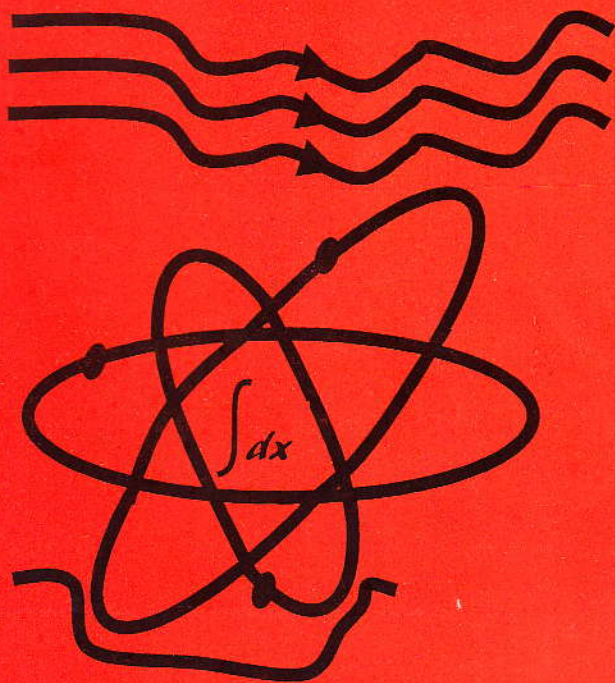


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Integrated Collocation Approximation Methods for Solving High Order Ordinary Differential Equations

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Abstract

This work deals with integrated collocation approximation methods for solving high order ordinary differential equations by Chebyshev polynomials and power series bases functions. The assumed approximate solution in terms of Chebyshev polynomials and power series are substituted into the general high order ordinary differential equations considered. Thus, after simplification, the resulting equation is then collocated for both cases and thus resulted into algebraic linear system of equations which are then solved by mathematica 5.2 to obtain the unknown constants involved in the approximate solution. The methods are illustrated on three examples. The results obtained in term of absolute errors are tabulated for comparison. From the tables of results, we observed that Chebyshev polynomials approximation gives a better result for all cases of N (i.e. the degree of approximant) in the work.

Keywords: Integrated Collocation, Chebyshev Polynomial, Power Series.

1. Introduction

The techniques for solving differential equations based on numerical approximations were developed long before programmable computers existed. Scores of mathematicians have tried to provide easier and less rigorous ways of solving high order ordinary differential equations numerically since the exact solution of some high order differential equations are not achievable.

Carl Runge [1], published a paper which extended the approximation method of Euler to a more elaborate scheme, which was capable of greater accuracy.

The idea of Runge was to base the approximate solution on such improved formulas as the mid-point and the trapezoidal rules. The requirement of evaluating the derivative at the mid or end point of a step not yet completed, was achieved by first performing an Euler type of calculation to obtain a preliminary approximation to the solution at one of the points. Runge was able to establish the scheme up to the third order.

In the year 1900, K. Heun took the order condition as far as four and introduced the method proposed amongst other methods (Heun's Method).

In the same vein in the year 1901, Kutta took the analysis of Kutta method as far as order five. The work made a complete classification of order four method and introduced the the famous Fourth order Runge-Kutta method.

The implicit Runge-Kutta were proposed by Kuntzmann and Butcher with the central example being based on Gaussian quadrature formulae. These methods stated form the basis on which many techniques of solving high order differential equations numerically are based.

So many people have worked on how to solve high order ordinary differential equations numerically. Although, most of them do not generalise the proposed methods, they are being specific on the order of the ODE.

Several Researchers [2-7] have worked on numerical solutions of fourth order ordinary differential equations (boundary value problems, specifically). Also, Taiwo and Evans [8] came up with collocation approximation for fourth order boundary value problem in which canonical polynomials are used as new basis function for a collocation solution via the perturbed collocation with and without exponential fitting.

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The use of Newton's linearization scheme of order four for non-linear cases is employed to linearize.

In the same vein, El-Gamel [9] used Sinc-Galerkin method, Shaowei [10] used homotopy perturbation method, Ghazala *et al.* [11] came up with solution of boundary value problems in reproducing kernel space and also, Ayyaz *et al.* [12] used Hermite wavelet method. All these methods are applied on fifth order ordinary differential equations (boundary value problems, specifically).

Researchers [13-19] solved order six problems with different methods reported therein.

Wang *et al.* [20] developed a scheme known as La-DQM which means Local Adaptive Differential Quadrature Method for the implementation of multi-boundary condition in solving high order ordinary differential equation. Both the formulation for weighting coefficients and the details of implementing multi-boundary conditions were presented.

Malek *et al.* [21] developed a method known as Novel Hybrid Method which was based on optimization techniques and neural networks methods for the solution of high order ordinary differential equations. In this method, neural network was considered as part of large field called neural or soft computing. This means that they proposed a new solution method for the approximated solution of high order ordinary differential equations using innovative mathematical tools and neural-like systems of computing.

In this work, we employed the use of integrated collocation approximation method to solve high order ordinary differential equation, showing its potency, efficacy and efficiency over other methods stated earlier using power series and Chebyshev polynomials as the bases functions.

