

Unsteady Non-Newtonian Magnetohydrodynamic Radiative Thin Film Flow in a Porous Medium Past an Oscillating Vertical Belt

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Abstract

This study examines the unsteady non-Newtonian magnetohydrodynamic radiative thin film flow in a porous medium past an oscillating vertical belt. Based on the physical model of the problem, the governing momentum and energy equations are non dimensionalized and solved analytically using Optimal Homotopy Asymptotic Method (OHAM). The velocity and temperature profiles are studied for different physical parameters such as the magnetic field, radiation parameter, Grashof number, Prandtl number, and Brinkman number for both lift and drainage problems. The analytical results are presented graphically and discussed.

Keywords: OHAM, Thin film, Unsteady, Second grade fluid, Buoyancy, Viscous dissipation.

Introduction

The study of non-Newtonian fluids of thin film flow has aroused the interest of researchers and received considerable attention because of its numerous applications in various branches of science, engineering and technology: particularly in chemical and nuclear industries, material processing, geophysics and bio-engineering. The importance of non-Newtonian fluid flow especially in a porous medium has led to the development of mathematical models in an extensive range. Its areas of application include petrochemical, bio-medical and environmental engineering such as polypropylene coalescence sintering, blood, geological and debris flows.

Literature Review

Several researchers have discussed the flows of second grade fluid in different dimensions, Amir and Gul (2015), studied unsteady magnetohydrodynamic flow of second grade fluid due to impulsive motion of plate. Ali, *et al.* (2014) investigated the closed form solution for unsteady second grade fluid near vertical oscillating belt. Samiulhaq, *et al.* (2014) studied unsteady free convection flow of a second grade fluid in a porous medium with ramped wall temperature. Kamran, *et al.* (2012) investigated the thin-film flow of Johnson-Segalman fluids for lifting and drainage problems.

Due to the non-linearity of the equations governing the flow, several approximate methods have been developed in Mathematics and Engineering sciences to solve the equations. Some of the common methods are Homotopy Analysis Method (HAM) by Liao, (2003) and Optimal Homotopy Asymptotic Method (OHAM). These methods deal with the nonlinear problems effectively. OHAM was built on convergence criteria similar to HAM but it differs from it in that, it does not require an initial guess as successfully applied by Marinca, *et al.* (2008) to problems in mechanics and has also showed its effectiveness and accuracy. Gul *et al.* (2014), used ADM and OHAM for solution of heat transfer analysis of magnetohydrodynamic (MHD) thin film flow of an unsteady second grade fluid past a vertical oscillating belt. An application of OHAM for solving non-linear equations arising in heat transfer was investigated by Marinca, *et al.* (2009) who applied OHAM to solve steady flow of a fourth-grade fluid past a porous plate. Gul, *et al.* (2013) investigated MHD thin film flow of a third grade fluid on a vertical belt with slip boundary conditions. Mabood *et al.* (2013) used OHAM for flow and heat transfer of a viscoelastic fluid in an axisymmetric channel with a porous wall.

The above studies neglected the combined effects of radiation, buoyancy force and porous medium on dissipative fluids. In the light of this, the interaction of radiation with heat transfer of a MHD thin film flow in a porous medium past an oscillating vertical belt is presented in this study. The governing momentum equation includes porosity and buoyancy force while the effect of radiation is taken into account in the energy equation.

Methodology

The problem of unsteady non-Newtonian magnetohydrodynamic radiative thin film flow of a second grade fluid in a porous medium past an oscillating vertical belt is considered. It is assumed that the flow is unsteady and laminar after a small distance above the liquid surface layer and the film thickness remains constant for both lift and drainage problems. The flow is only in y-direction (vertical flow) in the presence of magnetic field and heat dissipation and the problem is fully developed laminar flow. By using the Rosseland approximation, the radiative heat flux q_r for the given problem is

$$q_r = -\frac{4\sigma_s}{3k_e} \frac{\partial T'^4}{\partial x} \quad (1)$$

and the flow is governed by the following equations.

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$0 \quad (2)$$

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 u}{\partial x^2} + \alpha_1 \frac{\partial}{\partial t} \left(\frac{\partial^2 u}{\partial x^2} \right) - \rho g \beta_T (T - T_\delta) - \frac{\mu u}{k} - \sigma B_0^2 u \quad (3)$$

Energy equation:

$$\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + \mu \left(\frac{\partial u}{\partial x} \right)^2 - \frac{\partial q_r}{\partial x} + \alpha_1 \left(\frac{\partial u}{\partial x} \right) \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} \right) \quad (4)$$

In the formulation of the lift problem in this second grade fluid flow, a wide flat belt is considered to move vertically at time $t = 0^+$ and a layer of liquid of constant thickness is been carried by the belt. The belt oscillated and translated with constant speed u_o in the presence of

magnetic field which is placed transverse to the belt. Heat is applied to the fluid and atmospheric pressure is negligible. The y-axis is taken parallel to the belt while x-axis is perpendicular to it for the coordinate system. Velocity and temperature fields are defined as:

$$\mathbf{V} = (0, u(x, t), 0) \quad , \quad T = T(x, t) \quad (5)$$

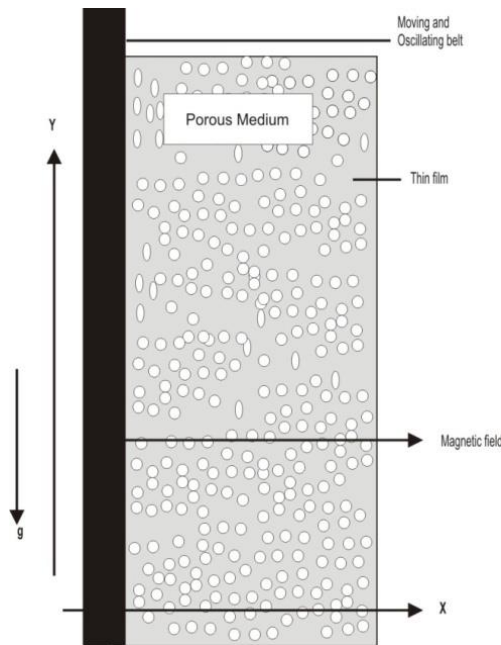


Figure 1: Geometry of the lift problem.

