xi^6}{180\tau^4} e^{-\eta} + \frac{6\xi^7}{126\tau^6} h''(\eta) + \frac{6\xi^8}{336\tau^6} e^{-\eta}$$

$$\text{Since } u_1(\xi, \tau, \eta) = \xi^2 e^{-\eta} + \frac{\xi^3}{3\tau^2} h''(\eta) - \frac{\xi^4}{6\tau^2} e^{-\eta}$$

This implies there exist noise terms in $u_1(\xi, \tau, \eta)$ and $u_2(\xi, \tau, \eta)$ terms, the exact solution is obtained as

$$u(\xi, \tau, \eta) = \xi^2 e^{-\eta} + \tau^2 e^{-\eta} + \xi h(\eta) + \frac{\xi^3}{3\tau^2} h''(\eta)$$

Applying the initial condition $u(\xi, \tau, 0) = \xi^2 + \tau^2$

$$u(\xi, \tau, 0) = \xi^2 + \tau^2 + \xi h(0) + \frac{\xi^3}{3\tau^2} h''(0) = \xi^2 + \tau^2$$

Comparing the coefficients of powers of ξ , we obtain

$$h(0) = 0, \quad h''(0) = 0$$

The Taylors expansion of $h(\eta) = 0$

Substitute $h(\eta) = 0$ in to $u(\xi, \tau, \eta)$

$$u(\xi, \tau, \eta) = \xi^2 e^{-\eta} + \tau^2 e^{-\eta} = (\xi^2 + \tau^2) e^{-\eta}$$

which is the exact solution.

Problem 2(Mohyud-Din, 2010)

Consider the following three-dimensional initial-boundary value problem

$$u_{\eta\eta} = \frac{1}{45} \xi^2 u_{\xi\xi} + \frac{1}{45} \tau^2 u_{\tau\tau} + \frac{1}{45} \vartheta^2 u_{\vartheta\vartheta} - u, \quad 0 < \xi, \tau < 1, \quad \eta < 0$$

Subject to Neumann boundary conditions:

$$u_\xi(0, \tau, \vartheta, t) = 0, \quad u_\xi(1, \tau, \vartheta, \eta) = 6\tau^6 \vartheta^6 \sinh \eta$$

$$u_\tau(\xi, 0, \vartheta, \eta) = 0, \quad u_\tau(\xi, 1, \vartheta, \eta) = 6\xi^6 \vartheta^6 \sinh \eta$$

$$u_\vartheta(\xi, \tau, 0, \eta) = 0, \quad u_\vartheta(\xi, \tau, 1, \eta) = 6\xi^6 \tau^6 \sinh \eta$$

and initial condition:

$$u(\xi, \tau, \vartheta, 0) = 0, \quad u_\eta(\xi, \tau, \vartheta, 0) = \xi^6 \tau^6 \vartheta^6$$

The exact solution for this problem is $u(\xi, \tau, \vartheta, \eta) = \xi^6 \tau^6 \vartheta^6 \sinh \eta$.

Solution

$$u_{\eta\eta} = \frac{1}{45} (\xi^2 u_{\xi\xi} + \tau^2 u_{\tau\tau} + \vartheta^2 u_{\vartheta\vartheta}) - u$$

The initial approximation $u_0(\xi, \tau, \vartheta, \eta)$ is obtained as

$$\begin{aligned} u_0(\xi, \tau, \vartheta, \eta) &= \psi_0(\xi, \tau, \vartheta, \eta) + \beta(\xi, \tau, \vartheta, \eta) \\ \psi_0(\xi, \tau, \vartheta, \eta) &= u(\xi, \tau, \vartheta, 0) + \eta u_\eta(\xi, \tau, \vartheta, 0) = (\xi^6 \tau^6 \vartheta^6) \eta, \\ \beta(\xi, \tau, \vartheta, \eta) &= 0 \\ u_0(\xi, \tau, \vartheta, \eta) &= (\xi^6 \tau^6 \vartheta^6) \eta \end{aligned}$$