General Representation

The general n th order ordinary differential equation considered is expressed as

$$\alpha_n y^n(x) + \alpha_1 y^{n-1}(x) + \alpha_2 y^{n-2}(x) + \dots + \alpha_n y(x) = f(x) \quad (1)$$

subject to the conditions

$$y(a) = z_0, y'(a) = z_1, y''(a) = z_2, \dots, y^{n-1}(a) = z_{n-1}$$

Where n is the order of the differential equation. The problem may either be homogenous or not depending on the formulation of the problem. Function f is of the form of exponential, trigonometric or even ordinary polynomials.

Discussion Of Integrated Approximation Method On General Problem Considered

Chebyshev Polynomials Method

In this method, we assumed an n th order approximate solution in term of Chebyshev polynomial function as

$$y_N^n(x) = \sum_{i=0}^N a_i T_i(x); \quad a \leq x \leq b \quad (2)$$

Here, equation (2) is integrated n times and we obtained

$$y_N^{n-1}(x) = \sum_{i=0}^N [a_i T_i(x) dx + c_1]$$

$$y_N^{n-2}(x) = \sum_{i=0}^N \int [a_i T_i(x) dx + c_1 x + c_2]$$

$$y_N(x) = \sum_{i=0}^N \int \dots \int a_i T_i(x) dx dx \dots dx_n + \frac{c_1 x^n}{n!} + \frac{c_2 x^{n-1}}{(n-1)!} + \dots + c_n$$

Thus, substituting all these into the given problem leads to

$$\alpha_n \sum_{i=0}^N a_i T_i(x) + \alpha_1 \sum_{i=0}^N [a_i T_i(x) dx + c_1] + \alpha_2 \sum_{i=0}^N \int [a_i T_i(x) dx dx + c_1 x + c_2] + \alpha_n \sum_{i=0}^N \int \dots \int a_i T_i(x) dx dx \dots dx_n + \frac{c_1 x^n}{n!} + \frac{c_2 x^{n-1}}{(n-1)!} + \dots + c_n = f(x)$$

The above equation is then simplified to get

$$\begin{aligned} & a_n (T_n(x) \alpha_n + \alpha_1 \int T_n(x) dx + \alpha_2 \int \int T_n(x) dx dx \dots + \alpha_n \int \dots \int T_n(x) dx dx \dots dx_n) + \\ & a_1 (T_1(x) \alpha_n + \alpha_1 \int T_1(x) dx + \alpha_2 \int \int T_1(x) dx dx \dots + \alpha_n \int \dots \int T_1(x) dx dx \dots dx_n) + \\ & + a_i (\alpha_n T_i(x) + \alpha_1 \int T_i(x) dx + c_1 + \alpha_2 \int \int T_i(x) dx dx + \dots + \alpha_n \int \dots \int T_i(x) dx dx \dots dx_n) \\ & + c_1 (1 + x + \frac{1}{2} x^2 + \dots + \frac{1}{n!} x^n) + c_2 (1 + x + \frac{1}{2} x^2 + \dots + \frac{x^{n-1}}{(n-1)!}) + \dots + c_n = f(x) \end{aligned} \quad (3)$$

Equation (3) is then collocated at point $x = x_i$, we obtain

$$\begin{aligned} & a_n (T_n(x_i) \alpha_n + \alpha_1 \int T_n(x_i) dx + \alpha_2 \int \int T_n(x_i) dx dx \dots + \alpha_n \int \dots \int T_n(x_i) dx dx \dots dx_n) + \\ & a_1 (T_1(x_i) \alpha_n + \alpha_1 \int T_1(x_i) dx + \alpha_2 \int \int T_1(x_i) dx dx \dots + \alpha_n \int \dots \int T_1(x_i) dx dx \dots dx_n) + \\ & + a_i (\alpha_n T_i(x_i) + \alpha_1 \int T_i(x_i) dx + c_1 + \alpha_2 \int \int T_i(x_i) dx dx + \dots + \alpha_n \int \dots \int T_i(x_i) dx dx \dots dx_n) \\ & + c_1 (1 + x_i + \frac{1}{2} x_i^2 + \dots + \frac{1}{n!} x_i^n) + c_2 (1 + x_i + \frac{1}{2} x_i^2 + \dots + \frac{x_i^{n-1}}{(n-1)!}) + \dots + c_n = f(x_i) \end{aligned} \quad (4)$$

where $x_i = a + \frac{(b-a)i}{N}$; $i = 1, 2, \dots, N-1$

Hence, equation (4) gives rise to $(N-1)$ algebraic system equations in $(N+1)$ unknown constants. Extra equations are then obtained from the given conditions and a total of $(N+1)$ algebraic linear equations are then obtained which are then solved by Mathematica software to obtain the values of the unknown constants

Demonstration Of Chebyshev Polynomials Approximate Solution On Some Numerical Examples

Example 1: Consider the second order ODE of the form

$$y'' + 4y' + 8y = 0, \quad y(0) = 2, y'(0) = 2 \quad [\text{Exact: } e^{-2x}(2\cos 2x + 2\sin 2x)] \quad (5)$$

Using Chebyshev Approximation given in eqn (2) for case N=2, we have

$$y_2''(x) = a_2 T_0(x) + a_1 T_1(x) + a_2 T_2(x) \quad (6)$$

Also, integrating eqn (6) to obtain $y_2'(x)$ and $y_2(x)$ leads to

$$y_2'(x) = \int a_2 T_0(x) dx + \int a_1 T_1(x) dx + \int a_2 T_2(x) dx + c_1 \quad (7)$$

