

# On a Proposed Test Statistic For Testing Equality of Two Means Under Unequal Population Variances

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## Abstract

In this paper, we propose a test statistic for testing equality of two independent sample means for unequal variances. When variances differ, the pooled sample variance ( $S_p^2$ ) is inadequate as a single value for the variances. This problem is commonly known as the Behrens – Fisher problem. Instead, the sample harmonic mean of variances ( $S_H^2$ ) is proposed, examined and found to represent the unequal variances better. The distribution of  $S_H^2$  which is known to be generalized Beta is further approximated by the chi – square distribution with the degrees of freedom related to that of degrees of freedom of the chi – square distribution of  $S_p^2$ . Consequently, it is used to replace the pooled sample variance in the resulting proposed  $t$  – test. An application of the test statistic using real-life data is provided.

**Keywords:** Harmonic mean of variances, ordered alternative, variance heterogeneity, chi- square distribution, modified  $t$  – test statistic

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## Introduction

Many times we often come across situation where means of populations are compared. In the literature there are appropriate known test statistics for situation when the populations are known to be normal with equal variance. We also have situations where the variances are unknown but are assumed to be equal. Then problem arise when the populations do not have equal variances. When the variances are unequal, the pooled sample variance over estimates the appropriate variance and the test statistic becomes conservative. This is the well known Behrens – Fisher problem. There is therefore the need to seek for an alternative to the pooled variance. Some researches on hypothesis testing for equality of two means, that is,  $\mu_1 - \mu_2$  include Abidoeye et. al (2016a), Abidoeye et. al (2016b), Abidoeye et. al (2016c), Jonckheere (1954), Dunnett (1964), Montgomery (1981), Dunnet and Tamhane (1997), Yahya and Jolayemi (2003),

Gupta et. al (2006). These researchers worked on the conventional test statistic on equality of two population means,  $H_0: \mu_1 = \mu_2$  against non-directional alternative,  $H_1: \mu_1 \neq \mu_2$ .

The interest in this work is to develop a suitable test procedure to address heterogeneity of variances, see Abidoye (2012) and Abidoye et. al (2016c)

## Methodology

We are interested in developing a suitable test procedure to test the hypothesis:

$H_0: \mu_1 = \mu_2$  against  $H_1: \mu_1 \neq \mu_2$

or  $H_0: \mu_1 - \mu_2 = 0$  vs  $H_1: \mu_1 - \mu_2 \neq 0$ .....(2.1)

when the error term  $e_{ij} \sim N(0, \sigma_i^2)$   $i = 1, 2$ , and  $j = 1, 2, \dots, n_i$

that is, under heterogeneity of variances.

Consequently  $X_{ij} \sim N(\mu_i, \sigma_i^2)$  where  $X_{ij}$  are the observed values.

The unbiased estimate of  $(\mu_1 - \mu_2) = (\bar{X}_1 - \bar{X}_2) = Y$

.....(2.2)

where  $\bar{X}_1, \bar{X}_2$  are the sample means for groups 1 and 2 respectively.

$V(Y) = \text{Var}[\bar{X}_1 - \bar{X}_2]$

$= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$ .....(2.3)

but

$Y = (\bar{X}_1 - \bar{X}_2) \sim N(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2})$ .....(2.4)

## Distribution of harmonic variance

Abidoye (2012) showed that  $\sigma_H^2$  harmonic mean of group variances better represents series of unequal group variances and is estimated by  $S_H^2$ . It was also

shown that the sample distribution of  $S_H^2$  is approximated by the chi - square distribution.

$Y = (\bar{X}_1 - \bar{X}_2) \sim N(\mu_1 - \mu_2, \sigma_H^2 (\frac{n_1 + n_2}{n_1 n_2}))$ .....(2.5)

Consequently, the test statistic for the hypotheses set in equation (2.1) is

$t = \frac{Y}{Z}$ .....(2.6)

where

$Y = (\bar{X}_1 - \bar{X}_2)$ .....(2.7)

and

$Z = \sqrt{S_H^2 \left( \frac{1}{n_1} + \frac{1}{n_2} \right)}$ .....(2.8)

Now p-value =  $P(t = t_r > t_{cal})$  .....(2.9)

where  $t_r$  is regular t - distribution and r is the appropriate degrees of freedom for the t - test .

