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Modelling the Projectile Motion of a Soccer Ball under Linear Drag Influence

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Abstract

The paper presents an investigation into the projectile motion of a soccer ball. The equation of motion that determines the trajectory motion of the soccer ball was developed. In order to check on the accuracy of the equation for the linear drag force, a perturbation expansion technique was employed to solve for the range of soccer ball in motion under specific conditions of initial velocity and launch angle. The perturbation expansion was solved up to the third order and the result obtained from previous research work was verified. Result shows that for a soccer game scenario where a player tries to score a goal from free kick at 20m; if the ball is kicked at 18.86m/s then the most suitable angles for scoring a goal are 52° and 55° . This is because the barrier formed by the players will surely be overcome and the ball will reach the goal post regardless of any obstacles. A follow up comparison between linear drag theory and experimental observation on the field of play is also suggested.

Keywords: Projectile, drag force, perturbation, soccer

1. Introduction

Projectile motion of objects has been subject of studies in engineering and physics for a long time. Dynamics is a branch of engineering mechanics that deals with the motion of objects which involves studying the effect of mass in relation to how and why objects accelerate. This interesting field is normally treated as an undergraduate course under three major headings which include (i) Dynamics of rigid bodies, (ii) Dynamics of a particle and (iii) Dynamics of deformable bodies. A particle is defined as an object which is small compared with space in which it moves, it occupies a single point in space.

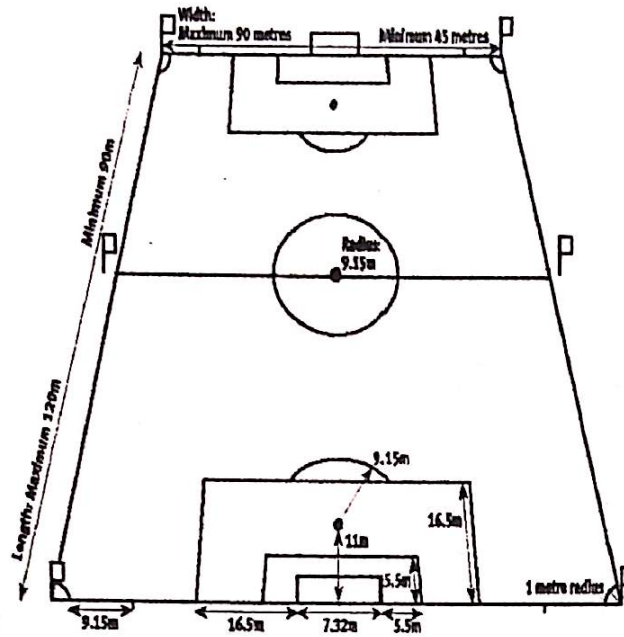


Figure 1: Metric Measurements of standard Soccer field
(Source:<http://www.psufc.com/pix/football-field-diagram.gif>)

Projectiles are considered as particles; hence the dynamics of particles will be relevant in this analysis. Projectile in motion through air can be studied under two major headings; projectile in the absence of drag and projectile in the presence of drag. When the effect of air resistance is neglected, the projectile motion becomes relatively easy and the results thus obtained are not valid for most real life problems. The inclusion of a quadratic drag force (that is, one that represents drag as a force proportional to the square of the projectile speed) into the equation of motion for two dimensional problems is desirable since it leads to a more realistic representation of the motion, though it will lead to more complex mathematical equations which are more difficult to analyze and solve Housnier (1959).

Housnier (1959) reported that solving practical problems in ballistics requires the effect of air resistance which must be considered. In making this consideration, one would have to solve problems by numerical integration or employ computing machines. In addition, he also reported that a few special problems can be treated by simpler mathematical procedures, such special problems include situation where the velocity range is between 30.5 m/s and 305 m/s. The drag force in this situation is approximately proportional to the square of the velocity.

Shames (1966) reported that for spherical objects which are so stabilized while in motion, their trajectory presents the same cross section in the direction of motion. If such objects move through still air which has no strong winds and updrafts, the frictional force (air resistance) developed during flights is at all times tangent to the trajectory and is always opposing the motion of the object.

Other interesting contributions are those of Coutis (1998) and Akanbi (2000). Coutis (1998) modeled the trajectory motion of a cricket ball subject to a linear drag force. He reported that for an interesting (although rare) cricket match scenario in which a batsman hits a straight drive back past a bowler for six, if the ball leaves the bat at 40 m/s, at an angle of 30° , it will clear a boundary line 90 meters away. Akanbi(2000) in his study developed a software package that modeled the equation of projectile in motion in the absence of drag (air resistance).Ken and David(2003) also modeled the flight of a soccer ball in a direct free kick. Lasode and Olaleye (2005) also published their work on computer aided analysis of projectile motion in air.