And

$$y_2(x) = \int \int a_2 T_0(x) dx dx + \int \int a_1 T_1(x) dx dx + \int \int a_2 T_2(x) dx dx + c_1 x + c_2 \quad (8)$$

Equations (6), (7) and (8) are then simplified to give

$$y_2''(x) = a_2 + a_1(2x-1) + a_2(8x^2-8x+1) \quad (9)$$

$$y_2'(x) = a_2 x + a_1(x^2 - x) + a_2\left(\frac{8}{3}x^3 - 4x^2 + x\right) + c_1 \quad (10)$$

$$y_2(x) = \frac{1}{2}a_2 x^2 + a_1\left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right) + a_2\left(\frac{2}{3}x^4 - \frac{4}{3}x^3 + \frac{1}{2}x^2\right) + c_1 x + c_2 \quad (11)$$

Putting these equations in the given example leads to

$$\begin{aligned} [a_2 + a_1(2x-1) + a_2(8x^2-8x+1)] &= 4[a_2 x + a_1(x^2 - x) + a_2\left(\frac{8}{3}x^3 - 4x^2 + x\right) + c_1] \\ &+ 8\left[\frac{1}{2}a_2 x^2 + a_1\left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right) + a_2\left(\frac{2}{3}x^4 - \frac{4}{3}x^3 + \frac{1}{2}x^2\right) + c_1 x + c_2\right] \\ &a[1 + 4x + 4x^2] + a_1[2x - 1 + 4(x^2 - x) + 8\left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right)] + a_2[8x^2 - 8x + 1 + 4\left(\frac{8}{3}x^3 - 4x^2 + x\right) \\ &- 8\left(\frac{2}{3}x^4 - \frac{4}{3}x^3 + \frac{1}{2}x^2\right)] + c_1[4 + 8x] + 8c_2 = 0 \end{aligned} \quad (12)$$

Collocating this equation at $x = x_k$; $k=1,2,3$

For $k=1$, $x_1 = \frac{1}{4}$, hence after simplification of eqn (12) gives

$$2.250000a_2 - 1.458330a_1 - 0.229167a_2 + 6.000000c_1 + 8.000000c_2 = 0 \quad (13)$$

For $k=2$, $x_2 = \frac{1}{2}$, hence after simplification of eqn (12) gives

$$4.000000a_2 - 1.666670a_1 - 1.666670a_2 + 8.000000c_1 + 8.000000c_2 = 0 \quad (14)$$

For $k=3$, $x_3 = \frac{3}{4}$, hence after simplification of eqn (12) gives

$$6.250000a_2 - 1.375000a_1 - 2.562500a_2 + 10.000000c_1 + 8c_2 = 0 \quad (15)$$

Then applying the conditions given in the example, we have

$$y_2(0) = 2$$

Implies

$$c_1 = 2 \quad (16)$$

And,

$$y_2'(0) = 2$$

Implies

$$c_2 = 2 \quad (17)$$

Solving equations (13), (14) and (15) after the substitution of equations (16) and (17) by Gaussian elimination method, we obtained

$$a_2 = -5.509770$$

$$a_1 = 11.579700$$

$$a_2 = -5.603200$$

Putting these values in eqn (11), after simplification, we obtained

$$y_2(x) = 2 + 2x - \frac{5.509770}{2}x^2 + 11.579700\left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right) - 5.603200\left(\frac{2}{3}x^4 - \frac{4}{3}x^3 + \frac{1}{2}x^2\right)$$

Which is the approximate solution for case N=2

Example:

We again consider the following high order ODE

$$2. \quad y'''' + y'' = 0; \quad y(0) = y'(0) = y''(0) = 0, y'''(0) = 1$$

$$3. \quad y'''' + 2y'' + y = 0; \quad y(0) = 1, y''(0) = 2, y'''(0) = -3$$

Remark

We have solved the above examples using the same method described for equation (5) for cases N (i.e., N=3, N=4, N=5 and N=6) and the results obtained are below

Power Series Integrated Approximation Method

In this method, we assumed an nth order approximate solution in term of Power Series function as

$$y_N''(x) = \sum_{n=0}^N a_n x^n; \quad a \leq x \leq b \quad (18)$$

Here, equation (18) is integrated n times, we obtain

$$\begin{aligned} y_N^{n-1}(x) &= \sum_{n=0}^N \int a_n x^n dx + c_1 \\ y_N^{n-2}(x) &= \sum_{n=0}^N \int \int a_n x^n dx dx + c_1 x + c_2 \\ y_N(x) &= \sum_{n=0}^N \int \int \int a_n x^n dx dx dx + \frac{c_1 x^2}{2!} + \frac{c_2 x^{n-1}}{(n-1)!} + \dots + c_n \end{aligned}$$

Thus, substituting all these into the given problem leads to

$$\begin{aligned} &\alpha_n \sum_{n=0}^N a_n x^n + \alpha_1 \sum_{n=0}^N \int a_n x^n dx + c_1 + \alpha_2 \sum_{n=0}^N \int \int a_n x^n dx dx + c_1 x + c_2 \\ &+ \alpha_n \sum_{n=0}^N \int \int \int a_n x^n dx dx dx + \frac{c_1 x^2}{2!} + \frac{c_2 x^{n-1}}{(n-1)!} + \dots + c_n = f(x) \end{aligned}$$