### Determination of degrees of freedom for the distribution of sample harmonic mean of variances simulation

Let  $X_{ij} \sim N(\mu_i, \sigma_i^2)$  ,  $i = 1, 2$  ;  $j = 1, 2, \dots, n_i$  and set  $\sigma_1^2 \neq \sigma_2^2$ . For various values of n -2, generate  $X_{ij}$  and compute  $S^2$  having ; degrees of freedom.

The degrees of freedom (r ) for the proposed test statistic in equation (2.7) was related to  $\sum n_i - 2 = n - 2$  degrees of freedom for the pooled variance ( $S_p^2$ ), where

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} .$$

The value of degree of freedom of r was obtained

through the best empirical chi- square which is equal to  $= \frac{v}{u} \bar{y}$  where

$v = s_H^2 + \bar{y}(Y_{(n)} - \bar{y})$  and  $u = Y_{(n)}(s_H^2)$ . from equation (2.6), Table 1 gives the values of n-2 the degrees of freedom for the pooled sample variance against the

estimated degrees of freedom for the chi - square distribution of  $S_H^2$ .

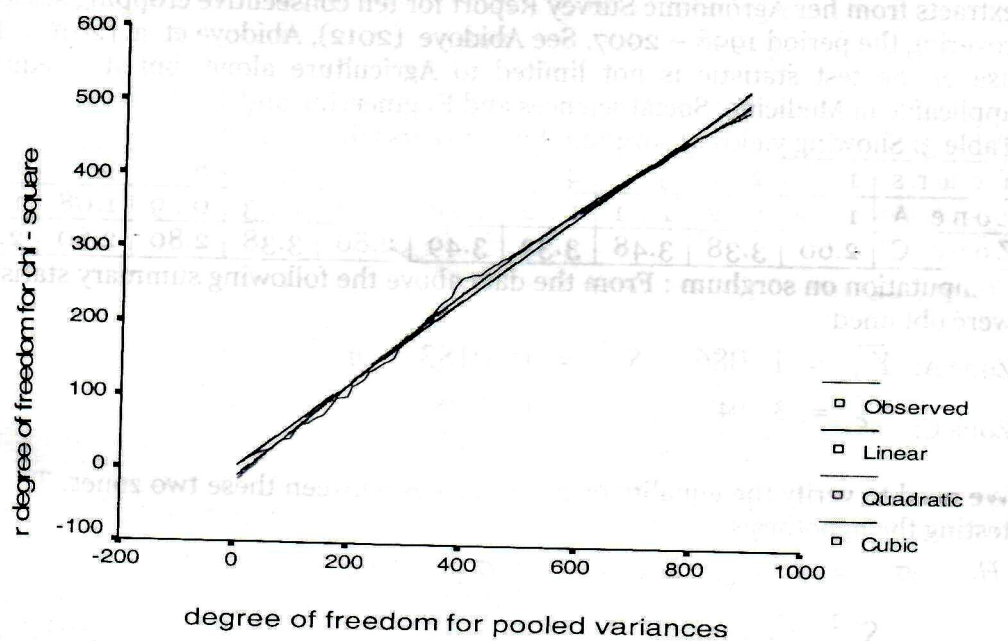
**Table 1: Simulated values of degrees of freedom for the proposed t - test statistic**