2. Aim

The problem analyzed here involves determining the speed at which the soccer ball will have to be kicked for the ball to gain enough distance and at the same time be at a height low enough for it to enter the net of a goal post. The aim of the present work is to formulate and solve the above problems using basic analytical.

3. Equation for Projectile Motion in the Absence of Drag

Considering a projectile fired at an initial velocity V_0 while making an angle θ with the horizontal axis of the rectangular coordinate system as shown in Figure 2.

The following assumptions were made:

- (i) The projectile is moving under a relatively short range with a shallow trajectory so that its gravitational acceleration, g may be taken as constant.
- (ii) The ground is a plane surface
- (iii) The attraction of gravity, g is normal to the plane representing the earth's surface and the rotation of the earth is neglected.
- (iv) The drag is at all times tangent to the trajectory.

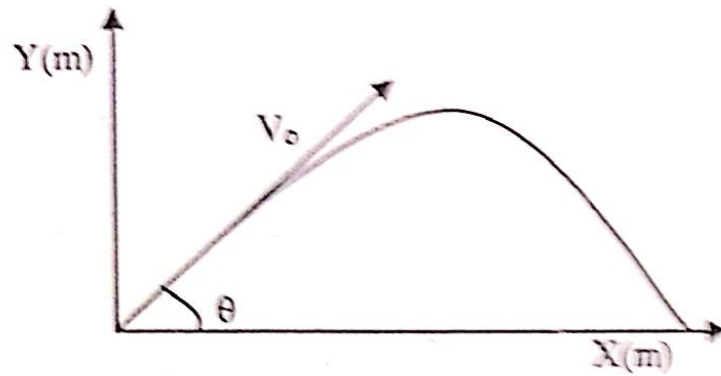


Figure 2 The trajectory of projectile in motion without air resistance (drag)

There is a unidirectional force of gravity on a particle during motion and the trajectory is coplanar. Thus, the equations of motion are

$$m \ddot{x} = 0 \quad (1)$$

$$m \ddot{y} = -mg \quad (2)$$

Where m is the mass, $\ddot{x}$, $\ddot{y}$ denote the acceleration of the particles in the horizontal (x-axis) and vertical axis (y-axis) respectively. If the initial

conditions are: At time $t = 0$, $\dot{x} = V_o \cos \theta$, $\dot{y} = V_o \sin \theta$, $x=0, y=0$

At any instant of time t ,

$$x = (V_o \cos \theta)t \quad (3)$$

$$y = (V_o \sin \theta)t - \frac{1}{2}gt^2 \quad (4)$$

Solving for t in equations (3) and (4) above and substituting into (4) gives the equation of trajectory

$$y = x \tan \theta - \frac{1}{2}g\left(\frac{x}{V_o \cos \theta}\right)^2 \quad (5)$$

The motion parameters like the time of flight, range and maximum height can

be obtained by using equations (3), (4) and or (5).

4. Equation for Projectile Motion in the Presence of Drag

Since drag retards projectile motion, solving the problem of projectile motion while considering drag is one that actually requires solving complex mathematical problems using numerical integration and powerful computational techniques. This is due to the fact that drag is a complicated function of the projectile velocity relative to air based on experimental findings Shames(1966).

Very realistic solutions which are more relevant to real life situations than the situation of zero drag are obtainable using simplified mathematical procedures.

In addition to the assumptions stated in section 3 above, the moving object is assumed to be spherical or is so stabilized in its trajectory that it presents the same cross section in the direction of motion, the lift force provided by the spin of the object is neglected, the only forces acting on the projectile is taken to be only gravity and drag where the drag is taken as linear function. The parameters governing the trajectory are the take-off speed, angle θ and the linear drag per unit mass k . Shown in the Figure 3 below is the motion of a particle in the presence of linearise drag force.