The above equation is then simplified to get

$$\begin{aligned} &a_1(\alpha_n + \alpha_1 \int dx + \alpha_2 \int \int dx dx + \dots + \alpha_n \int \int \int dx dx dx \dots dx_n) \\ &+ a_1(\alpha_n x + \alpha_1 \int x dx + \alpha_2 \int \int x dx dx + \dots + \alpha_n \int \int \int x dx dx dx \dots dx_n) \\ &+ a_n(\alpha_n x^n + \alpha_1 \int x^n dx + \alpha_2 \int \int x^n dx dx + \dots + \alpha_n \int \int \int x^n dx dx dx \dots dx_n) \\ &+ c_1(1+x+\frac{1}{2}x^2+\dots+\frac{x^n}{n!}) + c_2(1+x+\frac{1}{2}x^2+\dots+\frac{x^{n-1}}{(n-1)!})+\dots+c_n = f(x) \end{aligned} \quad (19)$$

Equation (19) is then collocated at point $x = x_i$, we obtain

$$\begin{aligned} &a_1(\alpha_n + \alpha_1 \int dx + \alpha_2 \int \int dx dx + \dots + \alpha_n \int \int \int dx dx dx \dots dx_n) + \\ &a_1(\alpha_n x_i + \alpha_1 \int x_i dx + \alpha_2 \int \int x_i dx dx + \dots + \alpha_n \int \int \int x_i dx dx dx \dots dx_n) \\ &+ a_n(\alpha_n x_i^n + \alpha_1 \int x_i^n dx + \alpha_2 \int \int x_i^n dx dx + \dots + \alpha_n \int \int \int x_i^n dx dx dx \dots dx_n) \\ &+ c_1(1+x_i+\frac{1}{2}x_i^2+\dots+\frac{x_i^n}{n!}) + c_2(1+x_i+\frac{1}{2}x_i^2+\dots+\frac{x_i^{n-1}}{(n-1)!})+\dots+c_n = f(x_i) \end{aligned} \quad (20)$$

where $x_i = a + \frac{(b-a)i}{N}$; $i = 1, 2, \dots, N-1$

Hence, equation (20) gives rise to (N-1) algebraic system equations in (N+1) unknown constants. Extra equations are then obtained from the given conditions and a total of (N+1) algebraic linear equations are then obtained which are then solved by Mathematica software to obtain the values of the unknown constants

Demonstration Of Power Series Approximate Solution On Some Numerical Examples

Example 1:

Consider the second order ODE of the form

$$y'' + 4y' + 8y = 0, \quad y(0) = 2, y'(0) = 2 \quad [\text{Exact: } e^{-2x}(2\cos 2x + 2\sin 2x)] \quad (21)$$

Using Power Series Approximation given in eqn (18) for case N=2, we have

$$y_2''(x) = a_0 + a_1 x + a_2 x^2 \quad (22)$$

Also, integrating eqn (22) to obtain $y_2'(x)$ and $y_2(x)$ leads to

$$y_2'(x) = \int a_0 dx + \int a_1 x dx + \int a_2 x^2 dx + c_1 \quad (23)$$

And

$$y_2(x) = \int \int a_0 dx dx + \int \int a_1 x dx dx + \int \int a_2 x^2 dx dx + c_1 x + c_2 \quad (24)$$

Equations (22), (23) and (24) are then simplified to give

$$y_2''(x) = a_0 + a_1 x + a_2 x^2 \quad (25)$$

$$y_2'(x) = a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3 + c_1 \quad (26)$$

$$y_2(x) = \frac{1}{2} a_0 x^2 + \frac{1}{6} a_1 x^3 + \frac{1}{12} a_2 x^4 + c_1 x + c_2 \quad (27)$$

Putting these equations in the given example leads to

$$[a_0 + a_1 x + a_2 x^2] + 4[a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3 + c_1] + 8[\frac{1}{2} a_0 x^2 + \frac{1}{6} a_1 x^3 + \frac{1}{12} a_2 x^4 + c_1 x + c_2] = 0$$

Hence, we have

$$a_0(1+4x+4x^2) + a_1(x+2x^2+\frac{4}{3}x^3) + a_2(x^2+\frac{4}{3}x^3+\frac{2}{3}x^4) + c_1(4+8x)+8c_2 = 0 \quad (28)$$

Collocating this equation at $x = x_k$; $k=1, 2, 3$

For $k=1$, $x_1 = \frac{1}{4}$, hence after simplification of eqn (28) gives

$$2.250000a_0 + 0.395833a_1 + 0.085938a_2 + 6.000000a_3 + 8.000000a_4 = 0$$

For $k=2$, $x_2 = \frac{1}{2}$, hence after simplification of eqn (28) gives

$$4.000000a_0 + 1.166670a_1 + 0.458333a_2 + 8.000000a_3 + 8.000000a_4 = 0$$

For $k=3$, $x_3 = \frac{3}{4}$, hence after simplification of eqn (28) gives

$$6.250000a_0 + 2.437500a_1 + 1.335940a_2 + 10.000000a_3 + 8.000000a_4 = 0$$

Then applying the conditions given in the example, we have

$$y_2(0) = 2$$

Implies

$$c_1 = 2$$

And,

$$y_2'(0) = 2$$

Implies

$$c_2 = 2$$

Solving equations (29), (30) and (31) after the substitution of equations (32) and (33) by Gaussian elimination method, we obtained

$$a_0 = -22.692200$$

$$a_1 = 67.981200$$

$$a_2 = -44.820900$$

Putting these values in eqn (27), after simplification, we obtained

$$y_2(x) = 2 + 2x - \frac{22.692200}{2}x^2 + \frac{67.9812}{6}x^3 - \frac{44.820900}{12}x^4$$