| n-2   |         | n-2   |           | n - 2 |           |
|-------|---------|-------|-----------|-------|-----------|
| 8     | 4 . 8   | 3 1 8 | 1 8 9 . 9 | 6 1 8 | 3 6 4 . 4 |
| 1 8   | 8 . 6   | 3 2 8 | 1 9 9 . 8 | 6 2 8 | 3 6 9 . 1 |
| 2 8   | 1 3 . 4 | 3 3 8 | 2 0 7 . 9 | 6 3 8 | 3 7 4 . 3 |
| 3 8   | 1 9 . 2 | 3 4 8 | 2 1 5 . 6 | 6 4 8 | 3 7 9 . 4 |
| 4 8   | 2 3 . 2 | 3 5 8 | 2 2 6 . 4 | 6 5 8 | 3 8 4 . 7 |
| 5 8   | 2 5 . 6 | 3 6 8 | 2 3 2 . 3 | 6 6 8 | 3 8 9 . 6 |
| 6 8   | 2 6 . 8 | 3 7 8 | 2 3 8 . 9 | 6 7 8 | 3 9 4 . 9 |
| 7 8   | 3 6 . 3 | 3 8 8 | 2 5 3 . 2 | 6 8 8 | 3 9 9 . 8 |
| 8 8   | 3 9 . 6 | 3 9 8 | 2 6 4 . 4 | 6 9 8 | 4 0 4 . 2 |
| 9 8   | 4 1 . 8 | 4 0 8 | 2 6 5 . 2 | 7 0 8 | 4 0 9 . 8 |
| 1 0 8 | 5 1 . 3 | 4 1 8 | 2 6 7 . 5 | 7 1 8 | 4 1 4 . 2 |
| 1 1 8 | 5 8 . 9 | 4 2 8 | 2 6 9 . 3 | 7 2 8 | 4 1 9 . 4 |
| 1 2 8 | 6 2 . 1 | 4 3 8 | 2 7 4 . 3 | 7 3 8 | 4 2 4 . 7 |
| 1 3 8 | 6 7 . 7 | 4 4 8 | 2 7 9 . 4 | 7 4 8 | 4 2 9 . 2 |
| 1 4 8 | 7 1 . 5 | 4 5 8 | 2 8 4 . 4 | 7 5 8 | 4 3 4 . 4 |

|   |   |   |         |       |       |       |       |
|---|---|---|---------|-------|-------|-------|-------|
| 1 | 5 | 8 | 7 4 . 6 | 4 6 8 | 289.8 | 7 6 8 | 439.7 |
| 1 | 6 | 8 | 8 5 . 2 | 4 7 8 | 294.3 | 7 7 8 | 444.5 |
| 1 | 7 | 8 | 9 5 . 9 | 4 8 8 | 299.1 | 7 8 8 | 449.2 |
| 1 | 9 | 8 | 9 8 . 6 | 4 9 8 | 304.4 | 7 9 8 | 454.6 |
| 2 | 0 | 8 | 113.8   | 5 0 8 | 309.4 | 8 0 8 | 459.8 |
| 2 | 1 | 8 | 118.4   | 5 1 8 | 314.1 | 8 1 8 | 464.2 |
| 2 | 2 | 8 | 124.9   | 5 2 8 | 319.2 | 8 2 8 | 469.5 |
| 2 | 3 | 8 | 133.2   | 5 3 8 | 324.4 | 8 3 8 | 474.2 |
| 2 | 4 | 8 | 138.8   | 5 4 8 | 329.9 | 8 4 8 | 479.3 |
| 2 | 5 | 8 | 142.4   | 5 5 8 | 334.3 | 8 5 8 | 484.7 |
| 2 | 6 | 8 | 148.3   | 5 6 8 | 339.4 | 8 6 8 | 489.8 |
| 2 | 7 | 8 | 152.7   | 5 7 8 | 344.4 | 8 7 8 | 494.4 |
| 2 | 8 | 8 | 163.2   | 5 8 8 | 349.2 | 8 8 8 | 499.2 |
| 2 | 9 | 8 | 175.4   | 5 9 8 | 354.2 | 8 9 8 | 504.8 |
| 3 | 0 | 8 | 182.8   | 6 0 8 | 359.5 |       |       |

### Relationship between empirical degree of freedom (r) and n-2

The values of empirical degree of freedom (r) for chi-square from above simulated data in Tables 1 were plotted against  $\sum n_i - 2 = n - 2$ , where n = total number of all the observations in the groups. Clearly the contending models were, simple linear quadratic and cubic models, see Figure 1. Table 2 shows coefficient of determination  $R^2$  and adjusted  $R^2$



**Figure 1:** Plot of simulated degrees of freedom against the chi - square degrees of freedom

Table 2: Showing  $R^2$  and Adjusted  $R^2$  under Linear, Quadratic and Cubic

| M o d e l         | $R^2$         | Adjusted $R^2$ | C o e f f i c i e n t s                                                           |
|-------------------|---------------|----------------|-----------------------------------------------------------------------------------|
| L i n e a r       | 0 . 9 9 3 3 7 | 0 . 9 9 3 2 9  | $a_0 = 0.607389$<br>$a_1 = 0.581022$                                              |
| Q u a d r a t i c | 0 . 9 9 7 3 1 | 0 . 9 9 7 2 4  | $a_0 = 20.966752$<br>$a_1 = 0.722882$<br>$a_2 = -0.000157$                        |
| C u b i c         | 0 . 9 9 7 6 0 | 0 . 9 9 7 5 2  | $a_0 = 13.856149$<br>$a_1 = 0.629294$<br>$a_2 = 0.000101$<br>$a_3 = -0.000000189$ |