Oscillating boundary conditions are:

$$u(0, t) = U(1 + \xi \cos \omega t), \quad u(\delta, t) = 0 \quad (6)$$

$$T(0, t) = T_o, \quad T(\delta, t) = T_\delta \quad (7)$$

where U, ξ and δ are the flow mean velocity, amplitude and boundary layer parameter respectively. Substituting equation (5) into equation (2), continuity equation is satisfied.

In order to present the governing equations in dimensionless forms, the following non-dimensional variables are introduced.

$$\begin{aligned} \bar{u} = \frac{u}{u_o}, \quad \bar{x} = \frac{x}{\delta}, \quad \theta = \frac{(T-T_o)}{(T_\delta-T_o)}, \quad \alpha = \frac{\alpha_1}{\rho\delta^2}, \quad \frac{\partial p}{\partial y} = 0, \quad S = \frac{\delta^2}{k}, \quad M = \\ \frac{\sigma B_o^2 \delta^2}{\mu}, \quad G_r = \frac{\rho\delta^2 g \beta_T (T_\delta-T_o)}{\mu u_o}, \quad H = \left[1 + \frac{4}{3N}\right], \quad N = \frac{k_e k}{4\sigma_s T_\infty'^3}, \quad p_r = \frac{\mu c_p}{k}, \quad B_r = \\ \frac{\mu u^2}{k(T_\delta-T_o)}, \quad \bar{t} = \frac{\mu t}{\rho\delta^2} \end{aligned} \quad (8)$$

Inserting these non-dimensional variables in equations (3) and (4) and dropping bars, the momentum and energy equations becomes:

$$\begin{aligned} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial}{\partial t} \left(\frac{\partial^2 u}{\partial x^2} \right) - Gu \\ - G_r \theta \end{aligned} \quad (9)$$

$$\begin{aligned} p_r \frac{\partial \theta}{\partial t} = \left[1 + \frac{4}{3N}\right] \frac{\partial^2 \theta}{\partial x^2} + B_r \left(\frac{\partial u}{\partial x} \right)^2 \\ + B_r \alpha \left(\frac{\partial u}{\partial x} \right) \left(\frac{\partial^2 u}{\partial t \partial x} \right) \end{aligned} \quad (10)$$

The non-dimensional boundary conditions of equations (6) and (7) are:

$$\begin{aligned} u(0, t) = (1 + \xi \cos \omega t), \quad u(1, t) \\ = 0 \end{aligned} \quad (11)$$

$$\theta(0, t) = 0, \quad \theta(1, t) = 1 \quad (12)$$

In the formulation of the drainage problem, coordinate system of the geometry and assumptions of the problem are the same as in the lift problem. But in drainage problem, the belt is oscillating only. The fluid drains down the oscillating belt due to gravity and the gravity is opposing the lift. Therefore, the term with Grashof number is positive in this model as follows:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial}{\partial t} \left(\frac{\partial^2 u}{\partial x^2} \right) - Gu + G_r \theta \quad (13)$$

$$p_r \frac{\partial \theta}{\partial t} = \left[1 + \frac{4}{3N} \right] \frac{\partial^2 \theta}{\partial x^2} + B_r \left(\frac{\partial u}{\partial x} \right)^2 + B_r \alpha \left(\frac{\partial u}{\partial x} \right) \left(\frac{\partial^2 u}{\partial t \partial x} \right) \quad (14)$$

Dimensionless boundary conditions for drainage problem when the belt is only oscillating are:

$$u(0, t) = \xi \cos \omega t, \quad u(1, t) = 0 \quad (15)$$

$$\theta(0, t) = 0, \quad \theta(1, t) = 1 \quad (16)$$

By means of OHAM which is the method of solution adopted, a family of equations is given as (Marinca *et al.* 2008):

$$(1 - p)[L\phi(x, p, c_i)] + g(x) = h(x, p, c_i)[L\phi(x, p, c_i) + g(x) + N(\phi(x, p, c_i))] \quad (17)$$

where p is an embedding parameter, $g(x)$ is a known function, L is a linear operator, N is a non linear operator, $\phi(x, p, c_i)$ is an unknown function which is expanded in series form with respect to the parameter p , $h(x, p, c_i)$ is a non-zero auxiliary function chosen as

$$h(p, c_i) = pc_1 + p^2c_2 + \dots \quad (18)$$

where c_1, c_2, c_3, c_4 are called convergence control parameters that will be determined from the series solution of velocity and temperature profiles using the method of least squares.

Adopting least squares method, the option is to minimize the square residual error (R)

where residual is defined as

$$R(x, c_i) = L\phi(x, c_i) + g(x) + N\phi(x, c_i), i = 1, 2, 3, \dots \quad (19)$$

Minimizing the squared residual error, we have

$$J(c_1, c_2, \dots c_m) = \int_a^b R^2(x, c_1, c_2, \dots c_m) dx \quad (20)$$

where a and b are real numbers depending on the given problem.

The convergence parameters $c_1, c_2, \dots c_m$ can be identified from the conditions

$$\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = \frac{\partial J}{\partial c_m} = 0 \quad (21)$$

Finally, from these auxiliary constants, the series solution of velocity and temperature profiles is well-determined.