Which is the approximate solution for case $N=2$

Example:

We again consider the following high order ODE

$$2. \quad y'''' + y'' = 0; \quad y(0) = y'(0) = y''(0) = 0, \quad y'''(0) = 1$$

$$3. \quad y'''' + 2y'' + y = 0; \quad y(0) = 1, \quad y'(0) = 2, \quad y'''(0) = -3$$

Remark

We have solved the above examples using the same method described for equation (21) for cases N (i.e., $N=3$, $N=4$, $N=5$ and $N=6$) and the results obtained are below:

Table of Errors

Table 1: Table of Errors For Example 1 For Case $N=2$

x	Error of Chebyshev	Error of Power Series
0.0	0	0
0.1	4.70 E-03	4.71 E-03
0.2	1.29 E-02	1.29 E-02
0.3	1.89 E-02	1.89 E-02
0.4	2.10 E-02	2.11 E-02
0.5	2.01 E-02	2.01 E-02
0.6	1.82 E-02	1.82 E-02
0.7	1.70 E-02	1.70 E-02
0.8	1.60 E-02	1.60 E-02
0.9	1.09 E-02	1.09 E-02
1.0	7.52 E-03	7.51 E-03

Table 3: Table of Errors For Example 1 For Case $N=4$

x	Error of Chebyshev	Error of Power Series
0.0	0	0
0.1	4.90 E-04	5.60 E-04
0.2	1.10 E-03	1.25 E-03
0.3	1.40 E-03	1.60 E-03
0.4	1.51 E-03	1.72 E-03
0.5	1.49 E-03	1.71 E-03
0.6	1.37 E-03	1.61 E-03
0.7	1.20 E-03	1.45 E-03
0.8	1.02 E-03	1.25 E-03
0.9	8.48 E-04	1.04 E-03
1.0	3.22 E-04	5.77 E-04

(29)

(30)

(31)

(32)

(33)

Table 2: Table of Errors For Example 1 For Case $N=3$

x	Error of Chebyshev	Error of Power Series
0.0	0	0
0.1	6.80 E-04	6.80 E-04
0.2	1.59 E-03	1.59 E-03
0.3	2.11 E-03	2.11 E-03
0.4	2.30 E-03	2.31 E-03
0.5	2.27 E-03	2.28 E-03
0.6	2.10 E-03	2.11 E-03
0.7	1.86 E-03	1.87 E-03
0.8	1.59 E-03	1.59 E-03
0.9	1.28 E-03	1.29 E-03
1.0	5.60 E-04	5.90 E-04

Table 4: Table of Errors For Example 1 For Case $N=5$

x	Error of Chebyshev	Error of Power Series
0.0	0	0
0.1	1.20 E-03	9.00 E-05
0.2	2.58 E-03	2.00 E-04
0.3	3.09 E-03	2.40 E-04
0.4	2.96 E-03	2.60 E-04
0.5	2.64 E-03	2.50 E-04
0.6	2.27 E-03	2.30 E-04
0.7	1.86 E-03	2.07 E-04
0.8	1.46 E-03	1.75 E-04
0.9	1.15 E-03	1.35 E-04
1.0	2.65 E-04	1.88 E-04

Table 5: Table of Errors For Example 1 For Case N=0

x	Error of Chebyshev	Error of Power Series
0.0	0	0
0.1	0	1.00 E-05
0.2	3.00 E-04	1.00 E-05
0.3	6.00 E-04	2.00 E-05
0.4	7.40 E-04	1.00 E-05
0.5	8.00 E-04	2.00 E-05
0.6	7.70 E-04	2.00 E-05
0.7	6.70 E-04	9.00 E-06
0.8	5.48 E-04	7.00 E-06
0.9	3.13 E-04	5.00 E-06
1.0	4.41 E-04	1.70 E-05

Table 6: Table of Errors For Example 2 For Case N=2

x	Error of Chebyshev	Error of Power Series
0.0	0	0
0.1	0	0
0.2	0	0
0.3	4.00 E-06	4.00 E-06
0.4	8.00 E-06	8.00 E-06
0.5	1.80 E-05	1.80 E-05
0.6	3.30 E-05	3.30 E-05
0.7	5.60 E-05	5.60 E-05
0.8	8.50 E-05	8.50 E-05
0.9	1.22 E-04	1.22 E-04
1.0	1.70 E-04	1.70 E-04

Table 7: Table of Errors For Example 2 For Case N=3

x	Error of Chebyshev	Error of Power Series
0.0	0	0
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	1.00 E-06	1.00 E-06
0.7	2.00 E-06	2.00 E-06
0.8	3.00 E-06	3.00 E-06
0.9	4.00 E-06	4.00 E-06
1.0	6.00 E-06	6.00 E-06

