

## ON A LINEAR COMBINATION OF SOME GEOMETRIC EXPRESSIONS

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Component geometric expressions of a certain linear combination are generalized via some differential operators and the conditions for univalence of the resulting linear combinations investigated. Our result completely unify earlier ones.

### 1. Introduction

Let  $A$  be the class of functions

$$f(z) = z + a_2 z^2 + \cdots \quad (1)$$

which are analytic in the open unit disk  $E = \{z \in \mathbb{C} : |z| < 1\}$ . A function  $f \in A$  is said to be of bounded turning, starlike and convex respectively if and only if, for  $z \in E$ ,  $\operatorname{Re} f'(z) > 0$ ,  $\operatorname{Re} z f'(z)/f(z) > 0$  and  $\operatorname{Re} (1 + z f''(z)/f'(z)) > 0$ . By usual notations we denote these classes of functions respectively by  $R$ ,  $S^*$  and  $C$ . These classes of functions all basically consist only of univalent functions in  $E$ . In [17], Yamaguchi studied functions which satisfy  $\operatorname{Re} f(z)/z > 0$  in  $E$ . We denote this class by  $R_0$  in this paper.

In 1975, Rucheweyh [12] introduced the operator  $\mathcal{D}^n$ ,  $n \in \mathbb{N}$  defined as:

$$\mathcal{D}^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = \frac{z \cdot z^{n-1} f(z)^{(n)}}{n!}.$$

and used it to generalize the concepts of starlikeness and convexity of functions in the unit disk by defining classes of functions for which the geometric quantities

$$\frac{\mathcal{D}^{n+1} f(z)}{\mathcal{D}^n f(z)}, \quad n \in \mathbb{N}$$

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have some positive real part (see also [14]). Similarly, Salagean [11] in 1983 defined a new operator  $D^n$ ,  $n \in \mathbb{N}$  as:

$$D^n f(z) = D(D^{n-1} f(z)) = z[D^{n-1} f(z)]'$$

with  $D^0 f(z) = f(z)$ . He also used it to generalize the same concepts of starlikeness and convexity of functions defining classes of functions for which the geometric quantities

$$\frac{D^{n+1} f(z)}{D^n f(z)}, \quad n \in \mathbb{N}$$

also have some positive real part in  $E$ .

Goel and Sohi [8] following Rucheweyh, generalized  $R$  by introducing classes of functions for which

$$\frac{\mathcal{D}^{n+1} f(z)}{z}, \quad n \in \mathbb{N} \quad (2)$$

have positive real parts in  $E$ , while Abdulhalim [1] followed Salagean by generalizing both  $R_0$  and  $R$  in the classes of functions with

$$\frac{D^n f(z)^\alpha}{z^\alpha}, \quad n \in \mathbb{N} \quad (3)$$

having positive real parts in  $E$ , the index  $\alpha > 0$  meaning principal determinations only. It is known that for  $n \geq 1$ , functions satisfying (3) also consists only of univalent functions in  $E$ . They are subclasses of Bazilevic functions having logarithmic growth in  $E$  [1, 4, 10, 13].

In the articles [2, 3, 15, 16] an interesting subject of univalence of analytic function for which the real part of the linear combination of  $f'(z)$  and  $(1 + zf''(z)/f'(z))$  is greater than a real number  $\beta$ , has been considered. That is: if  $f \in A$  satisfies

$$\operatorname{Re} \left[ (1 - \lambda)f'(z) + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] > \beta, \quad z \in E \quad (4)$$