OHAM Solution of Lifting Problem

Constructing a homotopy for equations (9) and (10), using the standard form of OHAM in equation (17), the zero (p^0), first (p^1) and second (p^2) components problems are obtained respectively as:

$$\begin{aligned} p^0: & \frac{\partial^2 u_o}{\partial x^2} \\ & = 0, \end{aligned} \quad (22)$$

$$\begin{aligned} & \left[1 + \frac{4}{3N} \right] \frac{\partial^2 \theta_o}{\partial x^2} \\ & = 0, \end{aligned} \quad (23)$$

$$\begin{aligned}
 p^1: & \quad \frac{\partial^2 u_1}{\partial x^2} \\
 & = \frac{\partial^2 u_o}{\partial x^2} + c_1 \frac{\partial^2 u_o}{\partial x^2} - c_1 \frac{\partial u_o}{\partial t} + \alpha c_1 \frac{\partial}{\partial t} \left(\frac{\partial^2 u_o}{\partial x^2} \right) - c_1 S u_o - c_1 M u_o \\
 & - c_1 G_r \theta_o
 \end{aligned} \tag{24}$$

$$\begin{aligned}
 & H \frac{\partial^2 \theta_1}{\partial x^2} \\
 & = H \frac{\partial^2 \theta_o}{\partial x^2} + c_3 H \frac{\partial^2 \theta_o}{\partial x^2} - p_r c_3 \frac{\partial \theta_o}{\partial t} + c_3 B_r \left(\frac{\partial u_o}{\partial x} \right)^2 \\
 & + c_3 B_r \alpha \left(\frac{\partial u_o}{\partial x} \right) \left(\frac{\partial^2 u_o}{\partial x \partial t} \right),
 \end{aligned} \tag{25}$$

$$\begin{aligned}
 p^2: \quad \frac{\partial^2 u_2}{\partial x^2} & = \frac{\partial^2 u_1}{\partial x^2} + c_1 \frac{\partial^2 u_1}{\partial x^2} - c_1 \frac{\partial u_1}{\partial t} + c_2 \frac{\partial^2 u_o}{\partial x^2} - c_2 \frac{\partial u_o}{\partial t} \\
 & + \alpha c_1 \frac{\partial}{\partial t} \left(\frac{\partial^2 u_1}{\partial x^2} \right) - c_1 S u_1 - c_1 M u_1 - c_1 G_r \theta_1 \\
 & + \alpha c_2 \frac{\partial}{\partial t} \left(\frac{\partial^2 u_o}{\partial x^2} \right) - c_2 S u_o - c_2 M u_o \\
 & - c_2 G_r \theta_o
 \end{aligned} \tag{26}$$

$$\begin{aligned}
 & H \frac{\partial^2 \theta_2}{\partial x^2} \\
 & = H \frac{\partial^2 \theta_1}{\partial x^2} + c_3 H \frac{\partial^2 \theta_1}{\partial x^2} - p_r c_3 \frac{\partial \theta_1}{\partial t} + c_3 B_r \alpha \left(\frac{\partial u_1}{\partial x} \right) \left(\frac{\partial^2 u_1}{\partial x \partial t} \right) \\
 & + c_4 H \frac{\partial^2 \theta_o}{\partial x^2} - p_r c_4 \frac{\partial \theta_o}{\partial t} + c_4 B_r \alpha \left(\frac{\partial u_o}{\partial x} \right) \left(\frac{\partial^2 u_o}{\partial x \partial t} \right) + c_3 B_r \left(\frac{\partial u_1}{\partial x} \right)^2 \\
 & + c_3 B_r \alpha \left(\frac{\partial u_1}{\partial x} \right) \left(\frac{\partial^2 u_1}{\partial x \partial t} \right)
 \end{aligned} \tag{27}$$

By solving equations (22) - (25) subject to the boundary conditions given in equations (11) and (12), the solutions are:

$$\begin{aligned}
 & u_o[x, t] \\
 & = -(-1 + x)(1 \\
 & + \xi \text{Cos}[t\omega]),
 \end{aligned} \tag{28}$$

$$\begin{aligned}\theta_0[x, t] \\ = x,\end{aligned}\tag{29}$$

$$\begin{aligned}u_1[x, t] \\ = 1 + \xi \text{Cos}[t\omega] \\ + x \left(-1 - \xi \text{Cos}[t\omega] + c_1 \left(\frac{2M}{3} + \frac{S}{3} + \frac{2}{3} M \xi \text{Cos}[t\omega] + \frac{1}{3} S \xi \text{Cos}[t\omega] \right) \right. \\ \left. - \frac{1}{3} \xi \omega \text{Sin}[t\omega] + \frac{G_r}{6} \right) \\ + x^2 \left(-M - \frac{S}{2} - M \xi \text{Cos}[t\omega] - \frac{1}{2} S \xi \text{Cos}[t\omega] + \frac{1}{2} \xi \omega \text{Sin}[t\omega] \right) c_1 \\ + x^3 c_1 \left(\frac{M}{3} + \frac{S}{6} + \frac{1}{3} M \xi \text{Cos}[t\omega] + \frac{1}{6} S \xi \text{Cos}[t\omega] - \frac{1}{6} \xi \omega \text{Sin}[t\omega] \right. \\ \left. - \frac{G_r}{6} \right)\end{aligned}\tag{30}$$

$$\begin{aligned}\theta_1[x, t] \\ = \frac{1}{2(3 + 4N)} (6x + 8Nx - 3x B_r c_3 + 3x^2 B_r c_3 - 6x \xi \text{Cos}[t\omega] B_r c_3 \\ + 6x^2 \xi \text{Cos}[t\omega] B_r c_3 - 3x \xi^2 \text{Cos}[t\omega]^2 B_r c_3 + 3x^2 \xi^2 \text{Cos}[t\omega]^2 B_r c_3 \\ + 3x \alpha \xi \omega \text{Sin}[t\omega] B_r c_3 - 3x^2 \alpha \xi \omega \text{Sin}[t\omega] B_r c_3 \\ + 3x \alpha \xi^2 \omega \text{Cos}[t\omega] \text{Sin}[t\omega] B_r c_3 \\ - 3x^2 \alpha \xi^2 \omega \text{Cos}[t\omega] \text{Sin}[t\omega] B_r c_3)\end{aligned}\tag{31}$$

Equations (28)-(31) are substituted into eqns. (26) and (27) and solved for u_2 and θ_2 .