Table 8: Table of Errors For Example 2 For Case N=4

x	Error of Chebyshev	Error of Power Series
0.0	0	0
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	1.00 E-06	1.00 E-06
0.6	1.00 E-06	1.00 E-06
0.7	0	0
0.8	0	1.00 E-06
0.9	1.00 E-06	1.00 E-06

Table 9: Table of Errors For Example 2 For Case N=5

x	Error of Chebyshev	Error of Power Series
0.0	0	0
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	1.00 E-06	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0
1.0	0	0

Table 10: Table of Errors For Example 2 For Case N=0

x	Error of Chebyshev	Error of Power Series
0.0	0	0
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	1.00 E-06	0
0.8	0	0
0.9	1.00 E-06	0
1.0	2.00 E-06	0

Table 11: Table of Errors For Example 3 For Case N=2

x	Error of Chebyshev	Error of Power Series
0.0	0	0
0.1	1.00 E-06	1.00 E-06
0.2	7.00 E-06	7.00 E-06
0.3	3.20 E-05	3.20 E-05
0.4	8.70 E-05	8.70 E-05
0.5	1.84 E-04	1.84 E-04
0.6	3.35 E-04	3.35 E-04
0.7	5.48 E-04	5.48 E-04
0.8	8.30 E-04	8.30 E-04
0.9	1.19 E-03	1.19 E-03
1.0	1.64 E-03	1.64 E-03

Table 12: Table of Errors For Example 3 For Case N=3

x	Error of Chebyshev	Error of Power Series
0.0	0	0
0.1	0	0
0.2	0	0
0.3	1.00 E-06	5.00 E-06
0.4	3.00 E-06	3.20 E-06
0.5	6.00 E-05	1.21 E-04
0.6	1.10 E-05	3.56 E-04
0.7	1.70 E-05	8.84 E-04
0.8	3.00 E-05	1.94 E-03
0.9	4.00 E-05	3.86 E-03
1.0	5.00 E-05	7.17 E-03

Table 13: Table of Errors For Example 3 For Case N=4

x	Error of Chebyshev	Error of Power Series
0.0	0	0
0.1	0	0
0.2	0	0
0.3	0	6.00 E-06
0.4	1.00 E-06	3.20 E-05
0.5	3.00 E-06	1.18 E-04
0.6	1.10 E-05	3.45 E-03
0.7	2.90 E-05	5.73 E-03
0.8	7.00 E-05	1.85 E-03
0.9	1.30 E-04	3.65 E-03
1.0	2.60 E-04	6.69 E-03

Table 14: Table of Errors For Example 3 For Case N=5

x	Error of Chebyshev	Error of Power Series
0.0	0	0
0.1	0	0
0.2	0	0
0.3	0	6.00 E-06
0.4	2.00 E-06	3.20 E-05
0.5	4.00 E-06	1.18 E-04
0.6	1.30 E-05	3.45 E-03
0.7	3.10 E-05	5.73 E-03
0.8	7.00 E-05	1.85 E-03
0.9	1.30 E-04	3.65 E-03
1.0	2.40 E-04	6.69 E-03

Table 15: Table of Errors For Example 3 For Case N=6

x	Error of Chebyshev	Error of Power Series
0.0	0	0
0.1	0	0
0.2	0	0
0.3	0	3.00 E-06
0.4	1.00 E-06	2.10 E-05
0.5	5.00 E-06	9.90 E-05
0.6	1.40 E-05	3.54 E-04
0.7	3.50 E-05	1.04 E-03
0.8	8.00 E-05	2.65 E-03
0.9	1.60 E-04	6.04 E-03
1.0	3.00 E-04	1.26 E-02

Graph of Comparison

This section contains the solution curves of Examples considered using the Chebyshev Approximation Method and the Power Series Approximation Method for values of $N = 2, 3, 4, 5, 6$.

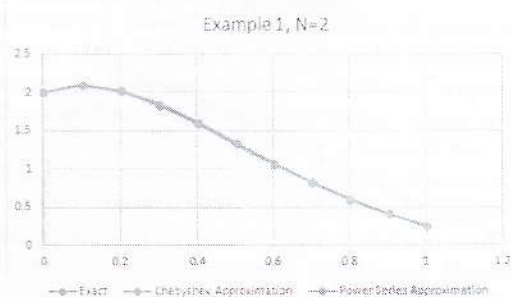


Fig 1: Comparison Graph of Example 1 for N = 2.

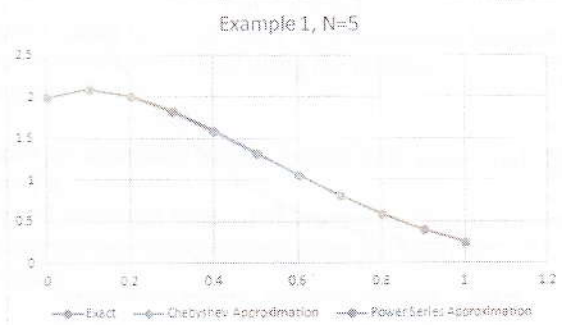


Fig 4: Comparison Graph of Example 1 for N = 5.

