

Chapter 3

ON CERTAIN ANALYTIC FUNCTIONS OF BOUNDED BOUNDARY ROTATION

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Abstract

In this paper we introduce certain analytic functions of boundary rotation bounded by $k\pi$ which are of Caratheodory origin. With them we study two classes of analytic and univalent functions in the unit disk $E = \{z \in \mathbb{C}: |z| < 1\}$, which are also of bounded boundary rotation.

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1. Introduction

Let $P(\beta)$ denote the Caratheodory family of functions:

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots \quad (1.1)$$

which are analytic in E and satisfy $\operatorname{Re} p(z) > \beta$, $0 \leq \beta < 1$, $z \in E$. For $\beta = 0$, we write P in place of $P(0)$. The function $L_{0,\beta}(z) = \beta + (1-\beta)L_0(z)$ plays the extremal role in $P(\beta)$ as does $L_0(z) = (1+z)/(1-z)$ in P . Functions in $P(\beta)$ have Herglotz representations

$$\int_0^{2\pi} \frac{1 + (1-2\beta)ze^{-is}}{1 - ze^{-is}} d\mu(s)$$

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for $d\mu(s) \geq 0$ and $\int_0^{2\pi} d\mu(s) = 1$. Denote by M_k , $k \geq 2$, the class of real-valued functions $m(s)$ of bounded variation on $[0, 2\pi]$ which satisfy the conditions:

$$\int_0^{2\pi} dm(s) = 2, \quad \int_0^{2\pi} |dm(s)| \leq k.$$

Clearly, M_2 is the class of nondecreasing functions on $[0, 2\pi]$ which satisfy $\int_0^{2\pi} dm(s) = 2$.

If $m \in M_k$ with $k \geq 2$, we can write $m(s) = a(s) - b(s)$ for some nonnegative, nondecreasing functions $a(s)$, $b(s)$ on $[0, 2\pi]$ satisfying

$$\int_0^{2\pi} da(s) \leq \frac{k}{2} + 1, \quad \int_0^{2\pi} db(s) \leq \frac{k}{2} - 1. \quad (1.2)$$

Furthermore, let $P_k(\beta)$ be the class of analytic functions in E which have the representation

$$h(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1 - 2\beta)ze^{-is}}{1 - ze^{-is}} dm(s), \quad (1.3)$$

where $m \in M_k$.

If we set $a(s) = \frac{k+2}{4}\mu_1(s)$ and $b(s) = \frac{k-2}{4}\mu_2(s)$, where $\int_0^{2\pi} d\mu_j(s) = 1$, $j = 1, 2$. Then it follows from (1.2) and (1.3) that

$$\begin{aligned} h(z) &= \frac{k+2}{4} \int_0^{2\pi} \frac{1 + (1 - 2\beta)ze^{-is}}{1 - ze^{-is}} d\mu_1(s) \\ &\quad - \frac{k-2}{4} \int_0^{2\pi} \frac{1 + (1 - 2\beta)ze^{-is}}{1 - ze^{-is}} d\mu_2(s) \\ &\equiv \frac{k+2}{4} p(z) - \frac{k-2}{4} q(z) \end{aligned} \quad (1.4)$$

where $p, q \in P(\beta)$.

Pinchuk [10] defined P_k and proved the following:

Lemma 1.1 ([10]). *All the functions in P_k have positive real parts for*

$$|z| = r < \frac{k - \sqrt{k^2 - 4}}{2}.$$

Furthermore, there exist functions in P_k which do not have positive real parts in any larger disk.

If we choose $h \in P_k(\beta)$ and set $h(z) = \beta + (1 - \beta)h_1(z)$, where $h_1 \in P_k$, we have the following:

Lemma 1.2. *All functions in $P_k(\beta)$ have positive real parts for*

$$|z| = r(k, \beta) < \begin{cases} \frac{(1-\beta)k - \sqrt{(1-\beta)^2 k^2 - 4(1-2\beta)}}{2(1-2\beta)} & \text{if } \beta \neq \frac{1}{2}, \\ \frac{2}{k} & \text{if } \beta = \frac{1}{2}. \end{cases} \quad (1.5)$$

The function

$$H(z) = \frac{k+2}{4}L_{0,\beta}(-z) - \frac{k-2}{4}L_{0,\beta}(z) \quad (1.6)$$

shows that there exist functions in $P_k(\beta)$ which do not have positive real parts in any larger disk.