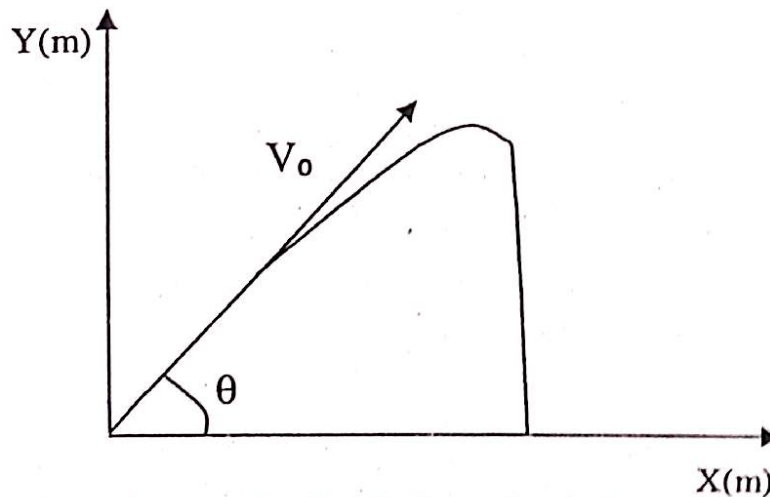


Figure 3: The trajectory of projectile in motion in the presence of linearise drag force.

The equation of motion in both x and y directions are

$$m \ddot{x} = -c \dot{x} \quad (6)$$

$$m \ddot{y} = -c \dot{y} - mg \quad (7)$$

Where c is a constant and m, y, x and g retain their standard definitions.

Dividing equation (6) and (7) by m and rearranging would yield

$$\ddot{x} + k \dot{x} = 0 \quad (8)$$

$$\ddot{y} + k \dot{y} = -g \quad (9)$$

$k=c/m$ is the drag force.

5. Calculation of the Drag Force k for Spherical Objects

The drag force which gives the overall effect of air resistance. Actually, the drag force, $k = h(v)$ has a very complex variation with instantaneous velocity v which can only be determined experimentally. However, a good approximation is to use [4]

$$h(v) = (c_1 + c_2 |v|) V_0 \quad (10)$$

Where v is the instantaneous projectile velocity ($x_i + y_j$), c_1 and c_2 are constants given by Coutis (1998), $c_1 = 1.55 \times 10^{-4}$, $c_2 = 0.22d^2$ in which d is the diameter of the projectile, $c_2 \gg c_1$ hence $c_1 \approx 0$

Also, the drag force expression in equation (10) denotes a non-linear nature, to linearise it we set

$$(c_1 + c_2 |v|)v = (c_1 + c_2 (\frac{2}{3} V_0))v \quad (11)$$

The choice of $(\frac{2}{3}V_0)$ is arbitrary. Although it over estimated it is chosen because the excess implied caters for the effect of the inhibitors to the projectile motion like spin, changes in the properties of air like density, temperature, pressure, wind speed and so on. Finally, drag coefficient can be represented as

$$k = \frac{(\frac{2}{3}V_0)^2 c_d}{m} \quad (11a)$$

or,

$$k = \frac{0.22d^2 (\frac{2}{3}V_0)^2}{m} \quad (11b)$$

It can be seen that the drag force is directly proportional to the square of the projectiles diameter and inversely proportional to its mass. Hence, a spherical object like a shot put or golf ball both having a relatively small diameter and large mass will be less affected by drag unlike an object like a beach ball which has large diameter but small mass.

One solution to equation (1.8) is

$$x(t) = A + Be^{-kt} \quad (13a)$$

Where A and B are constants which can be obtained from the initial conditions $x(0) = 0$ and

$$\dot{x}(0) = V_0 \cos \theta$$

Therefore,

$$x(t) = \frac{V_0 \cos \theta (1 - e^{-kt})}{k} \quad (13b)$$

Similarly, for y using the same method of undetermined coefficient yields

$$y(t) = C + De^{-kt} - \frac{gt}{k} \quad (14)$$

Initial conditions $y(0) = 0$, and $\dot{y}(0) = V_0 \sin \theta$

The expression for $y(t)$ becomes

$$y(t) = \left(\frac{V_0 \sin \theta}{k} + \frac{g}{k^2} \right) (1 - e^{-kt}) - \frac{gt}{k} \quad (15)$$

Rearranging equation (1.13) gives,

$$t = -\frac{1}{k} \ln \left(1 - \frac{kx}{V_0 \cos \theta} \right) \quad (16)$$

Substituting equation (1.16) into (1.15) gives

$$y = x \left(\tan \theta + \frac{g}{kV_0 \cos \theta} \right) + \frac{g}{k^2} \ln \left(1 - \frac{kx}{V_0 \cos \theta} \right) \quad (17)$$

Equation (1.17) gives the trajectory of a projectile moving in the presence of air resistance which is expressed as a linear function of the projectile initial velocity V_0 [4]. Figure 4 shows the theoretical prediction of a soccer ball struck with and without drag.