for some real numbers  $\lambda$  and  $\beta$ , then under what condition is  $f(z)$  univalent in  $E$ . Of course, for nonnegative  $\beta$ , the univalence is quite obvious for two cases  $\lambda = 0$  and  $\lambda = 1$ , which respectively coincides with  $R$  and  $C$ . For  $\beta = 0$ , Al-Amiri and Reade [3] showed that  $f(z)$  univalent for all  $\lambda \leq 0$  by proving that for each  $f \in A$  satisfying (4), then  $\operatorname{Re} f'(z) > 0$ , that is  $f \in R$ . Ahuja and Silverman [2] considered a subclass of  $A$  consisting of functions with neagtive coefficients and satisfy (4). They used simple inclusion relation to prove that for all real  $\lambda$ , such functions are not only univalent but also starlike in the unit disk. For all  $f \in A$ , Singh, Singh

and Gupta [16] established the univalence for two cases: (i)  $0 < \lambda = \beta < 1$  and (ii)  $\beta = 1/2$  and all  $\lambda \geq 0$ . And very recently, Singh, Gupta and Singh [15] proved that for all  $\lambda$  such that  $\lambda \leq \beta < 1$ , each  $f \in A$  satisfying (4) is univalent in  $E$ .

For  $f \in A$ , apart from the trivial cases  $\lambda = 0$  and  $\lambda = 1$ , the above results can be summarized into two cases: (i)  $\lambda \leq \beta < 1$  and (ii)  $\beta = 1/2$ ,  $\lambda$  real (since  $\lambda < 0$  is included in (i)).

In the present paper we extend these results to linear combination of more general geometric expressions defined by the operators  $\mathcal{D}^n$  and  $D^n$  as follows:

$$\operatorname{Re} \left[ (1 - \lambda) \frac{\mathcal{D}^n f(z)}{z} + \lambda \frac{\mathcal{D}^{n+1} f(z)}{\mathcal{D}^n f(z)} \right] > \beta, \quad z \in E \quad (5)$$

and

$$\operatorname{Re} \left[ (1 - \lambda) \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} + \lambda \frac{D^{n+1} f(z)}{D^n f(z)} \right] > \beta, \quad z \in E, \quad (6)$$

We shall prove that if  $f(z)$  satisfies (5) and (6) respectively, then  $\operatorname{Re} [\mathcal{D}^n f(z)/z] > 0$  and  $\operatorname{Re} [D^n f(z)^\alpha/z^\alpha] > 0$  in  $E$  for all  $\lambda$  such that  $\lambda \leq \beta < 1$ . For  $n \geq 1$ , univalence is implied by these results. Furthermore we shall prove that if  $\beta = 1/2$ , then for all  $\lambda \geq 0$ ,  $\operatorname{Re} [D^n f(z)^\alpha/z^\alpha] > 1/2$ ,  $0 < \alpha \leq 1$ . The main results are presented in Section 3. In the next section two preliminary lemmas are stated.

## 2. Lemmas

Let  $P$  denote the class of functions  $p(z) = 1 + c_1 z + c_2 z^2 + \dots$  which are regular in  $E$  and satisfy  $\operatorname{Re} p(z) > 0$ ,  $z \in E$ . We shall need the following lemmas.

**LEMMA 1** [4]. *Let  $u = u_1 + u_2 i$ ,  $v = v_1 + v_2 i$  and  $\psi(u, v)$  a complex-valued function satisfying:*

- (a)  $\psi(u, v)$  is continuous in a domain  $\Omega$  of  $\mathbb{C}^2$ ,
- (b)  $(1, 0) \in \Omega$  and  $\operatorname{Re} \psi(1, 0) > 0$ ,
- (c)  $\operatorname{Re} \psi(\xi + (1 - \xi)u_2 i, v_1) \leq \xi$  when  $(\xi + (1 - \xi)u_2 i, v_1) \in \Omega$  and  $2v_1 \leq -(1 - \xi)(1 + u_2^2)$  for real number  $0 \leq \xi < 1$ .

*If  $p \in P$  such that  $(p(z), zp'(z)) \in \Omega$  and  $\operatorname{Re} \psi(p(z), zp'(z)) > \xi$  for  $z \in E$ , then  $\operatorname{Re} p(z) > \xi$  in  $E$ .*

The case  $\xi = 0$  of the above lemma can be found in many related literatures.