In the next section, we introduce certain iterations of $P_k(\beta)$ and with a brief discussion that will lead to new classes of analytic functions having boundary rotations bounded by $k\pi$.

2. Functions of the Classes $\Phi_{\sigma,n,k}^j(\beta)$, $j = 1, 2$

In [2, 3], the authors identified the following iterated integral transformation of functions in the class P .

Definition 2.1. Let $p \in P$. Let $\sigma > 0$ be real number $n \geq 1$. Then, for any $z \in E$, define

$$\phi_{\sigma,n}^j(p(z)) = \int_0^z \lambda_{\sigma,n}^j(z,t) \phi_{\sigma,n-1}^j(p(t)) dt, \quad j = 1, 2 \quad (2.1)$$

where

$$\lambda_{\sigma,n}^1(z,t) = \frac{\sigma t^{\sigma-1}}{z^\sigma}$$

and

$$\lambda_{\sigma,n}^2(z,t) = \frac{(\sigma - (n-1))t^{\sigma-n}}{z^{\sigma-(n-1)}}, \quad \sigma - (n-1) > 0,$$

with $\phi_{\sigma,0}^j(p(z)) = p(z)$.

Changes in notations (cf. [2, 3]) became necessary only in order to unify our discussions of the two transformations. For any $p \in P$ defined by (1.1), it is known [2, 3] that the transformations $\phi_{\sigma,n}^j(p(z))$ have series representations

$$\phi_{\sigma,n}^j(p(z)) = 1 + (1-\beta) \sum_{l=1}^{\infty} c_{l,n}^j z^l, \quad j = 1, 2, \quad (2.2)$$

where

$$c_{l,n}^1 = \left(\frac{\sigma}{\sigma+l} \right)^n$$

and

$$c_{l,n}^2 = \frac{\sigma(\sigma-1)\dots(\sigma-(n-1))}{(\sigma+l)(\sigma+l-1)\dots(\sigma+l-(n-1))}.$$

Furthermore, these transformations preserve many geometric structures of the family P ; particularly the positivity of the real parts, compactness, convexity and subordination. They are iterative and are closely associated with certain families analytic and univalent functions involving the well known Salagean and Rucheweyh derivatives (see [2, 3]) and have been used effectively and elegantly too, to characterize them. Also, in [4], we have used the transformation $\phi_{\sigma,n}^1(p(z))$ together with a method of Nehari and Netanyahu to determine the

best possible coefficient bounds for some classes of functions. Further characterizations can be found in the articles.

Since for $0 \leq \beta < 1$, $\operatorname{Re} p(z) > \beta$ implies $\operatorname{Re} \phi_{\sigma,n}^j(p(z)) > \beta$ (see [2, 3]), we would suppose $p \in P(\beta)$ and denote by $\Phi_{\sigma,n}^j(\beta)$ the classes of these transformations. If, in (1.4), we replace $p(z)$ and $q(z)$ by their respective iterations (i.e., $\phi_{\sigma,n}^j(p(z))$ and $\phi_{\sigma,n}^j(q(z))$), we come to a more general class of functions $\Phi_{\sigma,n,k}^j(\beta)$ consisting of functions of the form:

$$\phi_{\sigma,n}^j(h(z)) = \frac{k+2}{4} \phi_{\sigma,n}^j(p(z)) - \frac{k-2}{4} \phi_{\sigma,n}^j(q(z)). \quad (2.3)$$

Let us note the equivalent classes:

$$\begin{aligned} \Phi_{\sigma,0,k}^j(\beta) &\equiv P_k(\beta), & \Phi_{\sigma,0,k}^j(0) &\equiv P_k, \\ \Phi_{\sigma,0,2}^j(\beta) &\equiv P(\beta), & \Phi_{\sigma,0,2}^j(0) &\equiv P. \end{aligned}$$

Theorem 2.2. *All the functions in $\Phi_{\sigma,n,k}^j(\beta)$ have integral representation*

$$\phi_{\sigma,n}^j(h(z)) = \frac{1}{2} \int_0^{2\pi} \left[1 + 2(1-\beta) \sum_{l=1}^{\infty} c_{l,n}^j z^l e^{-ls} \right] dm(s), \quad j = 1, 2, \quad (2.4)$$

where $c_{l,n}^j$ are those defined for series (2.2).