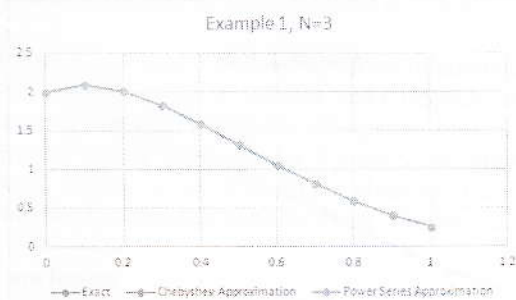


Fig 2: Comparison Graph of Example 1 for N = 3.

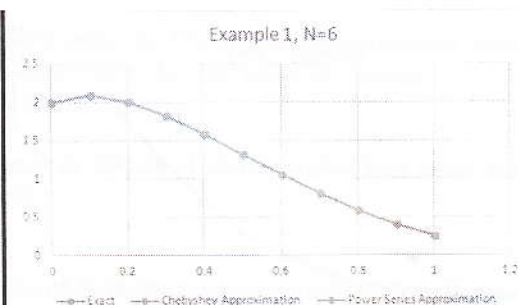


Fig 5: Comparison Graph of Example 1 for N = 6.

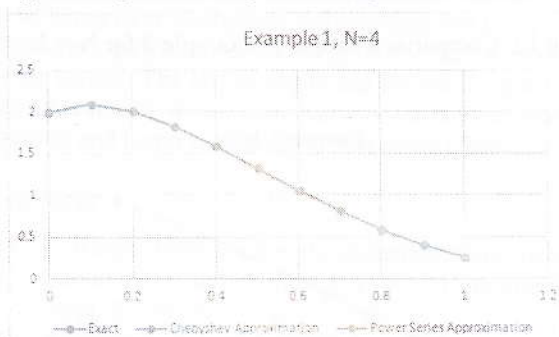


Fig 3: Comparison Graph of Example 1 for N = 4

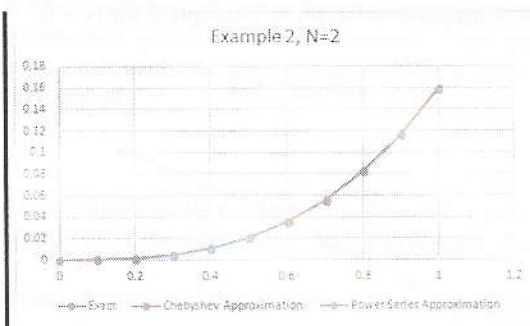


Fig 6: Comparison Graph of Example 2 for N = 2.

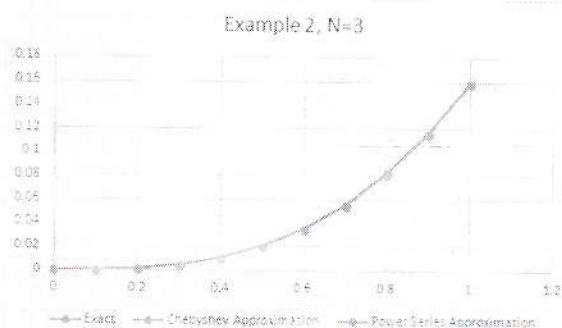


Fig 7: Comparison Graph of Example 2 for N = 3.

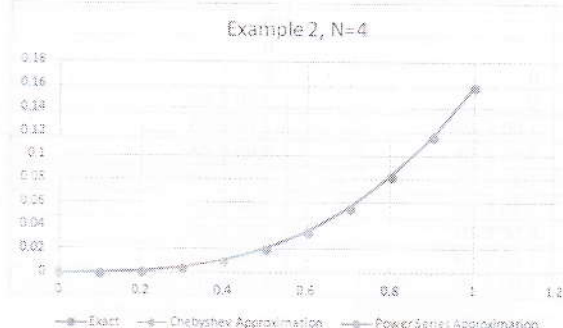


Fig 8: Comparison Graph of Example 2 for N = 4.

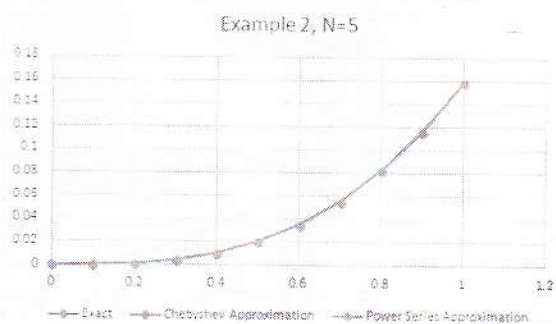


Fig 9: Comparison Graph of Example 2 for N = 4.

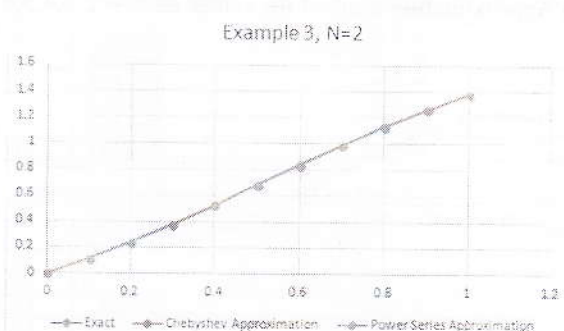


Fig 11: Comparison Graph of Example 3 for N = 2.

