

A Sharp Inclusion Theorem for a Certain Class of Analytic Functions Defined by the Salagean Derivative

K. O. Babalola, T. O. Opoola

Department of Mathematics
University of Ilorin
Ilorin, Nigeria

email: khayrah.babalola@gmail.com, opoola_stc@yahoo.com

(Received July 31, 2007, Accepted August 21, 2007)

Abstract

In this short note we employ the Briot-Bouquet differential subordination to determine the best possible inclusion relation within a certain family of analytic functions defined by the Salagean derivative.

1 Introduction

Let A denote the class of functions:

$$f(z) = z + a_2 z^2 + \dots$$

which are analytic in the unit disk $E = \{z \in \mathbb{C}: |z| < 1\}$. Also let P be the class of functions:

$$p(z) = 1 + p_1 z + \dots \quad (1.1)$$

which are also analytic in the unit disk E and have positive real part. In [3], Opoola introduced the subclass $T_n^\alpha(\beta)$ consisting of functions $f \in A$ which satisfy:

$$\operatorname{Re} \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta \quad (1.2)$$

where $\alpha > 0$ is real, $0 \leq \beta < 1$, $D^n (n \in N_0 = \{0, 1, 2, \dots\})$ is the Salagean derivative operator defined as: $D^n f(z) = D(D^{n-1} f(z)) = z[D^{n-1} f(z)]'$ with $D^0 f(z) = f(z)$ and powers in (1.2) meaning principal determinations only.

Key words and phrases: Analytic functions, Salagean derivative

AMS (MOS) Subject Classifications: Primary 30C45; Secondary 30C80

ISSN 1814-0424 © 2007, <http://ijmcs.futureintech.net>

The geometric condition (1.2) slightly modifies the one given originally in [3] (see [1]).

In earlier works [1, 3], inclusion relations have been discussed for the family $T_n^\alpha(\beta)$. In particular, it has been shown that members of the family are related by the inclusion.

Theorem 1 ([1, 3]).

$$T_{n+1}^\alpha(\beta) \subset T_n^\alpha(\beta), \quad n \in N_0.$$

The object of the present paper is to sharpen the above result. Our result is the following:

Theorem 2.

$$T_{n+1}^\alpha(\beta) \subset T_n^\alpha(\delta(\alpha, \beta)), \quad n \in N_0$$

where

$$\delta(\alpha, \beta) = 1 + 2(1 - \beta) \sum_{k=1}^{\infty} \frac{\alpha}{\alpha + k} (-1)^k. \quad (1.3)$$

The result is sharp.

We will make use of the powerful technique of Briot-Bouquet differential subordination to prove the above result. This technique has been employed frequently in recent times to sharpen and improve many results in geometric function theory. A function $p(z)$ given by (1.1) is said to satisfy the Briot-Bouquet differential subordination if

$$p(z) + \frac{zp'(z)}{\eta p(z) + \gamma} \prec h(z), \quad z \in E \quad (1.4)$$

where η and γ are complex constants and $h(z)$ a complex function satisfying $h(0) = 1$, and $\operatorname{Re}(\eta h(z) + \gamma) > 0$ in E . It is well known that if $p(z)$ given by (1.1) satisfies the Briot-Bouquet differential subordination, then $p(z) \prec h(z)$ [2].

A univalent function $q(z)$ is said to be a dominant of (1.4) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.4). If $\tilde{q}(z)$ is a dominant of (1.4) and $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (1.4), then $\tilde{q}(z)$ is said to be the best dominant of (1.4). The best dominant is unique up to rotation. Furthermore, it is known [2] that if the differential equation

$$q(z) + \frac{zq'(z)}{\eta q(z) + \gamma} = h(z), \quad q(0) = 1 \quad (1.5)$$

has univalent solution $q(z)$ in E , then $p(z) \prec q(z) \prec h(z)$ and $q(z)$ is the best dominant.

Now let $f \in A$. If we let

$$p(z) = \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}$$

Then we find that

$$p(z) + \frac{zp'(z)}{\alpha} = \frac{D^{n+1} f(z)^\alpha}{\alpha^{n+1} z^\alpha}$$

Thus it follows that

(a) if $\frac{D^{n+1} f(z)^\alpha}{\alpha^{n+1} z^\alpha} \prec h(z)$, then $\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} \prec h(z)$.