The series solutions of velocity and temperature for lift problem are obtained as

$$\begin{aligned}u[x, t] \\ = u_0[x, t] + u_1[x, t] \\ + u_2[x, t]\end{aligned}\tag{32}$$

$$\begin{aligned}\theta[x, t] \\ = \theta_0[x, t] + \theta_1[x, t] \\ + \theta_2[x, t]\end{aligned}\tag{33}$$

The arbitrary constants c_1, c_2, c_3 , and c_4 for both equations (32) and (33) are found by using the procedures in equations (19) - (21).

The arbitrary constants in velocity solution (equation. (32)) are obtained as

$$c_1 = -0.00028480877852541417, c_2 = -2.9977468464837203, \\ c_3 = 0.9864541962047597.$$

For temperature solution (equation. (33)), the arbitrary constants are

$$c_1 = -0.04264932656376229, c_2 = 0.15164362549144306, \\ c_3 = -0.8683837554484702, c_4 = -5.921227467947385.$$

OHAM Solution of Drainage Problem

Constructing a homotopy for equations (13) and (14), using the standard form of OHAM in equation (17), the zero p^0 , first (p^1) and second (p^2) components problems, respectively, are obtained as:

$$p^0: \frac{\partial^2 u_o}{\partial x^2} \\ = 0 \quad (34)$$

$$\left[1 + \frac{4}{3N}\right] \frac{\partial^2 \theta_o}{\partial x^2} \\ = 0 \quad (35)$$

$$p^1: \frac{\partial^2 u_1}{\partial x^2} \\ = \frac{\partial^2 u_o}{\partial x^2} + c_1 \frac{\partial^2 u_o}{\partial x^2} - c_1 \frac{\partial u_o}{\partial t} + \alpha c_1 \frac{\partial}{\partial t} \left(\frac{\partial^2 u_o}{\partial x^2} \right) - c_1 S u_o - c_1 M u_o \\ + c_1 G_r \theta_o \quad (36)$$

$$\begin{aligned}
 & H \frac{\partial^2 \theta_1}{\partial x^2} \\
 &= H \frac{\partial^2 \theta_o}{\partial x^2} + c_3 H \frac{\partial^2 \theta_o}{\partial x^2} - p_r c_3 \frac{\partial \theta_o}{\partial t} + c_3 B_r \left(\frac{\partial u_o}{\partial x} \right)^2 \\
 &+ c_3 B_r \alpha \left(\frac{\partial u_o}{\partial x} \right) \left(\frac{\partial^2 u_o}{\partial x \partial t} \right)
 \end{aligned} \tag{37}$$

$$\begin{aligned}
 p^2: \frac{\partial^2 u_2}{\partial x^2} &= \frac{\partial^2 u_1}{\partial x^2} + c_1 \frac{\partial^2 u_1}{\partial x^2} - c_1 \frac{\partial u_1}{\partial t} + c_2 \frac{\partial^2 u_o}{\partial x^2} - c_2 \frac{\partial u_o}{\partial t} \\
 &+ \alpha c_1 \frac{\partial}{\partial t} \left(\frac{\partial^2 u_1}{\partial x^2} \right) - c_1 S u_1 - c_1 M u_1 + c_1 G_r \theta_1 \\
 &+ \alpha c_2 \frac{\partial}{\partial t} \left(\frac{\partial^2 u_o}{\partial x^2} \right) - c_2 S u_o - c_2 M u_o \\
 &+ c_2 G_r \theta_o
 \end{aligned} \tag{38}$$

$$\begin{aligned}
 & H \frac{\partial^2 \theta_2}{\partial x^2} \\
 &= H \frac{\partial^2 \theta_1}{\partial x^2} + c_3 H \frac{\partial^2 \theta_1}{\partial x^2} - p_r c_3 \frac{\partial \theta_1}{\partial t} + c_3 B_r \alpha \left(\frac{\partial u_1}{\partial x} \right) \left(\frac{\partial^2 u_1}{\partial x \partial t} \right) \\
 &+ c_4 H \frac{\partial^2 \theta_o}{\partial x^2} - p_r c_4 \frac{\partial \theta_o}{\partial t} + c_4 B_r \alpha \left(\frac{\partial u_o}{\partial x} \right) \left(\frac{\partial^2 u_o}{\partial x \partial t} \right) + c_3 B_r \left(\frac{\partial u_1}{\partial x} \right)^2 \\
 &+ c_3 B_r \alpha \left(\frac{\partial u_1}{\partial x} \right) \left(\frac{\partial^2 u_1}{\partial x \partial t} \right)
 \end{aligned} \tag{39}$$