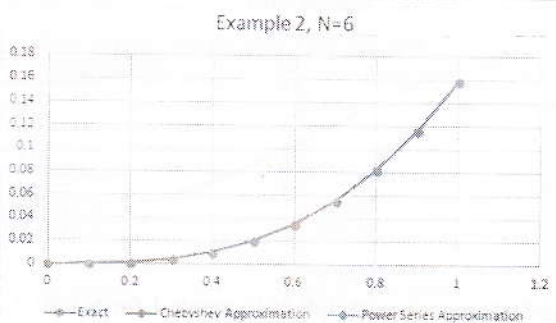


Fig 10: Comparison Graph of Example 2 for N = 6.

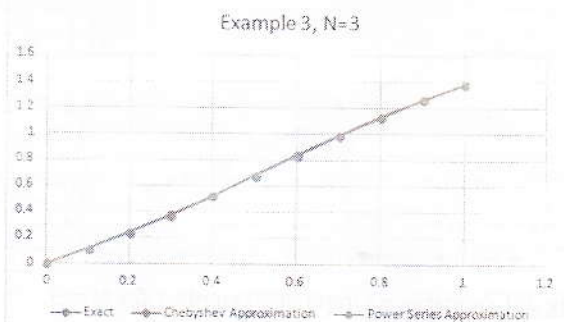


Fig 12: Comparison Graph of Example 3 for N = 3.

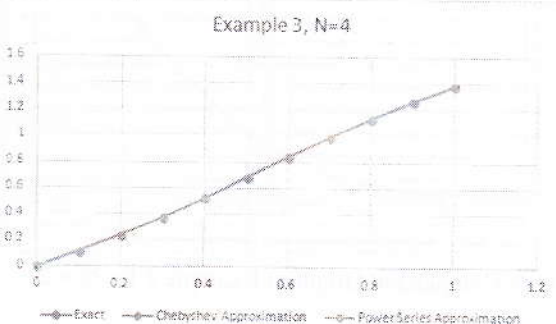


Fig 13: Comparison Graph of Example 3 for N = 4.

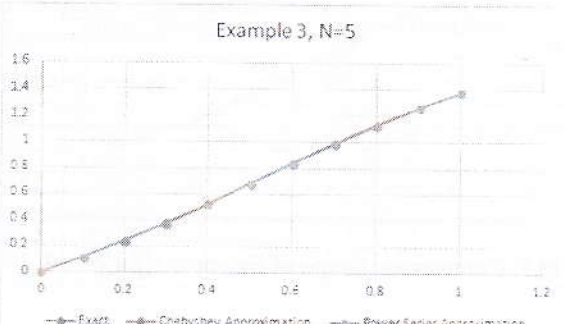


Fig 14: Comparison Graph of Example 3 for N = 5.

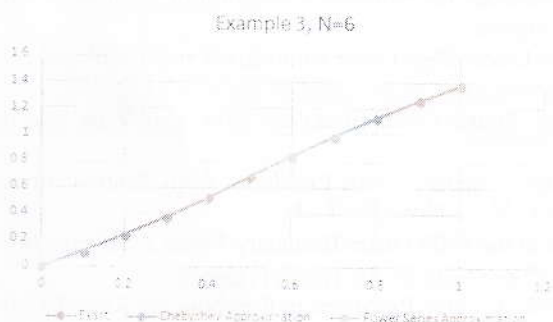


Fig 15: Comparison Graph of Example 3 for $N = 6$.

Discussion of Results

Tables 1-5 show the absolute errors of computation for Example 1 using the Chebyshev Approximation Method and Power Series Method with $N = 2, 3, 4, 5$, and 6 respectively. The absolute errors were obtained using the relation:

Error of Chebyshev = |Exact- Chebyshev Approximation Method| and

Error of Power Series = |Exact- Power Series Method|;

From Tables 1-5, it could be observed that these two methods as discussed above exhibit less noise and thus we say compared favourably with the exact solutions at different values of x .

Tables 6-10 show the absolute errors of computation for Example 2 using the Chebyshev Approximation Method and Power Series Method with $N = 2, 3, 4, 5$, and 6 respectively. The absolute errors were also obtained using the relation:

Error of Chebyshev = |Exact- Chebyshev Approximation Method| and

Error of Power Series = |Exact- Power Series Method|;

From Tables 6-10, it could be observed that these two methods as discussed above exhibit less noise and thus we say compared favourably with the exact solutions at different values of x .

Tables 11-15 show the absolute errors of computation for Example 3 using the Chebyshev Approximation Method and Power Series Method with $N = 2, 3, 4, 5$, and 6 respectively. The absolute errors were obtained using the relation:

Error of Chebyshev = |Exact- Chebyshev Approximation Method| and

Error of Power Series = |Exact- Power Series Method|;

From Tables 11-15, it could be observed that these two methods as discussed above exhibit less noise and thus we say compared favourably with the exact solutions at different values of x .

Conclusion

In the table of results presented, we observed that at lower values of N in the examples considered, both methods have the same degree of accuracy. As N increases, better results are achieved. In the first example considered, power series has proven to be better over Chebyshev polynomials due to its proximity to the exact solution. In the second example, it is observed that Chebyshev polynomials converge faster than the power series at lower values of N but at higher values, power series gives better results. The last example has shown the potency of Chebyshev over power series as its rate of convergence is higher than that power series. Conclusively, as the value of N increases in both methods, the answers converge more to the exact solution and better results are gotten using Chebyshev polynomials method.

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