And the recurrence relation is derived as

$$\begin{aligned} u_{n+1}(\xi, \tau, \vartheta, \eta) &= \int_0^\eta \int_0^s \left(\frac{1}{45} (\xi^2 (u_n)_{\xi\xi} + \tau^2 (u_n)_{\tau\tau} + \vartheta^2 (u_n)_{\vartheta\vartheta}) - u_n \right) d\epsilon ds, \quad n = 0, 1, 2, \dots \\ u_1(\xi, \tau, \vartheta, \eta) &= \int_0^\eta \int_0^s \left(\frac{1}{45} (\xi^2 (u_0)_{\xi\xi} + \tau^2 (u_0)_{\tau\tau} + \vartheta^2 (u_0)_{\vartheta\vartheta}) - u_0 \right) d\epsilon ds, \\ u_1(\xi, \tau, \vartheta, \eta) &= (\xi^6 \tau^6 \vartheta^6) \frac{\eta^3}{6} \\ u_2(\xi, \tau, \vartheta, \eta) &= (\xi^6 \tau^6 \vartheta^6) \frac{\eta^5}{120} \end{aligned}$$

The solution is given by

$$\begin{aligned} u(\xi, \tau, \vartheta, \eta) &= u_0(\xi, \tau, \vartheta, \eta) + u_1(\xi, \tau, \vartheta, \eta) + u_2(\xi, \tau, \vartheta, \eta) + \dots \\ u(\xi, \tau, \vartheta, \eta) &= (\xi^6 \tau^6 \vartheta^6) \eta + (\xi^6 \tau^6 \vartheta^6) \frac{\eta^3}{6} + (\xi^6 \tau^6 \vartheta^6) \frac{\eta^5}{120} \\ u(\xi, \tau, \vartheta, \eta) &= (\xi^6 \tau^6 \vartheta^6) \sinh \eta \end{aligned}$$

which is the exact solution.

Problem 3 (Mohyud-Din, 2010)

Consider the two-dimensional non-linear inhomogeneous initial-boundary value problem

$$u_{\eta\eta} = 2\xi^2 + 2\tau^2 + \frac{15}{2} (\xi u_{\xi\xi}^2 + \tau u_{\tau\tau}^2), \quad 0 < \xi, \tau < 1, \quad \eta > 0$$

with boundary conditions:

$$\begin{aligned} u(0, \tau, \eta) &= \tau^2 \eta^2 + \tau \eta^6, & u(1, \tau, \eta) &= (1 + \tau^2) \eta^2 + (1 + \tau) \eta^6 \\ u(\xi, 0, \eta) &= \xi^2 \eta^2 + \xi \eta^6, & u(\xi, 1, \eta) &= (1 + \xi^2) \eta^2 + (1 + \xi) \eta^6 \end{aligned}$$

and initial condition

$$u(\xi, \tau, 0) = 0, \quad u_\eta(\xi, \tau, 0) = 0.$$

The exact solution for this problem is

$$u(\xi, \tau, \eta) = (\xi^2 + \tau^2)\eta^2 + (\xi + \tau)\eta^6.$$

Solution

The initial approximation $u_0(\xi, \tau, \eta)$ is obtained as

$$u_0(\xi, \tau, \eta) = (\xi^2 + \tau^2)\eta^2$$

And the recurrence relation is derived as

$$u_{n+1}(\xi, \tau, \eta) = \frac{15}{2} \int_0^\eta \int_0^s (\xi A_n(\varepsilon) + \xi B_n(\varepsilon)) d\varepsilon ds, \quad n = 0, 1, 2, 3 \dots$$

where $A_n(\varepsilon)$ and $B_n(\varepsilon)$ are the Adomian polynomials of the nonlinear terms $u_{\xi\xi}^2$ and $u_{\tau\tau}^2$ respectively

$$\begin{aligned} u_1(\xi, \tau, \eta) &= (\xi + \tau)\eta^6 \\ u_2(\xi, \tau, \eta) &= 0 \end{aligned}$$

The result is given

$$u(\xi, \tau, \eta) = (\xi^2 + \tau^2)\eta^2 + (\xi + \tau)\eta^6.$$

Problem 4(Mohyud-Din, 2010)

Consider the three-dimensional nonlinear in homogeneous initial-boundary value problem

$$u_{\eta\eta} = (2 - \eta^2) + u - (e^{-\xi} u_{\xi\xi}^2 + e^{-\tau} u_{\tau\tau}^2 + e^{-\vartheta} u_{\vartheta\vartheta}^2), \quad 0 < \xi, \tau < 1, \quad \eta > 0$$