Proof. From (1.3) and (2.1), we have

$$\begin{aligned} \phi_{\sigma,n}^j(h(z)) &= \int_0^z \lambda_{\sigma,n}^j(z, t) \phi_{\sigma,n-1}^j(h(t)) dt, \\ &= \int_0^z \lambda_{\sigma,n}^j(z, t) \phi_{\sigma,n-1}^j \left[\frac{1}{2} \int_0^{2\pi} \frac{1 + (1-2\beta)te^{-is}}{1 - te^{-is}} dm(s) \right] dt \\ &= \frac{1}{2} \int_0^{2\pi} \left[\int_0^z \lambda_{\sigma,n}^j(z, t) \phi_{\sigma,n-1}^j \left(\frac{1 + (1-2\beta)te^{-is}}{1 - te^{-is}} \right) dt \right] dm(s). \end{aligned} \quad (2.5)$$

If we write the kernel of the inner integral in series form and apply the transformations successively, we obtain the desired expression. This completes the proof. $\square$

Remark 2.3. The representations (2.4) are the Herglotz's for functions in $\Phi_{\sigma,n,k}^j(\beta)$, $j = 1, 2$.

Theorem 2.4. *For any $h \in P(\beta)$, let $\phi_{\sigma,n}^j(h(z)) \in \Phi_{\sigma,n,k}^j(\beta)$. Then we have the best possible lower bound*

$$\operatorname{Re} \phi_{\sigma,n}^j(h(z)) \geq 1 + (1-\beta) \sum_{l=1}^{\infty} \left(2c_{2l,n}^j r - kc_{2l-1,n}^j \right) r^{2l-1},$$

where $c_{2l,n}^j$ and $c_{2l-1,n}^j$ are appropriately defined from those for series (2.2).

Proof. Let $z = re^{i\theta}$ and $t = \rho e^{i\theta}$ with $0 < \rho < r < 1$ in the second equation of (2.5). Then we have

$$\phi_{\sigma,n}^j(h(re^{i\theta})) = \int_0^r \lambda_{\sigma,n}^j(r, \rho) \phi_{\sigma,n-1}^j(\Omega(\rho)) d\rho,$$

where

$$\begin{aligned}\Omega(\rho) &= \frac{1}{2} \int_0^{2\pi} \frac{1 + (1 - 2\beta)\rho e^{i(\theta-s)}}{1 - \rho e^{i(\theta-s)}} dm(s) \\ &= \frac{1}{2} \int_0^{2\pi} \left(\beta + (1 - \beta) \frac{1 + \rho e^{i(\theta-s)}}{1 - \rho e^{i(\theta-s)}} \right) dm(s).\end{aligned}$$

Thus we have

$$\operatorname{Re} \phi_{\sigma,n}^j(h(re^{i\theta})) = \int_0^r \lambda_{\sigma,n}^j(r, \rho) \phi_{\sigma,n-1}^j(\operatorname{Re} \Omega(\rho)) d\rho. \quad (2.6)$$

Assume $\int_0^{2\pi} |dm(s)| = k$, then we have

$$\begin{aligned}\operatorname{Re} \Omega(\rho) &= \frac{1}{2} \int_0^{2\pi} \left(\beta + (1 - \beta) \frac{1 - \rho^2}{1 - 2\rho \cos(\theta - s) + \rho^2} \right) [da(s) - db(s)] \\ &\geq \beta + (1 - \beta) \left[\left(\frac{k}{2} + 1 \right) \frac{1 - \rho^2}{1 + 2\rho + \rho^2} - \left(\frac{k}{2} - 1 \right) \frac{1 - \rho^2}{1 - 2\rho + \rho^2} \right] \\ &= \beta + (1 - \beta) \left(\frac{\rho^2 - k\rho + 1}{1 - \rho^2} \right) \\ &= \frac{(1 - 2\beta)\rho^2 - (1 - \beta)k\rho + 1}{1 - \rho^2}.\end{aligned} \quad (2.7)$$

In series form, we have

$$\operatorname{Re} \Omega(\rho) \geq 1 + (1 - \beta) \sum_{l=1}^{\infty} (2\rho - k) \rho^{2l-1}.$$