Solving equations (34)-(37) subject to the boundary conditions given in equations (15) and (16), the solutions are:

$$\begin{aligned}
 & u_0[x, t] \\
 &= -(-1 + x) \xi \text{Cos}[t\omega]
 \end{aligned} \tag{40}$$

$$\begin{aligned}
 & \theta_0[x, t] \\
 &= x
 \end{aligned} \tag{41}$$

$$\begin{aligned}
 & u_1[x, t] \\
 &= \xi \cos[t\omega] + x^2 \left(-M\xi \cos[t\omega] - \frac{1}{2}S\xi \cos[t\omega] + \frac{1}{2}\xi\omega \sin[t\omega] \right) c_1 \\
 &+ x \left(-\xi \cos[t\omega] + c_1 \left(\frac{2}{3}M\xi \cos[t\omega] + \frac{1}{3}S\xi \cos[t\omega] \right) \right. \\
 &\quad \left. - \frac{1}{3}\xi\omega \sin[t\omega] - \frac{G_r}{6} \right) \\
 &+ x^3 c_1 \left(\frac{1}{3}M\xi \cos[t\omega] + \frac{1}{6}S\xi \cos[t\omega] - \frac{1}{6}\xi\omega \sin[t\omega] \right. \\
 &\quad \left. + \frac{G_r}{6} \right) \tag{42}
 \end{aligned}$$

$$\begin{aligned}
 & \theta_1[x, t] \\
 &= \frac{1}{2(3+4N)} \left(6x + 8Nx - 3x\xi^2 \cos[t\omega]^2 B_r c_3 + 3x^2 \xi^2 \cos[t\omega]^2 B_r c_3 + \right. \\
 &\quad \left. 3x\alpha \xi^2 \omega \cos[t\omega] \sin[t\omega] B_r c_3 - 3x^2 \alpha \xi^2 \omega \cos[t\omega] \sin[t\omega] B_r c_3 \right) \tag{43}
 \end{aligned}$$

The series solutions of velocity and temperature for drainage problem are expressed respectively as:

$$\begin{aligned}
 u[x, t] &= u_0[x, t] + u_1[x, t] \\
 &\quad + u_2[x, t] \tag{44}
 \end{aligned}$$

$$\begin{aligned}
 \theta[x, t] &= \theta_0[x, t] + \theta_1[x, t] \\
 &\quad + \theta_2[x, t] \tag{45}
 \end{aligned}$$

The arbitrary constants c_1, c_2, c_3 , and c_4 for both equations (44) and (45) are found by using the procedures in equations (19) - (21).

The arbitrary constants in velocity solution (equation (44)) are obtained as

$$\begin{aligned}
 c_1 &= -0.9684126046268788, \quad c_2 = -0.9479219250666509, \\
 c_3 &= -38.323428491839174.
 \end{aligned}$$

For temperature solution (equation (44)), the calculated arbitrary constants are

$$\begin{aligned}
 c_1 &= -0.09152659686668704, \quad c_2 = 0.09581570123493369, \\
 c_3 &= -3.777664639888141, \quad c_4 = -5.352328033713726.
 \end{aligned}$$

Results and Findings

If the parameters that were introduced to the governing equations are relaxed in these problems, a plot of 3-dimensional graphs with $\alpha = 0.02$, $\xi = 0.4$, $\omega = 0.2$, $M = 0.5$, $Gr = 0.5$, $Br = 4$, $pr = 0.6$, $S = 0$, $N \rightarrow \infty$ obtained in Figures (2) - (4) are in agreement with the work of Gul *et al.* (2014).

The numerical results for velocity and temperature profile are presented with different values of radiation parameters, heat source parameter, Grashof number, Prandtl number, Hartman number and Brinkman number. The 3-dimensional graphs are produced with a fixed Porous medium shape factor (S) = 0.3, radiation parameter (N) = 2, prandtl number (p_r) = 0.6, brinkman number (B_r) = 4, Grashof number G_r = 0.5, Hartman number (M) = 0.5, frequency of oscillating belt (ω) = 0.2, amplitude of oscillation (ξ) = 0.4, and non-Newtonian effect (α) = 0.02.

The 3-dimensional lift and drainage velocity profiles of time level variation were plotted in Figures (5) and (6). It was observed that lift and drainage velocity profiles are oscillating with time. The wavelengths for drainage and lift velocity profiles appear constant with time. It could be due to the fact that the belt is oscillating. Figures (7) and (8) show that there is a sharper variation in drainage temperature with time when compared with lift temperature: it is due to the fact that a layer of fluid with zero temperature was being carried by the moving belt in lift problem.

Figures (9) and (10) present lift and drainage velocity profiles plotted against frequency parameter (ω) for different values of M and fixed values of $\alpha = 0.02$, $\xi = 0.4$, $G_r = 0.5$, $B_r = 4$, $p_r = 0.6$, $S = 0.6$, $K = 2$, $x = 0.7$, $t = 5$. An increase in magnetic parameter M , decreases the velocity profile of the fluid in lift problem while the velocity profile in drainage problem increases within the intervals [0.3, 1.0], [1.75, 2.0] and decreases within the intervals [0, 0.3], [1.0, 1.75] with an increase in M . Figures (11) and (12) show the influence of Brinkman number B_r with a fixed values of $\alpha = 0.02$, $\xi = 0.6$, $\omega = 0.4$, $M = 0.5$, $G_r = 0.5$, $p_r = 0.6$, $S = 2$, $t = 10$, $K = 0.4$ for both lift and drainage temperature profiles. The

temperature distribution increases with an increase in B_r for both lift and drainage temperature problems.

Increasing Grashof number (G_r) decreases the velocity in lift flow while a rise in Grashof number increases the velocity in drainage problem as shown in Figures (13) and (14). An increase in Radiation parameter N decreases the temperature profiles in lift and drainage problems as shown in Figures (15) and (16). Figures (17) and (18) present the temperature profiles of lift and drainage problems. Increasing Prandtl number (p_r) accelerates the temperature distributions in lift and drainage problems. Figures (19) and (20) present the effect of Porosity on lift and drainage velocity profiles. Increasing the porous medium shape factor decreases the velocity distributions in both lift and drainage problems.