**LEMMA 2** [5]. *Let  $f \in A$  and  $\alpha > 0$  be real. If  $D^{n+1}f(z)^\alpha/D^n f(z)^\alpha$  is independent of  $n$  for  $z \in E$ , then*

$$\frac{D^{n+1}f(z)^\alpha}{D^n f(z)^\alpha} = \alpha \frac{D^{n+1}f(z)}{D^n f(z)}.$$

That the above lemma is also true if  $D^{n+1}f(z)/D^n f(z)$  is independent of  $n$ ,  $z \in E$  was proved in [6].

### 3. Main Result

First we prove:

**THEOREM 1.** *Let  $\lambda, \beta$  be real numbers such that  $\lambda \leq \beta < 1$ . If  $f \in A$  satisfies (6), then  $\operatorname{Re} [D^n f(z)^\alpha/z^\alpha] > 0$ . The result is sharp in the sense that  $\beta$  cannot be replaced by any constant smaller than  $\lambda$ . Furthermore if  $\beta = 1/2$ , then for  $0 < \alpha \leq 1$ ,  $\operatorname{Re} [D^n f(z)^\alpha/(\alpha^n z^\alpha)] > 1/2$  for all  $\lambda \geq 0$ .*

**PROOF.** Let  $f \in A$  satisfy (6). For  $p \in P$ , define

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = p(z).$$

Then we find that

$$\frac{D^{n+1}f(z)^\alpha}{D^n f(z)^\alpha} = \alpha + \frac{zp'(z)}{p(z)}.$$

Using Lemma 2, we obtain

$$\frac{D^{n+1}f(z)}{D^n f(z)} = 1 + \frac{zp'(z)}{\alpha p(z)}.$$

So, the condition (6) can now be written as

$$\operatorname{Re} \left[ (1 - \lambda)p(z) + \lambda \left( 1 + \frac{zp'(z)}{\alpha p(z)} \right) \right] > \beta, \quad z \in E. \quad (7)$$

Now write the above equation as

$$\operatorname{Re} \left[ (1 - \lambda)p(z) + \lambda \left( 1 + \frac{zp'(z)}{\alpha p(z)} \right) \right] - \beta > 0.$$

and define  $\psi(u, v) = (1 - \lambda)u + \lambda[1 + v/(\alpha u)] - \beta$  on a domain  $\Omega = [\mathbb{C} - \{0\}] \times \mathbb{C}$  of  $\mathbb{C}^2$ . Obviously  $\psi(u, v)$  passes the conditions (a) and (b) of Lemma 1. Thus (with  $\xi = 0$  in Lemma 1) we have

$$\psi(u_2 i, v_1) = (1 - \lambda)u_2 i + \frac{\lambda v_1}{\alpha u_2 i} + \lambda - \beta$$

so that  $\operatorname{Re} \psi(u_2 i, v_1) = \lambda - \beta \leq 0$  if and only if  $\lambda \leq \beta < 1$ . Furthermore suppose  $\beta = 1/2$  in (7) and define  $\psi(u, v) = (1 - \lambda)u + \lambda[1 + v/(\alpha u)]$  also on  $\Omega = [\mathbb{C} - \{0\}] \times \mathbb{C}$  of  $\mathbb{C}^2$ . This  $\psi(u, v)$  also satisfies the conditions (a) and (b) of Lemma 1. Then set  $\xi = 1/2$  in the lemma, so that

$$\psi\left(\frac{1}{2} + \frac{1}{2}u_2 i, v_1\right) = \frac{(1 - \lambda)(1 + u_2 i)}{2} + \lambda\left(1 + \frac{2v_1}{\alpha(1 + u_2 i)}\right)$$

so that

$$\begin{aligned} \operatorname{Re} \psi\left(\frac{1}{2} + \frac{1}{2}u_2 i, v_1\right) &= \frac{(1 - \lambda)}{2} + \lambda + \frac{2\lambda v_1}{\alpha(1 + u_2^2)} \\ &\leq \frac{1}{2} + \frac{\lambda}{2} - \frac{\lambda}{2\alpha} \quad \text{if } 2v_1 \leq -\frac{1}{2}(1 + u_2^2) \\ &\leq \frac{1}{2} \quad \text{if } 0 < \alpha \leq 1. \end{aligned}$$