Now, applying (2.6) appropriately, we have the results. And, if $\int_0^{2\pi} |dm(s)| = k' < k$, then we have $\operatorname{Re} \Omega(\rho) \geq \frac{(1-2\beta)\rho^2 - (1-\beta)k'\rho + 1}{1-\rho^2} > \frac{(1-2\beta)\rho^2 - (1-\beta)k\rho + 1}{1-\rho^2}$. The extremal functions are $\phi_{\sigma,n}^j(H(z))$, where $H(z)$ is given by (1.6). This completes the proof. $\square$

Two important corollaries ($j = 1, 2$) which follow from the proof of the above theorem are results similar to Lemma 1.2, which are as follows:

Corollary 2.5. *All the functions in $\Phi_{\sigma,n,k}^j(\beta)$ have positive real parts for $|z| = r(k, \beta)$ given by (1.5). This disk $|z| = r(k, \beta)$ is the largest possible.*

Proof. The proof easily follows from (2.7) and the functions $\phi_{\sigma,n,k}^j(H(z))$, where $H(z)$ is given by (1.6), show that there exist functions in $\Phi_{\sigma,n,k}^j(\beta)$ which do not have positive real parts in any larger disk. $\square$

Following from (2.3) (and similar arguments as in [2, 3]) it is not difficult to see the inclusions

Theorem 2.6. $\Phi_{\sigma,n+1,k}^j(\beta) \subset \Phi_{\sigma,n,k}^j(\beta)$ for each $n \in \mathbb{N}$.

Theorem 2.7. $\Phi_{\sigma,n,k}^j(\beta) \subset P_k(\beta)$ for each $n \in \mathbb{N}$.

Finally, we make the following remarks:

Remark 2.8. For any real numbers $\sigma_1, \sigma_2 > 0$ and $n_1, n_2 \in \mathbb{N}$ satisfying Definition 1.1 and $h \in P(\beta)$, we have

$$\phi_{\sigma_1, n_1}^j[\phi_{\sigma_2, n_2}^j(h(z))] = \phi_{\sigma_2, n_2}^j[\phi_{\sigma_1, n_1}^j(h(z))].$$

Remark 2.9. For any $h \in P(\beta)$, we have

$$\phi_{\sigma, n}^1(h(z)) + \frac{z(\phi_{\sigma, n}^1(h(z)))'}{\sigma} = \phi_{\sigma, n-1}^1(h(z))$$

and

$$\phi_{\sigma, n}^2(h(z)) + \frac{z(\phi_{\sigma, n}^2(h(z)))'}{\sigma - (n-1)} = \phi_{\sigma, n-1}^2(h(z)).$$

3. Functions of Bounded Boundary Rotation

Let A be the class of functions

$$f(z) = z + a_2 z^2 + \dots$$

which are analytic in E . Throughout, σ and n shall have their definitions as in Section 1. Then via the classes of analytic functions $P_k(\beta)$, we define the following classes of functions.

Definition 3.1. We say that a function $f \in A$ is in the class $T_n^\sigma(k, \beta)$ if and only if

$$\frac{D^n f(z)^\sigma}{\sigma^n z^\sigma} \in P_k(\beta), \quad z \in E.$$

Definition 3.2. We say that a function $f \in A$ is in the class $B_n^\sigma(k, \beta)$ if and only if

$$\frac{L_n^\sigma f(z)}{z} \in P_k(\beta), \quad z \in E.$$

The operator $D^n : A \rightarrow A$ is the Salagean derivative operator defined as $D^n f(z) = D(D^{n-1} f(z)) = z[D^{n-1} f(z)]'$ with $D^0 f(z) = f(z)$. The index $\sigma > 0$ means principal determinations only. Whereas, $L_n^\sigma : A \rightarrow A$ was defined in [3] (using the convolution $*$) as follows:

$$L_n^\sigma f(z) = (\tau_\sigma * \tau_{\sigma, n}^{(-1)} * f)(z),$$

where

$$\tau_{\sigma, n}(z) = \frac{z}{(1-z)^{\sigma-(n-1)}}, \quad \sigma - (n-1) > 0,$$