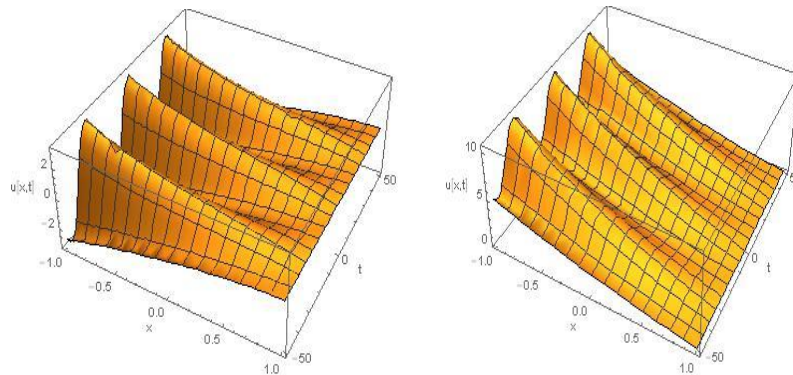


Figure 2: Influence of different time level on drainage velocity profile.
Figure 3: Influence of different time level on lift velocity profile

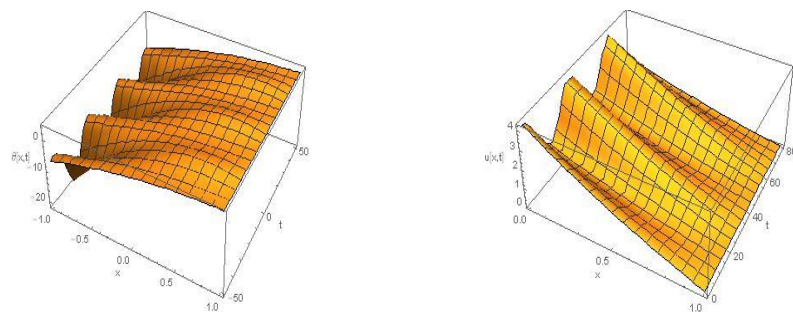


Figure 4: Influence of different time level on lift temperature profile.
Figure 5: Time level variation on lift velocity profile.

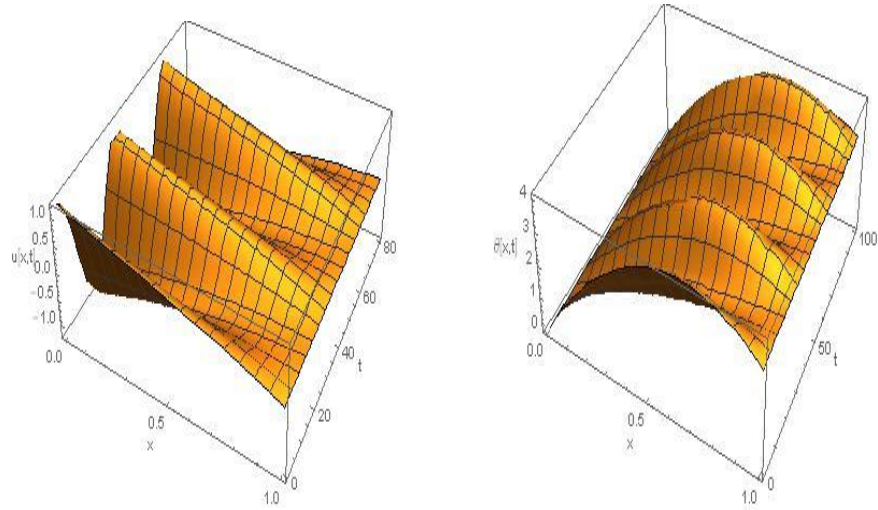


Figure 6: Time level variation on drainage velocity profile.

Figure 7: Time level variation on lift temperature.

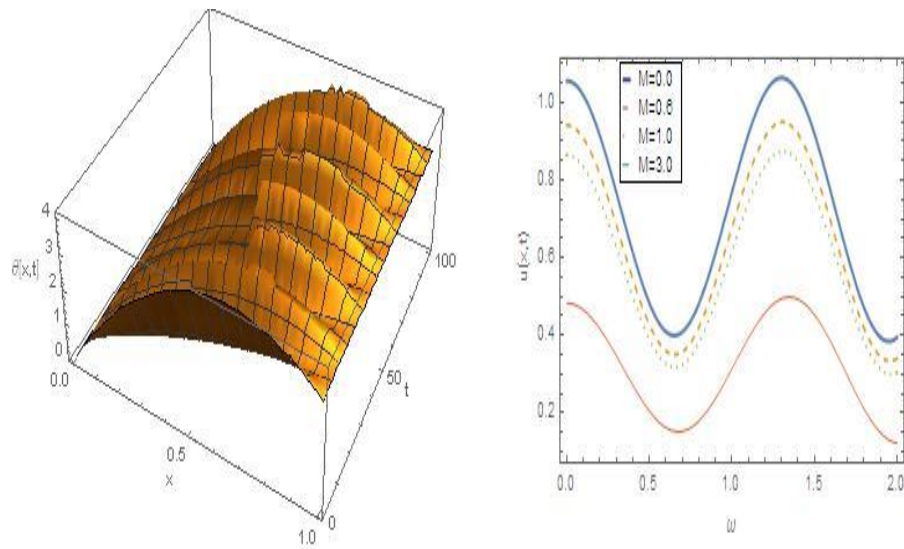


Figure 8: Time level variation on drainage temperature profile.

Figure 9: The effect of M and ω on lift velocity.

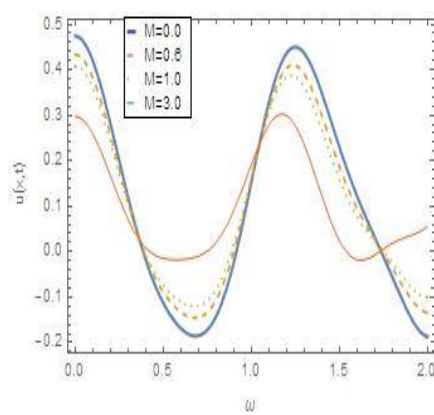


Figure 10: The effect of M and ω on drainage velocity.