Subject to Neumann boundary conditions:

$$\begin{aligned} u_\xi(0, \tau, \vartheta, \eta) &= 1, & u_\xi(1, \tau, \vartheta, \eta) &= e \\ u_\tau(\xi, 0, \vartheta, \eta) &= 0, & u_\tau(\xi, 1, \vartheta, \eta) &= e. \\ u_\vartheta(\xi, \tau, 0, \eta) &= 1, & u_\vartheta(\xi, \tau, 1, \eta) &= e \end{aligned}$$

and initial conditions:

$$u(\xi, \tau, \vartheta, 0) = e^\xi + e^\tau + e^\vartheta, \quad u_\eta(\xi, \tau, \vartheta, 0) = 0$$

The exact solution for this problem is

$$u(\xi, \tau, \vartheta, \eta) = (e^\xi + e^\tau + e^\vartheta) + \eta^2.$$

Solution

The initial approximation $u_0(\xi, \tau, \vartheta, \eta)$ is obtained as

$$u_0(\xi, \tau, \vartheta, \eta) = e^\xi + e^\tau + e^\vartheta + \eta^2 - \frac{\eta^4}{12}$$

And the recurrence relation is derived as

$$u_{n+1}(\xi, \tau, \vartheta, \eta) = \int_0^\eta \int_0^s \left(u_n - e^{-\xi} A_n(\varepsilon) - e^{-\tau} B_n(\varepsilon) - e^{-\vartheta} C_n(\varepsilon) \right) d\varepsilon ds, \quad n = 0, 1, 2, 3 \dots$$

$$u_1(\xi, \tau, \vartheta, \eta) = \frac{\eta^4}{12} - \frac{\eta^6}{360}$$

Since there exist noise terms in u_1 and u_0 . Therefore, the exact solution is

$$u(\xi, \tau, \vartheta, \eta) = (e^\xi + e^\tau + e^\vartheta) + \eta^2$$

Problem 5 (Wazwaz, 2009)

Solve the one-dimensional initial-boundary value problem:

$$\begin{aligned} u_\eta &= u_{\xi\xi} - u, & 0 < \xi < \pi, \eta > 0, \\ u(0, \eta) &= 0, & u(\pi, \eta) &= 0, & u(\xi, 0) &= \sin \xi. \end{aligned}$$

Solution

Solving in the ξ -direction,

$$u_{\xi\xi} = u_\eta + u$$

The initial approximation $u_0(\xi, \eta)$ is obtained as

$$u_0(\xi, \eta) = \xi h(\eta),$$

And the recurrence relation is derived as

$$u_{n+1}(\xi, \eta) = \int_0^\xi \int_0^s ((u_n)_\eta + u_n) d\varepsilon ds, \quad n = 0, 1, 2, 3 \dots$$

$$u_1(\xi, \eta) = \frac{\xi^3}{6} h'(\eta) + \frac{\xi^3}{6} h(\eta)$$

$$u_2(\xi, \eta) = \frac{\xi^5}{120} h''(\eta) + \frac{\xi^5}{60} h'(\eta) + \frac{\xi^5}{120} h(\eta)$$

The result is obtained as

$$u(\xi, \eta) = \xi h(\eta) + \frac{\xi^3}{3!} (h'(\eta) + h(\eta)) + \frac{\xi^5}{5!} (h''(\eta) + 2h'(\eta) + h(\eta))$$

Applying initial condition $u(\xi, 0) = \sin \xi$, we have,

$$\begin{aligned} u(\xi, 0) &= \xi h(0) + \frac{\xi^3}{3!} (h'(0) + h(0)) + \frac{\xi^5}{5!} (h''(0) + 2h'(0) + h(0)) \\ &= \xi - \frac{\xi^3}{3!} + \frac{\xi^5}{5!} - \dots \end{aligned}$$

Comparing the coefficients of powers of ξ , we obtain

$$h(0) = 1, \quad h'(0) = -2, \quad h''(0) = 4$$

The Taylor expansion of $h(\eta)$ gives

$$h(\eta) = e^{-2\eta}$$

Substituting in to $u(\xi, \eta)$,

$$\begin{aligned} u(\xi, \eta) &= e^{-2\eta} \left(\xi - \frac{\xi^3}{3!} + \frac{\xi^5}{5!} \right) \\ u(\xi, \eta) &= e^{-2\eta} \sin \xi \end{aligned}$$

which is the exact solution.