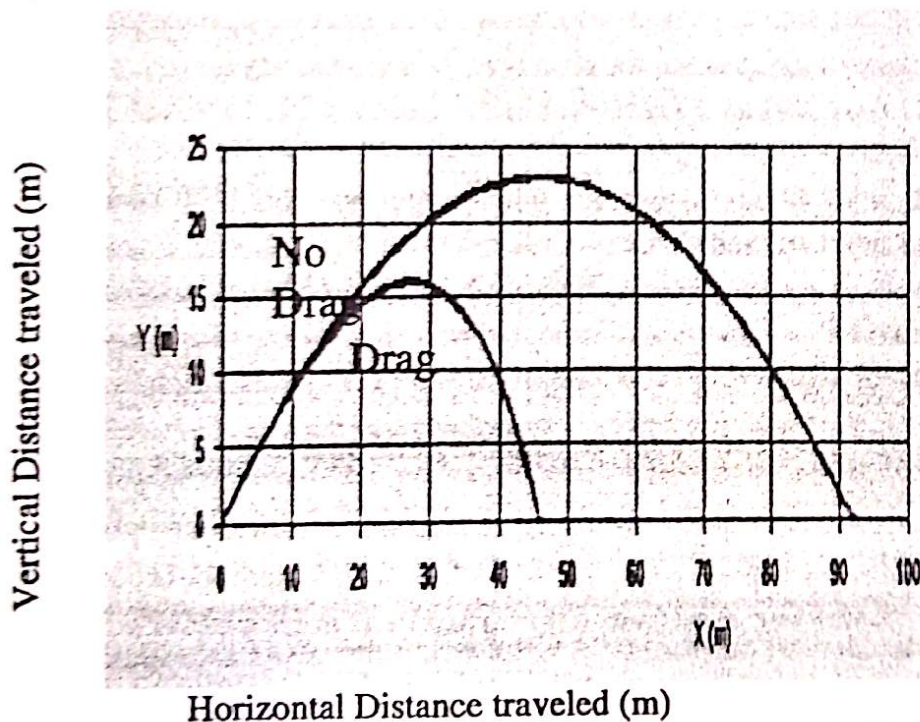


Figure 4: Theoretical prediction of the trajectories of a soccer ball struck with a speed of 30m/s at an angle of 45° for the case of zero drag and in the presence of linearise drag force.

6. Solving the Trajectory Equation for a Projectile under the Influence of Drag

The trajectory equation of equation (17) can be solved to a high reasonable accuracy using the approximate solution technique. Consider the logarithm term on the right hand side of equation (17), The series below

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots |u| < 1 \quad (18)$$

upon integration equation (18) becomes

$$\ln(1-u) = -u - \frac{u^2}{2} - \frac{u^3}{3} - \dots |u| < 1 \quad (19)$$

Let $u = \frac{kx}{(V_0 \cos \theta)}$ and $y=0$ equation (17) becomes

$$\begin{aligned} x \tan \theta = & \frac{gx^2}{2V_0^2 \cos^2 \theta} + \left(\frac{gx^3}{3V_0^3 \cos^3 \theta} \right) k \\ & + \left(\frac{gx^4}{4V_0^4 \cos^4 \theta} \right) k^2 + \dots \end{aligned} \quad (20)$$

The standard method for solving equations involving series expansion is to employ the perturbation expansion technique, let x be represented in the form

$$x = x_0 + kx_1 + k^2x_2 + k^3x_3 + \dots \quad (21)$$

Substituting equation (21) into (20) and equating coefficient of like power of k this will give the following solutions for x_0, x_1, x_2 and x_3

$$x_0 = \frac{V_0^2 \sin 2\theta}{g} \quad (22)$$

$$x_1 = \frac{-4V_0^3 \sin 2\theta \sin \theta}{3g^2} \quad (23)$$

$$x_2 = \frac{14V_0^4 \sin 2\theta \sin^2 \theta}{9g^3} \quad (24)$$

$$x_3 = \frac{-104V_0^5 \sin 2\theta \sin^3 \theta}{45g^4} \quad (25)$$

The solution generated is slightly different from that of Coutis (1998). In his work, he stopped at second order component of equation (21) while the second order component of equation (21) which was reported in equation (24) of the present work was reported as

$$x_2 = \frac{28V_0^4 \sin 2\theta \sin^2 \theta}{9g^3} \quad (26)$$

More terms of the series (21) can be included for very accurate result to be obtained. It is now possible to calculate the range of a soccer ball struck towards the boundary. The range is given by equation (21) where x_0, x_1, x_2 and x_3 are as given in equation (22) to (25). It is worthy of note that the series was evaluated up to the third order and the percentage contribution of the third order component is 9%.