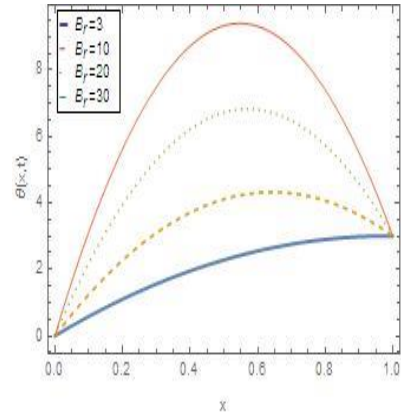


Figure 11: Influence of B_r on lift temperature profile.

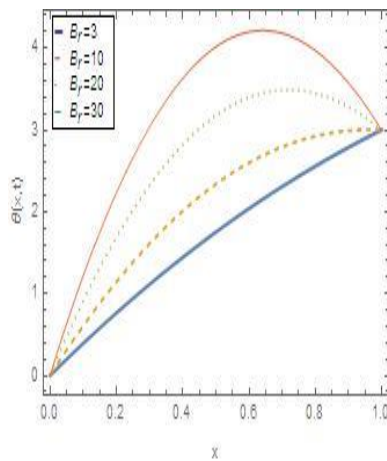


Figure 12: Influence of B_r on drainage temperature profile.

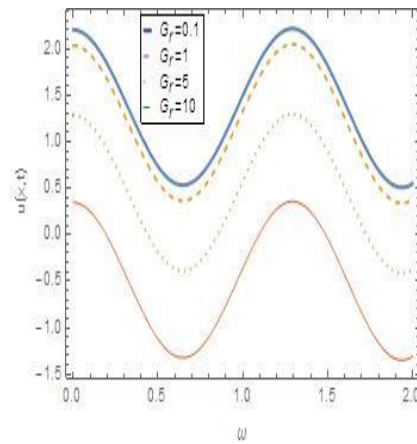


Figure 13: Effect of G_r and ω on lift velocity profile.

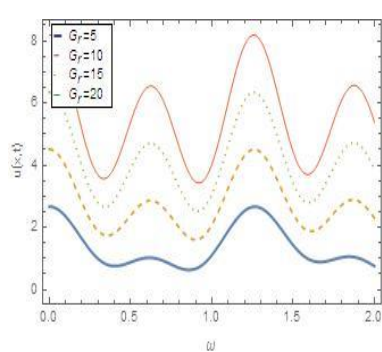


Figure 14: Effect of G_r and ω on drainage velocity.

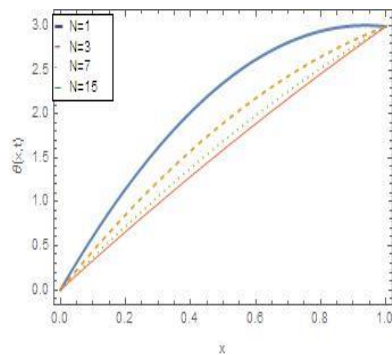


Figure 15: Influence of N on lift temperature.

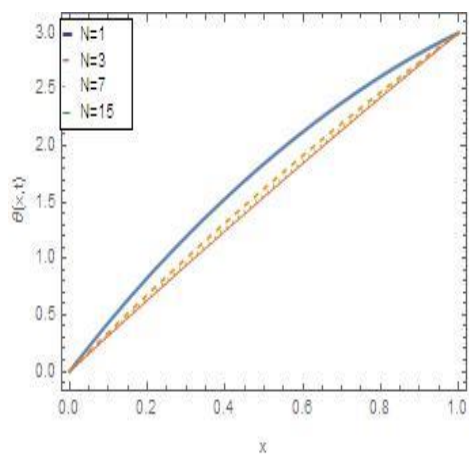


Figure 16: Influence of N on drainage temperature.

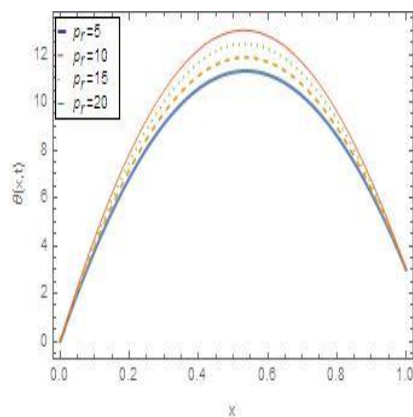


Figure 17: Influence of p_r on lift temperature.

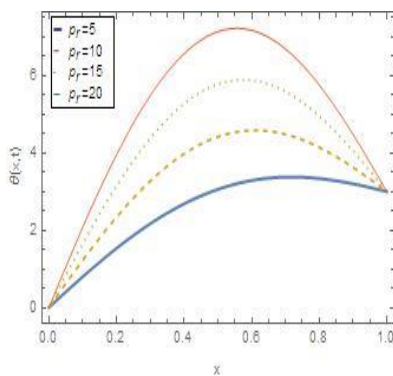


Figure 18: Influence of p_r on drainage temperature.

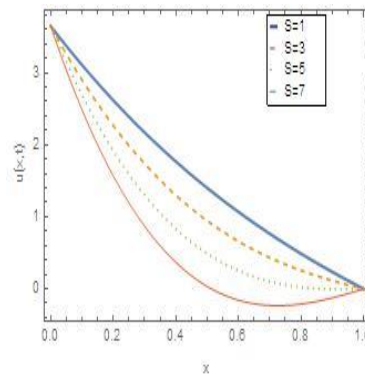


Figure 19: Effects of S on lift velocity.