Thus applying Lemma 1 with  $\xi = 1/2$ , we have  $\operatorname{Re} [D^n f(z)^\alpha / (\alpha^n z^\alpha)] > 1/2$  for  $\beta = 1/2$  and  $\lambda \geq 0$ .

To show that  $\beta$  on the right hand of (5) is the best possible, consider  $f_0(z)$  defined by

$$\frac{D^n f_0(z)^\alpha}{\alpha^n z^\alpha} = \frac{1 + z}{1 - z}.$$

By simple calculation, using Lemma 2, we find that

$$\frac{D^{n+1} f_0(z)}{D^n f_0(z)} = 1 + \frac{1}{\alpha} \left( \frac{z}{1 + z} + \frac{z}{1 - z} \right).$$

And it is easily verified that for all  $z$  on the boundary of  $E$ , except  $z = \pm 1$ ,

$$\operatorname{Re} \left[ (1 - \lambda) \frac{D^n f_0(z)^\alpha}{\alpha^n z^\alpha} + \lambda \frac{D^{n+1} f_0(z)}{D^n f_0(z)} \right] = \lambda$$

Observe also that at  $z = 0$  we have

$$\operatorname{Re} \left[ (1 - \lambda) \frac{D^n f_0(z)^\alpha}{\alpha^n z^\alpha} + \lambda \frac{D^{n+1} f_0(z)}{D^n f_0(z)} \right] = 1,$$

so that the rest of the argument follows as in [15], and the proof is complete.  $\square$

**COROLLARY 1.** *Let  $\lambda, \beta$  be real numbers such that  $\lambda \leq \beta < 1$ . Assume  $f \in A$  satisfies*

$$\operatorname{Re} \left[ (1 - \lambda) \frac{f(z)^{\alpha-1} f'(z)}{z^{\alpha-1}} + \lambda \left( 1 + \frac{z f''(z)}{f'(z)} \right) \right] > \beta, \quad z \in E.$$

Then  $\operatorname{Re} [f(z)^{\alpha-1} f'(z)/z^{\alpha-1}] > 0$  in  $E$ . So,  $f$  is close-to-convex, Bazilevic univalent of type  $\alpha$  in  $E$ . The result is sharp in the sense that  $\beta$  cannot be replaced by any constant smaller than  $\lambda$ . Furthermore if  $\beta = 1/2$  and  $0 < \alpha \leq 1$ , then  $\operatorname{Re} [f(z)^{\alpha-1} f'(z)/z^{\alpha-1}] > 1/2$  in  $E$  for all  $\lambda \geq 0$ . So,  $f$  is close-to-convex, Bazilevic univalent of type  $\alpha$  in  $E$  for all real number  $\lambda$  if  $\beta = 1/2$ .

If  $\alpha = 1$  in the above corollary, then we have the following result which completely unify all earlier ones for  $f \in A$  (cf: [3, 15, 16]).