$\tau_\sigma = \tau_{\sigma, 0}$ and $\tau_{\sigma, n}^{(-1)}$ is such that

$$(\tau_{\sigma, n} * \tau_{\sigma, n}^{(-1)})(z) = \frac{z}{1-z}.$$

It is remarkable to mention that $L_n^\sigma \equiv \mathcal{D}^n$ where $\mathcal{D}^n$ is the well known Rusheweyh derivative operator. $T_n^\sigma(2, \beta)$ and $B_n^\sigma(2, \beta)$ respectively coincide with the classes $T_n^\sigma(\beta)$

and $B_n^\sigma(\beta)$, which are generalizations of several other classes of functions [1, 5, 6, 12, 13]. Also $T_1^1(k, 0)$ and $B_1^1(k, 0)$ both coincide with B_k studied in [11]. The classes $T_n^\sigma(k, \beta)$ and $B_n^\sigma(k, \beta)$ consist of analytic and univalent functions in A having boundary rotations bounded by $k\pi$ and respectively generalize the classes $T_n^\sigma(\beta)$ and $B_n^\sigma(\beta)$ in the same manner. U_k and V_k generalize the well known classes of starlike and convex functions in the open unit disk (see [8, 10, 11]).

In the sequel, we study these classes of functions. The conciseness of our proofs is due to the following lemmas, which relate functions of these classes with those of $\Phi_{\sigma,n,k}^j(\beta)$ of the last section. The lemmas are consequences of Lemmas 4.2 and 2 of [2, 3], respectively.

Lemma 3.3. *Let $f \in A$. Then the following are equivalent:*

- (i) $f \in T_n^\sigma(k, \beta)$,
- (ii) $\frac{D^n f(z)^\sigma}{\sigma^n z^\sigma} \in P_k(\beta)$,
- (iii) $f(z)^\sigma / z^\sigma \in \Phi_{\sigma,n,k}^1(\beta)$.

Lemma 3.4. *Let $f \in A$. Then the following are equivalent:*

- (i) $f \in B_n^\sigma(k, \beta)$,
- (ii) $\frac{L_n^\sigma f(z)}{z} \in P_k(\beta)$,
- (iii) $f(z)/z \in \Phi_{\sigma,n,k}^2(\beta)$.

Our first results are the inclusions as follows:

Theorem 3.5. *For each $n \in \mathbb{N}$,*

$$T_{n+1}^\sigma(k, \beta) \subset T_n^\sigma(k, \beta), \quad B_{n+1}^\sigma(k, \beta) \subset B_n^\sigma(k, \beta).$$

Proof. Let $f \in T_{n+1}^\sigma(k, \beta)$. By Lemma 3, $f(z)^\sigma / z^\sigma \in \Phi_{\sigma,n+1,k}^1(\beta)$. Hence by Theorem 3, we have $f(z)^\sigma / z^\sigma \in \Phi_{\sigma,n,k}^1(\beta)$. This implies $f \in T_n^\sigma(k, \beta)$ as required. The proof of the second part is similar. $\square$

It is known that, for $n \geq 1$, the classes $T_n^\sigma(\beta)$ and $B_n^\sigma(\beta)$ both consist only of univalent functions in the open unit disk. Based on this, the next results show that $|z| = r(k, \beta)$ given by (1.5) is the radius of univalence for both $T_n^\sigma(k, \beta)$ and $B_n^\sigma(k, \beta)$ when $n \geq 1$.

Theorem 3.6. *If $f \in T_n^\sigma(k, \beta)$, then*

$$\operatorname{Re} \frac{D^n f(z)^\sigma}{z^\sigma} > 0 \quad (3.1)$$

for $|z| = r(k, \beta)$ given by (1.5). For any larger disk, there exist functions in $T_n^\sigma(k, \beta)$ which do not satisfy (3.1). Similarly, if $f \in B_n^\sigma(k, \beta)$, then

$$\operatorname{Re} \frac{L_n^\sigma f(z)}{z} > 0 \quad (3.2)$$

for $|z| = r(k, \beta)$ given by (1.5). For any larger disk, there exist functions in $T_n^\sigma(k, \beta)$ which do not satisfy (3.2).

Proof. The proofs follow easily from Lemmas 2.2, 3.3 and 3.4. The functions $f(z)$ defined by $f(z)^\sigma / z^\sigma = \phi_{\sigma,n}^1(H(z))$ and $f(z)/z = \phi_{\sigma,n}^2(H(z))$, where $H(z)$ is given by (1.6), respectively, show that the results are sharp. $\square$

Define the integral

$$F(z)^\kappa = \frac{c + \kappa}{z^c} \int_0^z t^{c-1} f(t)^\kappa dt, \quad c + \kappa > 0.$$