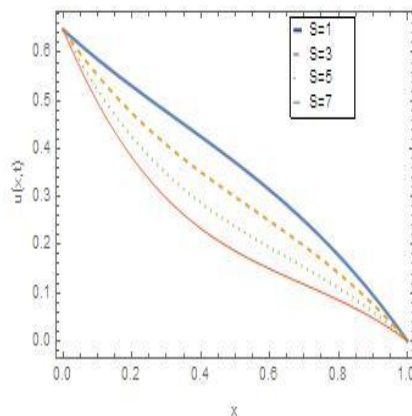


Figure 20: Effects of S on drainage velocity.

Conclusion

In this paper, the unsteady non-Newtonian magnetohydrodynamic radiative thin film flow in a porous medium past an oscillating vertical belt has been investigated. An Optimal Homotopy Asymptotic Method was employed in solving the resulting coupled partial differential equations. The effects of different physical parameters have been observed. The graphical results of the present work compared well with the work of Gul *et al.* (2014). From the study, it is concluded that:

the fluid temperature decreases with an increase in radiation parameter N in both cases of lift and drainage problems;

the magnitude of the velocity profile decreases as the porous medium shape factor (S) increases in lift velocity profile and drainage velocity profile increases with an increase in S ;

an increase in magnetic parameter M for lift flow, increases the velocity profile.

Nomenclature

T	Fluid temperature	σ_e	Stefan bolzmann-constant
α_1	Non-Newtonian effect	k_e	Conductivity of the fluid
T_o	Wall temperature	θ	Fluid temperature
U	Flow mean velocity	ξ	Amplitude
T	Time	ω	Frequency of oscillation
P	Pressure	α	Non-Newtonian effect
Q	Radiative heat flux	β_T	Coefficient of thermal expansion
S	Porous medium factor	T_δ	Temperature at boundary layer
K	Thermal conductivity	M	Hartman number
B_o	Electro magnetic induction	$\bar{u}$	Axial velocity
p_r	Prandtl number	μ	Fluid viscosity
G	Gravitational force	δ	Boundary layer
B_r	Brinkman number	ρ	Fluid density
G_r	Grashof number	μ_e	Magnetic permeability
N	Radiation parameter	ν	Kinematic viscosity

References

- Ali F, Khan I, Shafie S, (2014). Closed Form Solutions for Unsteady Free Convection Flow of a Second Grade Fluid Over an Oscillating Vertical Plate. *plos one* 9(2): e85099.doi:10.1371/journal.pone.0085099.
- Amir, K. and Gul, Z. (2015). Unsteady magnetohydrodynamic flow of a second grade fluid due to impulsive motion of plate. *Electronic Journal of Mathematical Analsis and Applications*. 3(1): 215-227.
- Griffiths R.W. (2000). The Dynamics of Lava Flows. *Annu. Rev. Fluid Mech*. 32: 477–518.
- Gul T, Shah, R.A, Islam, S. Arif, M. (2013). MHD Thin Film Flows of a Third Grade Fluid on a Vertical Belt With Slip Boundary Conditions. *Journal of Applied Mathematics* Article ID 707286, <http://dx.doi.org/10.1155/2013/707286>
- Gul T, Islam S, Shah R.A, Khan I, Shafie S, (2014). Thin Film Flow in MHD Third Grade Fluid on a Vertical Belt with Temperature Dependent Viscosity. *Plos One* 9(6). e97552. doi:10.1371/journal.pone.0097552.
- Gul, T., Islam S., Shah, R.A., Khan, I, Khalid A., Shafie S. (2014). Heat Transfer analysis of MHD Thin Film Flow of an Unsteady Second Grade Fluid Past a Vertical Oscillating Belt. *Plos One* 9(11): e103843.doi:10.1371/journal.pone.0103843.
- Gul, T., Islam, S., Shah, R.A., Khalid A., Khan, I., Shafie S, (2015). Unsteady MHD Thin Film Flow of an Oldroyd-B Fluid over an Oscillating Inclined Belt. *Plos One* 10(7): e0126698. doi:10.1371/journal.pone.0126698.
- Kamran, A.M., Rahim, M.T., Haroon, T., Islam S., Siddiqui, A.M. (2012). Thin-Film Flow of Johnson-Segalman Fluids for Lifting and Drainage Problems. *Appl. Math. Comput* 218: 10413–10428.
- Liao, S.J. (2003). Beyond Perturbation. Introduction to Homotopy Analysis Method. Chapman and Hall/CRC Press, Boca Raton.

Mabood, F., Khan, W.A., Ismail, A.I. (2013). Flow and heat transfer of a viscoelastic fluid in an axisymmetric channel with a porous medium. *Plos One* 8 (12): e83581. doi:10.1371/journal.pone.0083581.

Marinca, V. Herisanu, N., Nemes, I. (2008). Optimal homotopy asymptotic method with application to thin film flow. *Cent. Eur. J. Phys.* 6: 648–653.

Marinca, V., Herisanu, N. (2008). Application of Optimal Homotopy Asymptotic Method for Solving Non-Linear Equations Arising in Heat Transfer, *Int.Communications in heat and mass.Trans* 35: 710-715.

Marinca, V., Herisanu, N., Constantin, B., Bogdan, M. (2009). An Optimal Homotopy Asymptotic Method Applied to the Steady Flow of a Fourth Grad Fluid Past a Porous Plate. *Appl. Math. Lett.* 22: 245-251.

Samiulhaq, Sohail A., Dumitru V, Ilyas K, Sharidan S. (2014).Unsteady magnetohydrodynamic free convection flow of a second grade fluid in a porous medium with ramped wall temperature. *Plos One* 9(5):: e88766. doi: 10.1371/journal.pone.0088766