Problem 6(Wazwaz, 2010)

Solve the one-dimensional initial-boundary value problem:

$$\begin{aligned} u_\eta &= u_{\xi\xi} - 2\pi^2 u, \quad 0 < \xi < \pi, \quad \eta > 0, \\ u(0, \eta) &= 0, \quad u(1, \eta) = 0, \quad u(\xi, 0) = \sin(\pi\xi). \end{aligned}$$

Solution

Solving in the ξ -direction,

$$u_{\xi\xi} = u_\eta + 2\pi^2 u$$

The initial approximation $u_0(\xi, \eta)$ is obtained as

$$u_0(\xi, \eta) = \xi h(\eta).$$

And the recurrence relation is derived as

$$\begin{aligned} u_{n+1}(\xi, \eta) &= \int_0^\xi \int_0^s ((u_n)_\eta + 2\pi^2 u_n) d\epsilon ds, \quad n = 0, 1, 2, 3 \dots \\ u_1(\xi, \eta) &= \frac{\xi^3}{6} h'(\eta) + \frac{\pi^2 \xi^3}{3} h(\eta) \\ u_2(\xi, \eta) &= \frac{\xi^5}{120} h''(\eta) + \frac{2\pi^2 \xi^5}{60} h'(\eta) + \frac{2\pi^4 \xi^5}{60} h(\eta) \end{aligned}$$

The result is obtained as

$$u(\xi, \eta) = \xi h(\eta) + \frac{\xi^3}{3!} (h'(\eta) + 2\pi^2 h(\eta)) + \frac{\xi^5}{5!} (h''(\eta) + 4\pi^2 h'(\eta) + 4\pi^4 h(\eta))$$

Applying initial condition $u(\xi, 0) = \sin(\pi\xi)$, we have,

$$\begin{aligned} u(\xi, 0) &= \xi h(0) + \frac{\xi^3}{3!} (h'(0) + 2\pi^2 h(0)) + \frac{\xi^5}{5!} (h''(0) + 4\pi^2 h'(0) + 4\pi^4 h(0)) \\ &= \pi\xi - \frac{(\pi\xi)^3}{3!} + \frac{(\pi\xi)^5}{5!} - \dots \end{aligned}$$

Comparing the coefficients of powers of ξ , we obtain

$$h(0) = \pi, \quad h'(0) = -3\pi^3, \quad h''(0) = 9\pi^5$$

The Taylor expansion of $h(\eta)$ gives

$$h(\eta) = \pi e^{-3\pi^2\eta}$$

Substituting in to $u(\xi, \eta)$,

$$\begin{aligned} u(\xi, \eta) &= e^{-3\pi^2\eta} \left(\pi\xi - \frac{(\pi\xi)^3}{3!} + \frac{(\pi\xi)^5}{5!} \right) \\ u(\xi, \eta) &= e^{-3\pi^2\eta} \sin(\pi\xi) \end{aligned}$$

which is the exact solution.

Problem 7(Wazwaz, 2010)

Solve the one-dimensional initial-boundary value problem:

$$\begin{aligned} u_\eta &= u_{\xi\xi} + \sin(2\xi), \quad 0 < \xi < \pi, \quad \eta > 0, \\ u(0, \eta) &= 0, \quad u(\pi, \eta) = 0, \quad u(\xi, 0) = \sin(\xi) + \frac{1}{4} \sin(2\xi). \end{aligned}$$

Solution

Solving in the ξ -direction,

$$u_{\xi\xi} = u_\eta - \sin(2\xi)$$

The initial approximation $u_0(\xi, \eta)$ is obtained as,

$$u_0(\xi, \eta) = \xi h(\eta) + \frac{1}{4} \sin(2\xi)$$

And the recurrence relation is derived as

$$u_{n+1}(\xi, \eta) = \int_0^\xi \int_0^s (u_n)_\xi d\epsilon ds, \quad n = 0, 1, 2, 3 \dots$$

$$u_1(\xi, \eta) = \frac{\xi^3}{6} h'(\eta)$$

$$u_2(\xi, \eta) = \frac{\xi^5}{120} h''(\eta)$$

The result is obtained as,

$$u(\xi, \eta) = \xi h(\eta) + \frac{1}{4} \sin(2\xi) + \frac{\xi^3}{3!} h'(\eta) + \frac{\xi^5}{5!} h''(\eta)$$

Applying initial condition $u(\xi, 0) = \sin(\xi) + \frac{1}{4} \sin(2\xi)$, we have,

$$u(\xi, 0) = \xi h(0) + \frac{1}{4} \sin(2\xi) + \frac{\xi^3}{3!} h'(0) + \frac{\xi^5}{5!} h''(0) = \xi - \frac{\xi^3}{3!} + \frac{\xi^5}{5!} + \frac{1}{4} \sin(2\xi).$$