For $\kappa = \sigma$, we consider the integral F in $T_n^\sigma(k, \beta)$ while it is considered in $B_n^\sigma(k, \beta)$ for $\kappa = 1$. Our results are the following

Theorem 3.7. *Both classes $T_n^\sigma(k, \beta)$ and $B_n^\sigma(k, \beta)$ are preserved under F .*

Proof. Note that $\kappa = \sigma$ for $T_n^\sigma(k, \beta)$ and $\kappa = 1$ for $B_n^\sigma(k, \beta)$. Suppose f belong to either class. Then, by Lemmas 3.3 and 3.4, $f(z)^\kappa/z^\kappa$ belongs to $\Phi_{\sigma,n,k}^j(\beta)$, $j = 1, 2$; that is, there exists $h \in P(\beta)$ such that $f(z)^\kappa/z^\kappa = \phi_{\sigma,n}^j(h(z))$. Hence

$$\begin{aligned} F(z)^\kappa/z^\kappa &= \frac{c + \kappa}{z^{c+\kappa}} \int_0^z t^{c+\kappa-1} (f(t)^\kappa/t^\kappa) dt, \quad c + \kappa > 0. \\ &= \frac{c + \kappa}{z^{c+\kappa}} \int_0^z t^{c+\kappa-1} \phi_{\sigma,n}^j(h(t)) dt \\ &= \phi_{\gamma,1}^j[\phi_{\sigma,n}^j(h(t))], \quad \gamma = c + \kappa \\ &= \phi_{\sigma,n}^j[\phi_{\gamma,1}^j(h(t))], \end{aligned}$$

so that by Remark 2.8 and Lemmas 3.3 and 3.4 again we have the results. This completes the proof. $\square$

Theorem 3.8. *Let $c_{l,n}^j$, $j = 1, 2$, have their definitions as in (2.2). If $f \in T_n^\sigma(k, \beta)$, then*

$$\operatorname{Re} \frac{f(z)^\sigma}{z^\sigma} \geq 1 + (1 - \beta) \sum_{k=1}^{\infty} (2c_{2l,n}^1 r - kc_{2l-1,n}^1) r^{2l-1}.$$

Similarly, if $f \in B_n^\sigma(k, \beta)$, we have

$$\operatorname{Re} \frac{f(z)}{z} \geq 1 + (1 - \beta) \sum_{k=1}^{\infty} (2c_{2l,n}^2 r - kc_{2l-1,n}^2) r^{2l-1}.$$

The bounds are the best possible.

Proof. Take $f(z)^\sigma/z^\sigma = \phi_{\sigma,n}^1(h(z))$ and $f(z)/z = \phi_{\sigma,n}^2(h(z))$ in Theorem 2.6. The extremal functions are defined by $f(z)^\sigma/z^\sigma = \phi_{\sigma,n}^1(H(z))$ and $f(z)/z = \phi_{\sigma,n}^2(H(z))$, where $H(z)$ is given by (1.6). $\square$

Theorem 3.9. *Let $c_{l,n}^j$, $j = 1, 2$ have their definitions as in (2.2). If $f \in T_n^\sigma(k, \beta)$, then*

$$\operatorname{Re} \frac{f(z)^{\sigma-1} f'(z)}{z^{\sigma-1}} \geq 1 + (1 - \beta) \sum_{k=1}^{\infty} (2c_{2l,n-1}^1 r - kc_{2l-1,n-1}^1) r^{2l-1}.$$

Similarly, if $f \in B_n^\sigma(k, \beta)$, we have

$$\operatorname{Re} \frac{(\sigma - n) \frac{f(z)}{z} + f'(z)}{\sigma - (n - 1)} \geq 1 + (1 - \beta) \sum_{k=1}^{\infty} (2c_{2l,n-1}^2 r - kc_{2l-1,n-1}^2) r^{2l-1}.$$

The bounds are the best possible.

Proof. Take $f(z)^\sigma/z^\sigma = \phi_{\sigma,n}^1(h(z))$ and $f(z)/z = \phi_{\sigma,n}^2(h(z))$. On differentiation and using Remark 3, we have the results. The extremal functions are those defined by $f(z)^\sigma/z^\sigma = \phi_{\sigma,n}^1(H(z))$ and $f(z)/z = \phi_{\sigma,n}^2(H(z))$, $H(z)$ given by (1.6). $\square$

Remark 3.10. Variants of our results for special cases of the parameters σ , n , k and β can be derived by specifying them.

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