The ball has a circumference within the range of 0.68-0.71 which translates into diameter within the range of 0.216-0.226. It weighs between 0.396-0.453kg. For the present work, it is assumed that the soccer ball has an average mass of 0.425kg and an average diameter of 0.221m.

7. Results and Discussion

7.1 Dynamics in the Football Game

Case I: Kicking the ball from 5.5m box

Let us consider a hypothetical game in which a football player attempts to score a goal against his opponent by kicking the ball from the 5.5 m box. If the length of the field is 90 meters, the player would have to overcome a horizontal distance of 84.5 m. The average speed, V_0 at which the ball is kicked in addition to the angle at which the ball is fired will determine if this can be achieved.

Consider the series equation of (21), if we substitute the drag coefficient of equation (12) into (21) and if we replace x with R we would arrive at

$$R = \frac{(\sin 2\theta)V_0^2}{g} - \frac{(8c_2 \sin 2\theta \sin \theta)V_0^4}{9g^2m} + \frac{(56c_2^2 \sin 2\theta \sin^2 \theta)V_0^6}{81g^3m^2} - \frac{(832c_2^3 \sin 2\theta \sin^3 \theta)V_0^8}{1215g^4m^3} \quad (27)$$

Where $m=0.425$ Kg, $g = 9.81$ m/s², $c_2 = 0.22D^2$
 $c_2=1.075 \times 10^{-2}$ since $D=221$ mm

When the discriminant of the first two terms of equation (26) are considered, we would discover that solutions for V_0 exist only when

$$R \leq \frac{9m \cos \theta}{16c_2} \quad (28)$$

Though the above inequality is the result of an approximation, it has interesting implications. In the presence of a linear drag force, there is a particular range R_f beyond which the ball cannot be kicked no matter how fast it leaves the leg of the player. R_f was found to be 22.25m and this occurs as $\theta \rightarrow 0^\circ$

If we consider the trajectories of six footballs all struck at $\theta = 45^\circ$ to the horizontal and at different speeds; 20,40,60,80,100 and 120 m/s. From the figure 5 , 6 , and 7 plotted (which was developed using visual basic programming), it can be seen that the curves converges to point close to 41.25 m. This is from speed 60 m/s and above which is a speed practically impossible to attain by a player when kicking the ball. Maximum speed ever recorded for a player is for Mr. Beckham which was about 36m/s (Source: <http://soccer.epicsports.com/resources/ballsizing.htm>)

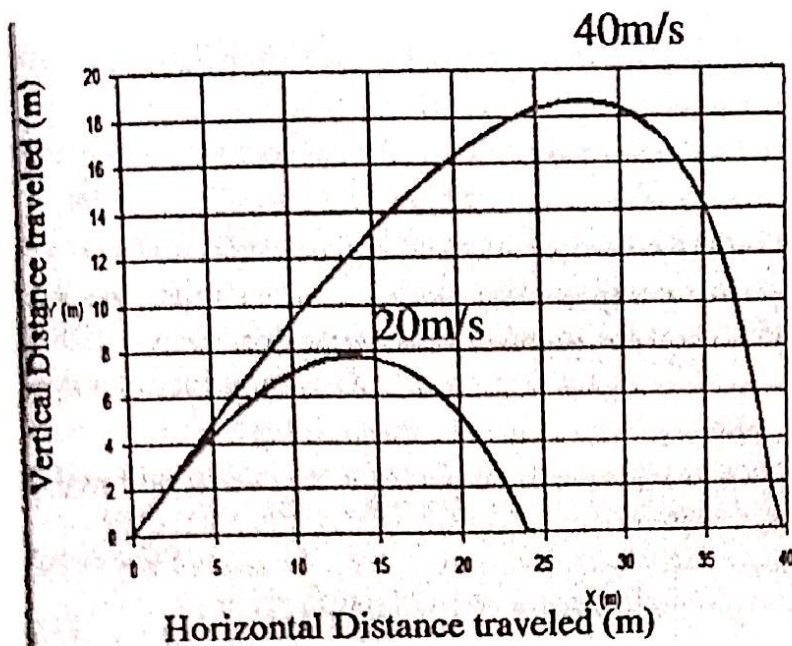


Figure 5: Trajectories of a soccer ball struck with a speed of 20m/s and 40m/s at an angle of 45° in the presence of linearise drag force