From the results the quadratic equation ( polynomial of order two ) was seen to perform best in term of  $R^2$  or adjusted  $R^2$  beating cubic model because of the associated complexity so that the degree of freedom for the approximated t – test is given by  $\hat{r} = 2\alpha$ .

### Application

In this study, the data used were secondary data, obtained from Kwara Agriculture Development Project (KWADP), Ilorin, Kwara State. They were extracts from her Agronomic Survey Report for ten consecutive cropping seasons, covering the period 1998 – 2007. See Abidoye (2012), Abidoye et. al (2016). The use of the test statistic is not limited to Agriculture alone, but it is equally applicable in Medicine, Social sciences and Engineering and Technology.

Table 3: Showing yields of sorghum for ten years (1998 – 2007).

| Y e a r s | 1     | 2     | 3     | 4     | 5    | 6     | 7     | 8     | 9    | 1 0   |
|-----------|-------|-------|-------|-------|------|-------|-------|-------|------|-------|
| Z o n e A | 1 . 0 | 1 . 2 | 1 . 1 | 1 . 2 | 1.18 | 0 . 9 | 1 . 3 | 0 . 9 | 1.08 | 1 . 0 |
| Z o n e C | 2.60  | 3.38  | 3.48  | 3.32  | 3.49 | 2.80  | 3.38  | 2.80  | 2.60 | 2.62  |

Computation on sorghum : From the data above the following summary statistics were obtained:

$$\text{Zone A: } \bar{Y}_A = 1.086, S_A^2 = 0.0183, n_A = 10$$

$$\text{Zone C: } \bar{Y}_C = 3.047, S_C^2 = 0.1538, n_C = 10$$

We need to verify the equality of the variances between these two zones. That is, testing the hypothesis;

$$H_0 : \sigma_1^2 = \sigma_2^2 \text{ vs } H_0 : \sigma_2^2 > \sigma_1^2$$

$$F = \frac{S_2^2}{S_1^2} \sim F_{(n_2-1, n_1-1, \alpha)}$$

$$F = \frac{0.1538}{0.0183} = 8.404$$

Since  $F_{cal} = 8.404 > F_{9,9,(0.05)} = 3.18$  we reject  $H_0$  and therefore conclude that the two variances are not equal. Hence, we cannot use the normal t- test statistic, thus we use our recommended modified t - test statistic.

$$\text{In the above data set, } n_i = 10, g = 2, n = \sum_{i=1}^2 n_i = 20 \quad , \quad S_H^2 = \left( \frac{1}{g} \sum \frac{1}{s_i^2} \right)^{-1}$$

$$S_H = 0.1809$$

The hypothesis to be tested is

$$H_0 : \mu_A = \mu_C \text{ Vs } H_1 : \mu_A < \mu_C ; \sigma_1^2 \neq \sigma_2^2$$

$$t^* = \frac{\bar{X}_1 - \bar{X}_2}{S_H \sqrt{\left( \frac{1}{n_1} + \frac{1}{n_2} \right)}} \sim t_r$$

$$= \frac{0.109}{0.1809 \sqrt{\left( \frac{1}{10} + \frac{1}{10} \right)}} = \frac{0.109}{0.080901} = 1.34733$$

$$\begin{aligned} \text{Now p-value} &= P(t^* = t_r > t_{cal}) = P(t_r > t_{cal}) \\ &= P(t_r > 1.34733) \\ &= 0.093393 > 0.05 \end{aligned}$$

This leads to the non-rejection of the null hypothesis,  $H_0$  and we conclude that the mean yields of sorghum in the two zones are not significantly different.

### Conclusion

In this work we have established that the sample harmonic mean of variances was found to have a good approximation against the pooled variances under unequal variances. Because the sample harmonic mean of variances has the chi - square distribution, the modified t - statistic is appropriate and eliminates the Beheren- Fisher's problem.

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