**COROLLARY 2** ([3, 15, 16]). *Let  $\lambda, \beta$  be real numbers such that  $\lambda \leq \beta < 1$ . Assume  $f \in A$  satisfies*

$$\operatorname{Re} \left[ (1 - \lambda) f'(z) + \lambda \left( 1 + \frac{z f''(z)}{f'(z)} \right) \right] > \beta, \quad z \in E.$$

*Then  $f \in R$  (that is close-to-convex univalent in  $E$ ). The result is sharp in the sense that  $\beta$  cannot be replaced by any constant smaller than  $\lambda$ . Furthermore if  $\beta = 1/2$ ,  $\operatorname{Re} f'(z) > 1/2$  in  $E$  for all  $\lambda \geq 0$ . So,  $f \in R$  (that is close-to-convex univalent in  $E$ ) for all real number  $\lambda$  if  $\beta = 1/2$ .*

Furthermore, we have:

**COROLLARY 3.** *Let  $\lambda, \beta$  be real numbers such that  $\lambda \leq \beta < 1$ . Assume  $f \in A$  satisfies*

$$\operatorname{Re} \left[ (1 - \lambda) \frac{f(z)}{z} + \lambda \frac{z f'(z)}{f(z)} \right] > \beta, \quad z \in E.$$

*Then  $\operatorname{Re} f(z)^\alpha / z^\alpha > 0$  in  $E$ . The result is sharp in the sense that  $\beta$  cannot be replaced by any constant smaller than  $\lambda$ . Furthermore if  $\beta = 1/2$  and  $0 < \alpha \leq 1$ , then  $\operatorname{Re} f(z)^\alpha / z^\alpha > 1/2$  in  $E$  for all  $\lambda \geq 0$ . So,  $f$  satisfies  $\operatorname{Re} f(z)^\alpha / z^\alpha > 0$  in  $E$  for all real number  $\lambda$  if  $\beta = 1/2$ .*

It is proved in [5] that if  $f \in A$  satisfies  $\operatorname{Re} f(z)^\alpha / z^\alpha > 0$  in  $E$ , the such  $f$  is univalent in the subdisk  $|z| < (\sqrt{1 + \alpha^2} - 1)/\alpha$ . By extension we say if  $f \in A$  satisfies the conditions of the last corollary, then  $f$  is univalent in the subdisk  $|z| < (\sqrt{1 + \alpha^2} - 1)/\alpha$ . See also [17] for the case  $\alpha = 1$ .

And finally,

**THEOREM 2.** *Let  $\lambda, \beta$  be real numbers such that  $0 \leq \beta < 1$ . If  $f \in A$  satisfies (5). Then  $\operatorname{Re} [\mathcal{D}^n f(z)/z] > 0$  provided  $\lambda \leq \beta < 1$ . The result is sharp in the sense that  $\beta$  cannot be replaced by any constant smaller than  $\lambda$ .*

**PROOF.** Let  $f \in A$  satisfy (5). For  $p \in P$ , define

$$\frac{\mathcal{D}^n f(z)}{z} = p(z).$$

Then we find that

$$z(\mathcal{D}^n f(z))' = \mathcal{D}^n f(z) + z^2 p'(z)$$

Now using the relation  $z(\mathcal{D}^n f(z))' = (n+1)\mathcal{D}^{n+1}f(z) - n\mathcal{D}^n f(z)$  (see [12, 14]) we obtain

$$\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)} = 1 + \frac{zp'(z)}{(n+1)p(z)}$$

so that the condition (5) can be written as

$$Re \left[ (1-\lambda)p(z) + \lambda \left( 1 + \frac{zp'(z)}{(n+1)p(z)} \right) \right] > \beta. \quad z \in E.$$

The rest of the proof follows as in Theorem 1.

We note that for  $\beta = 1/2$ , the proof is valid only for  $n = 0$ . This shows the generalization by Ruscheweyh derivative does not adequately capture all existing results.

As for sharpness, consider  $f_0(z)$  defined by

$$\frac{\mathcal{D}^n f_0(z)}{z} = \frac{1+z}{1-z}.$$

Then we find that

$$\frac{\mathcal{D}^{n+1}f_0(z)}{\mathcal{D}^n f_0(z)} = 1 + \frac{z}{1+z} + \frac{z}{1-z}.$$

The conclusion also follows as in Theorem 1.

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