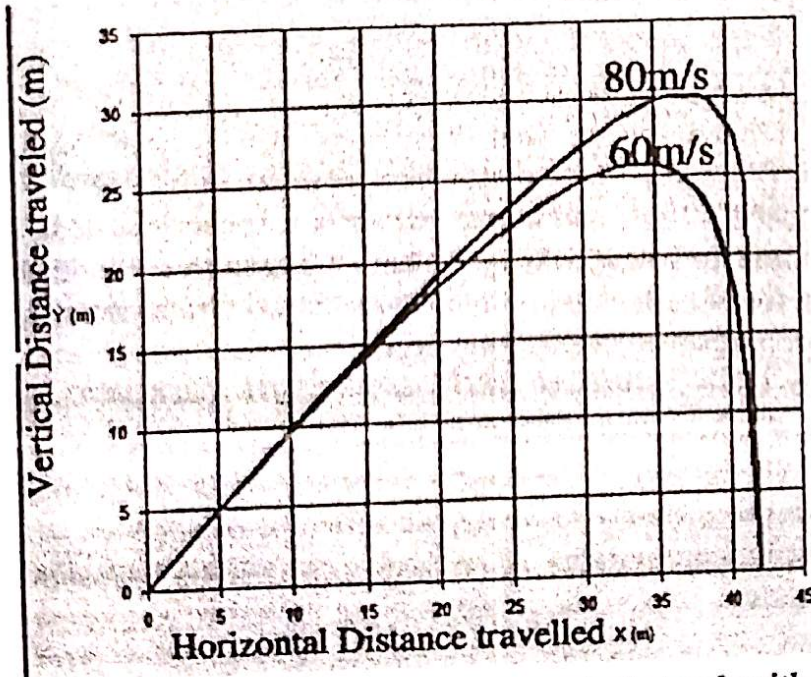


Figure 6: Trajectories of a soccer ball struck with a speed of 60m/s and 80m/s at an angle of 45° in the presence of linearise drag force 24

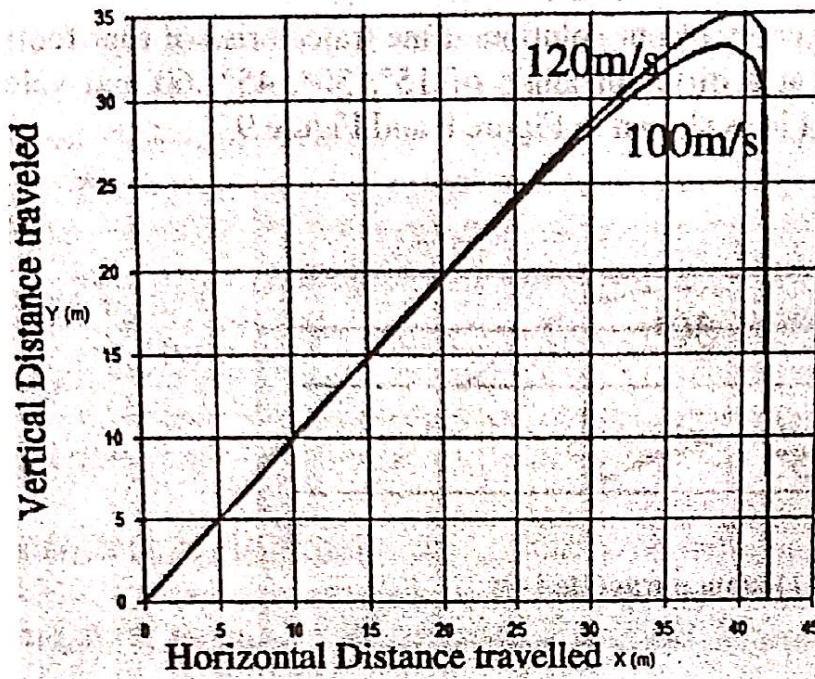


Figure 7: Trajectories of a soccer ball struck with a speed of 100m/s and 120m/s at an angle of 45° in the presence of linearise drag force

Case II

In an average football match, goals are usually scored when players kick the ball at a distance close to or within the 16.5 m. Let us consider a situation where the football is kicked at point that is at a distance of $R=20\text{m}$ from goal poles. From the inequality of equation (1.28)

when $\theta = 20^\circ$, $R \leq 20.9\text{ m}$, at $\theta = 25^\circ$, $R \leq 20.15\text{ m}$, at

$\theta = 30^\circ$, $R \leq 19.26\text{ m}$

Based on the above, if we assume that the ball is kicked at an angle $\theta = 25^\circ$, although this represent a flat trajectory, the series solution of (1.21) for x which is based on the assumption that $\cos\theta > \frac{kx}{V_0}$ is more applicable for flat

trajectories.