(b) if the differential equation $q(z) + \frac{zq'(z)}{\eta q(z) + \gamma} = h(z)$, $q(0) = 1$, has univalent solution $q(z)$ in E , then $\frac{D^{n+1} f(z)^\alpha}{\alpha^{n+1} z^\alpha} \prec h(z)$ implies $\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} \prec q(z) \prec h(z)$ and $q(z)$ is the best dominant.

In view of the above expositions we proceed to the proof of our result in the next section .

2 Proof of Theorem

Let $f \in T_{n+1}^\alpha(\beta)$. Define

$$h(z) = h_\beta(z) = \frac{1 + (1 - 2\beta)z}{1 - z}.$$

It is obvious that $h_\beta(z)$ maps the unit disk onto the plane $Re \omega > \beta$. Hence

$$f \in T_{n+1}^\alpha(\beta) \Leftrightarrow \frac{D^{n+1} f(z)^\alpha}{\alpha^{n+1} z^\alpha} \prec h_\beta(z)$$

and the differential equation

$$q(z) + \frac{zq'(z)}{\alpha} = h_\beta(z), \quad q(0) = 1$$

has univalent solution

$$q_\beta(z) = \frac{\alpha}{z^\alpha} \int_0^z t^{\alpha-1} h_\beta(t) dt.$$

This yields

$$q_\beta(z) = 1 + 2(1 - \beta) \sum_{k=1}^{\infty} \frac{\alpha}{\alpha + k} z^k.$$

That is,

$$q_\beta(z) = \beta + (1 - \beta) \left\{ 1 + 2 \sum_{k=1}^{\infty} \frac{\alpha}{\alpha + k} z^k \right\}.$$

In [1] we have shown that

$$\operatorname{Re} \left\{ 1 + 2 \sum_{k=1}^{\infty} \frac{\alpha}{\alpha + k} z^k \right\} \geq 1 + 2 \sum_{k=1}^{\infty} \frac{\alpha}{\alpha + k} (-r)^k, \quad |z| = r.$$

Hence taking limit as $r \rightarrow 1^-$, we have $\min_{|z| \leq 1} \operatorname{Re} q_\beta(z) = q_\beta(-1) = \delta(\alpha, \beta)$ where $\delta(\alpha, \beta)$ is given by (1.3). Hence we have

$$f \in T_{n+1}^\alpha(\beta) \Rightarrow \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} \prec q_\beta(z) \prec h_\beta(z)$$

That is,

$$f \in T_{n+1}^\alpha(\beta) \Rightarrow \operatorname{Re} \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} \geq \delta(\alpha, \beta)$$

i.e. $f \in T_n^\alpha(\delta(\alpha, \beta))$. The inclusion is sharp since $q_\beta(z)$ is the best dominant. This completes the proof.

If we choose $\alpha = 1$, we have

Corollary 1.

$$T_{n+1}^1(\beta) \subset T_n^1(2(1 - \beta) \ln 2 + 2\beta - 1), \quad n \in N_0.$$

In particular for $n = 0$, we have

Corollary 2. *Let $f \in A$. Then*

$$\operatorname{Re} f'(z) > \beta \Rightarrow \operatorname{Re} \frac{f(z)}{z} > 2(1 - \beta) \ln 2 + 2\beta - 1.$$

The result is sharp.

Finally, we note that the above result improves an earlier one due to Owa and Obradovic [4] in which they proved that if $f \in A$ satisfies $\operatorname{Re} f'(z) > \beta$ for $0 \leq \beta < 1$ and $z \in E$, then

$$\operatorname{Re} \frac{f(z)}{z} > \frac{1 + 2\beta}{3}.$$

References

- [1] K. O. Babalola, T. O. Opoola, Iterated integral transforms of Caratheodory functions and their applications to analytic and univalent functions, *Tamkang J. Math.*, **37** (4) (2006), 355–366.
- [2] P. Eenigenburg, S. S. Miller, P. T. Mocanu, M. O. Reade, On a Briot-Bouquet differential subordination, *Rev. Roumanie Math. Pures Appl.*, **29** (1984), 567–573.
- [3] T. O. Opoola, On a new subclass of univalent functions, *Matematica (Cluj)*, **36**, 59 (2)(1994), 195–200.
- [4] S. Owa, M. Obradovic, Certain subclass of Bazilevic functions of type α , *Int. J. Math. and Math. Sci.*, **9** (1986), 347–359.
- [5] G. S. Salagean, Subclasses of univalent functions, *Lecture Notes in Math.*, **1013** (1983), 362–372, Springer-Verlag, Berlin, Heidelberg, New York.