Comparing the coefficients of powers of ξ , we obtain

$$h(0) = 1, \quad h'(0) = -1, \quad h''(0) = 1$$

The Taylor expansion of $h(\eta)$ gives

$$h(\eta) = e^\eta$$

Substituting in to $u(\xi, \eta)$,

$$u(\xi, \eta) = e^{-\eta} \left(\xi - \frac{\xi^3}{3!} + \frac{\xi^5}{5!} \right) + \frac{1}{4} \sin(2\xi)$$

$$u(\xi, \eta) = e^{-\eta} \sin \xi + \frac{1}{4} \sin(2\xi)$$

which is the exact solution.

Problem 8(Wazwaz, 2009)

Solve the one-dimensional initial-boundary value problem:

$$u_\eta = u_{\xi\xi} - 6, \quad 0 < \xi < \pi, \quad \eta > 0,$$

$$u(0, \eta) = e^{-\eta}, \quad u(\pi, \eta) = 3\pi^2 - e^{-\eta}, \quad u(\xi, 0) = 3\xi^2 + \cos(\xi)$$

Solution

Solving in the ξ -direction,

$$u_{\xi\xi} = u_\eta + 6$$

The initial approximation $u_0(\xi, \eta)$ is

$$u_0(\xi, \eta) = e^{-\eta} + \xi h(\eta) + 3\xi^2$$

And the recurrence relation is derived as

$$u_{n+1}(\xi, \eta) = \int_0^\xi \int_0^s (u_n)_\eta d\epsilon ds, \quad n = 0, 1, 2, 3 \dots$$

$$u_1(\xi, \eta) = -\frac{\xi^2}{2} e^{-\eta} + \frac{\xi^3}{6} h'(\eta)$$

$$u_2(\xi, \eta) = \frac{\xi^4}{24} e^{-\eta} + \frac{\xi^5}{120} h''(\eta)$$

The result is obtained as,

$$u(\xi, \eta) = e^{-\eta} + \xi h(\eta) + 3\xi^2 - \frac{\xi^2}{2} e^{-\eta} + \frac{\xi^3}{6} h'(\eta) + \frac{\xi^4}{24} e^{-\eta} + \frac{\xi^5}{120} h''(\eta)$$

Applying initial condition $u(\xi, 0) = 3\xi^2 + \cos(\xi)$, we have,

$$\begin{aligned} u(\xi, 0) &= 1 + \xi h(0) + 3\xi^2 - \frac{\xi^2}{2} + \frac{\xi^3}{6} h'(0) + \frac{\xi^4}{24} + \frac{\xi^5}{120} h''(0) \\ &= 3\xi^2 + 1 - \frac{\xi^2}{2} + \frac{\xi^4}{4!} - \frac{\xi^6}{6!} \end{aligned}$$

Comparing the coefficients of powers of ξ , we obtain

$$h(0) = 0, \quad h'(0) = 0, \quad h''(0) = 0$$

The Taylor expansion of $h(\eta)$ gives

$$h(\eta) = 0.$$

and the solution give

$$\begin{aligned} u(\xi, \eta) &= 3\xi^2 + e^{-\eta} \left(1 - \frac{\xi^2}{2} + \frac{\xi^4}{4!} - \frac{\xi^6}{6!} \right) \\ u(\xi, \eta) &= 3\xi^2 + e^{-\eta} \cos(\xi) \end{aligned}$$

which is the exact solution.

Discussion of Results

The results obtained for the entire problems 1, 2, 3, and 4 are all exact solutions. The first two problems are linear initial-boundary value PDEs, while the following two are nonlinear. Adomian polynomials are used to overcome the nonlinear terms in problems 3 and 4. The four problems were earlier solved by (Mohyud-Din, 2010) when

he used variation of parameter approach. While our approach gives exact solutions for both linear and nonlinear problems, the approach proposed in the referenced literature is cumbersome and really gives exact solutions. Problems 5, 6, 7, and 8 are taken from (Abdul-Majid, 2009) where he used ADM to solve the problems. Our results tally with the author's results, being exact solutions in the four cases. The present method is ADM fused with Taylor's series of two variables has proven to be a powerful tool for providing solutions to problems of linear and nonlinear partial differential equations.

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