When $R=20\text{ m}$ and $\theta = 25^\circ$, equation (1.26) yields 21.647 m/s for the first two terms, 18.243 m/s for the first three terms and 18.855 m/s for all the

terms. This result implies that a footballer must kick the ball with a speed of about 18 m/s at $\theta = 25^\circ$ in order to clear a distance of 20 m from the post. As a check on the accuracy of our solution, if the trajectories of four footballs are considered each at a different angle of 15° , 30° , 45° , 60° and velocity $V_0 = 18.86$ m/s the plot is as shown in Figure 8 and Figure 9

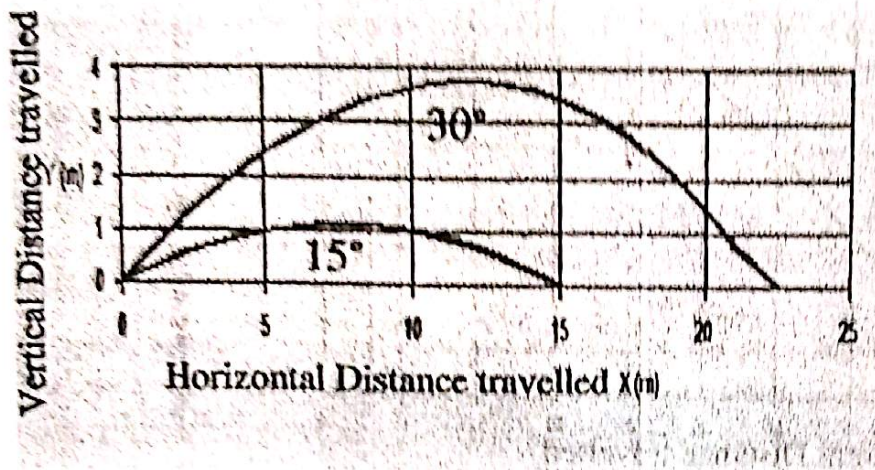


Figure 8: Trajectories of a soccer ball struck with speed of 18.86m/s at angles 15° and 30° in the presence of linearise drag

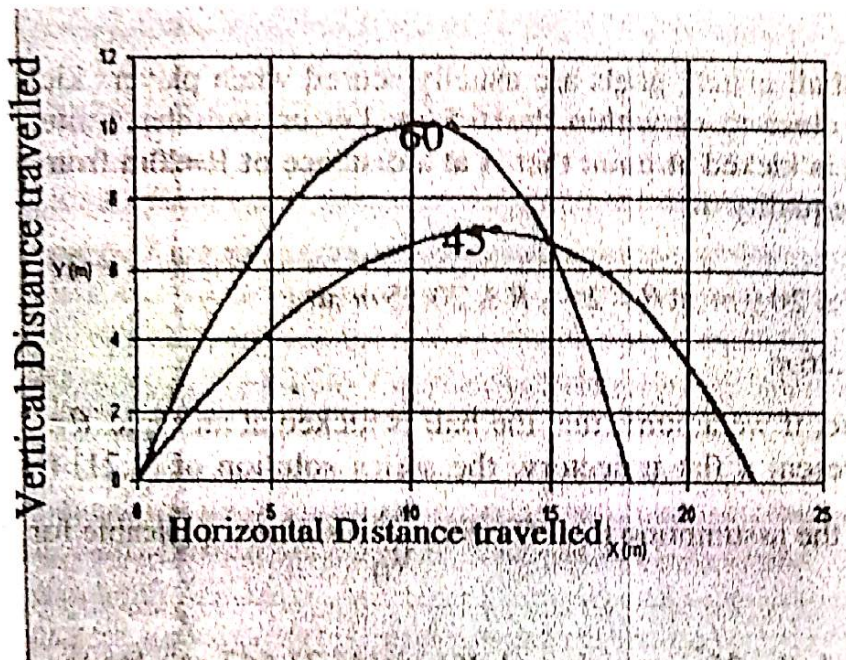


Figure 9: Trajectories of a soccer ball struck with speed of 18.86m/s at angles 45° and 60° in the presence of linearise drag²⁶

The trajectories shows that at $\theta = 30^\circ$ and $\theta = 45^\circ$, the range obtained is 22.5m which is a sharp contrast from the zero drag theory which states that the maximum range is obtainable only at $\theta = 45^\circ$. The optimum angle for a range of 20m is calculated when $V_0=18.86$ m/s using a visual Basic programme developed with Newton-Raphson's iteration technique; for an initial guess value of $\theta = 30^\circ$ after 205 iterations and tolerance of 0.001, the optimum angle $\theta = 31.9^\circ$.

Let us consider the approximate equation for range R of equation (1.26) with $\theta = 30^\circ$, $k=0.318$ and $V_0=18.86$ m/s we obtain
 $R = 31.4 - 12.8 + 4.6 - 2.1 = 21.1$ m

If equation (1.17) is solved numerically for $\theta = 30^\circ$, $k=0.318$, $V_0=18.86$ m/s and $y=0$. Using Newton-Raphson's iteration technique for an initial guess value of 21 m, after 4 iterations and a tolerance of 0.001, we obtain $R=24.1399$ m. Therefore, the first four terms of equation (1.26) have converged to 12.6% of the true range obtained from trajectory equation (1.17)

CASEIII: Attempt to score opponent from outside 16.5m box

This is the case where the football is kicked from 20 m from the goal post. An important factor to be considered in this analysis is the height of the goal post which is 2.44m. The trajectories of the soccer football at different angles were considered when $V_0=18.86$ m/s and $k=0.318$, the following results were obtained

Table 1: The trajectories of a football kicked at an average velocity $V_0=18.86$ m/s, drag coefficient $k= 0.318$ and at different angles

Angle θ°	Range (m)	Approx. Height at 20m (m)	Max. Height (m)	Time To reach max. height (s)	Time of flight to cover 20m(s)
25	21.00	0.30*	2.77	0.72	1.56
30	22.50	1.50*	3.78	0.84	1.81
32	22.50	1.80*	4.20	0.88	1.86

Table 1 contd.

35	23.25	2.45	4.86	0.95	2.05
37	23.25	2.80	5.30	0.99	2.12
40	23.25	3.20	5.97	1.04	2.25
42	23.25	3.26	6.42	1.08	2.36
45	22.50	3.30	7.08	1.13	2.42
50	21.75	2.45	8.17	1.21	2.66
52	21.00	1.47*	8.59	1.24	2.69
55	20.25	0.50*	9.20	1.28	2.84

*Heights less than the height of a standard goal post (2.44m)

From table 1 above, the suitable angle to which the football can be kicked successfully to attain the set range of 20 m and fit into the height of the goal post can be obtained. All relevant heights have been represented with asterisks (*) symbol.

From the result of the analysis, the most suitable angles are $\theta = 30^\circ$, $\theta = 32^\circ$ and $\theta = 52^\circ$. Angles $\theta = 30^\circ$ and $\theta = 32^\circ$ will yield the best results because at 0.84 and 0.88 seconds respectively after the ball is kicked, the maximum heights are already attained. This is a very short time interval which will give little time for the goalkeeper to react to the football that will be heading into the post 1.81 seconds and 1.86 seconds respectively after it is kicked. The goal keeper has a shorter time to make any adjustments necessary to prevent the ball from entering the post. In a situation where a free-kick is awarded to a team by the referee, players usually form a wall at a distance of about 9.14 meters (10yards) away from the ball. For such situation angles $\theta = 52^\circ$ and $\theta = 55^\circ$ might be considered as the best angles. This is because the wall formed by the players will surely be overcome and the ball will reach the goal post regardless of any obstacles.

One major obstacle with such large angles is that it becomes easier for the goal keeper to observe the movement of the ball, this in turn makes it easy for the goal keeper to reposition and prevent the ball from entering the goal post. Also, with these large angles are the possibilities of increase in wind speed, fluctuations in the density of air and spin to affect the range attained by the ball. Such alteration, if it results in reduction in range, will prevent the ball from entering the goal post at angle 55° . In other words, if the goal keeper is closer than 20m with that large angle, the initial speed has negligible effects because the player just tip-up the ball over the goal keeper at $<18.86\text{m/s}$.

8. Conclusion

The trajectory of a soccer ball in flight has been examined while employing the linear drag in the equation of motion. This method is more realistic than the approach of the zero drag theory. The method, which is readily available to undergraduate student in engineering, was used to determine the speed at which soccer ball was struck in order for it to score a goal. Results show that to score a goal from 20m while considering wall formed by opposing team, the best angles are 52 and 55 at velocity of 18.86 m/s in the presence of linearise drag. On the other hand, if there are no walls formed by the opposing team, then the most appropriate angles are 30 and 32 at 18.86m/s in the presence of linearise drag force. As a recommendation for future work, the results can be verified by comparing it with experimental observations on the field of play of a soccer game.

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