



RATIO AND PRODUCT TYPE EXPONENTIAL ESTIMATORS OF POPULATION VARIANCE UNDER TRANSFORMED SAMPLE INFORMATION OF STUDY AND SUPPLEMENTARY VARIABLES

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AUTHORS' CONTRIBUTIONS

This work was carried out in collaboration among the authors. Author AA formulates the estimators and established their properties. Authors RVKS and AAA help in checking the solutions gathered the initial data and help to interpret the results. All the authors read and approved the final manuscript.

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ABSTRACT

In this paper, we have suggested a class of modified ratio and product exponential estimators for finite population variance. The proposed class of exponential estimators was obtained by transforming both the sample variances and means of study and auxiliary variables of Singh et al. [1] and Asgha et al. [2] estimators. The mean square errors (MSEs) of the proposed estimators have been obtained and the conditions for their efficiency over some existing variance estimators have been established. The efficiency of proposed estimators based on optimal values of the constants, exhibit significant improvement over the estimators considered in the study. The numerical illustration was also conducted to corroborate the theoretical results. The result of the empirical study does not only show high efficiency of the proposed class of estimators over existing estimators but also save cost because their efficiency is high for small samples.

Keywords: Efficiency; mean square error; exponential estimator; finite population; variance.

1 Introduction

Judicious utilization of supplementary (auxiliary) information improves the efficiency of estimator for population parameters (characteristics) especially at the estimation stage. Population variance is one of the parameters which require efficient estimators because of its great significance in various fields such as in matters of health: variations in body temperature, pulse beat and blood pressure which are the basic guides of diagnosis when prescribed treatment is designed to control their variation. The population variance may also play significant role in the study of variations of produce or yields in the manufacturing industries and pharmaceutical laboratories.

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In this paper, an improved class of ratio estimators for estimating finite population variance has been proposed with objective to produce efficient estimators and their properties have established.

Let $\Omega = (1, 2, 3 \dots N)$ be a population of size N and Y, X be two real valued functions having values $(Y_i, X_i) \in \mathbb{R}^+ > 0$ on the i^{th} unit of $U(1 \leq i \leq N)$. Let S_y^2 and S_x^2 be the finite population variance of Y and X respectively and s_y^2 and s_x^2 be respective sample variances based on the random sample of size n drawn without replacement.

Consider the following definitions;

$$s_y^2 = \frac{1}{n-1} \sum_{i=1}^n (y_i - \bar{y})^2, \quad s_x^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2, \quad S_y^2 = \frac{1}{N-1} \sum_{i=1}^N (Y_i - \bar{Y})^2, \\ S_x^2 = \frac{1}{N-1} \sum_{i=1}^N (X_i - \bar{X})^2, \quad \bar{X} = \frac{1}{N} \sum_{i=1}^N X_i, \quad \bar{Y} = \frac{1}{N} \sum_{i=1}^N Y_i, \quad \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i, \quad \bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$$

The sample variance estimator of the finite population variance is defined as

$$\eta_0 = s_y^2 \tag{1}$$

which is an unbiased estimator of finite population variance S_y^2 and its variance is

$$Var(\eta_0) = \gamma S_y^2 (\lambda_{40} - 1) \tag{2}$$

$$\text{where } \gamma = \frac{1}{n}, \quad \lambda_{rs} = \frac{\mu_{rs}}{\mu_{20}^{r/2} \mu_{02}^{s/2}} \text{ and } \mu_{rs} = \frac{1}{N-1} \sum_{i=1}^N (Y_i - \bar{Y})^r (X_i - \bar{X})^s$$

Singh et al. [1] proposed ratio and product type exponential estimators for finite population variance respectively as

$$\eta_1 = s_y^2 \exp \left[\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right] \tag{3}$$

$$\eta_2 = s_y^2 \exp \left[\frac{s_x^2 - S_x^2}{s_x^2 + S_x^2} \right] \tag{4}$$

The Mean Square Error (MSE) of the estimator in equations (3) and (4) up to second degree approximation, are given respectively as

$$MSE(\eta_1) = \gamma S_y^4 \left[(\lambda_{40} - 1) + \frac{1}{4} (\lambda_{04} - 1) - (\lambda_{22} - 1) \right] \tag{5}$$

$$MSE(\eta_2) = \gamma S_y^4 \left[(\lambda_{40} - 1) + \frac{1}{4}(\lambda_{04} - 1) + (\lambda_{22} - 1) \right] \quad (6)$$

Asghar et al. [2] having considered the work of Bahl and Tuteja [3], suggested new ratio and product type exponential estimators for finite population variance respectively as

$$\eta_3 = s_y^2 \exp \left[\frac{\bar{X} - \bar{X}}{\bar{X} + \bar{X}} \right] \quad (7)$$

$$\eta_4 = s_y^2 \exp \left[\frac{\bar{X} - \bar{X}}{\bar{X} + \bar{X}} \right] \quad (8)$$

The Mean Square Error (MSE) of the new estimators in equations (7) and (8) up to second degree approximation, are given respectively as

$$MSE(\eta_3) = \gamma S_y^4 \left[(\lambda_{40} - 1) + \frac{1}{4}C_x^2 - \lambda_{21}C_x \right] \quad (9)$$

$$MSE(\eta_4) = \gamma S_y^4 \left[(\lambda_{40} - 1) + \frac{1}{4}C_x^2 + \lambda_{21}C_x \right] \quad (10)$$

Yadav & Kadilar [4], in his work titled improved exponential type ratio estimator of population variance also proposed exponential estimator for finite population variance in single-phase sampling as

$$\eta_5 = s_y^2 \exp \left[\frac{S_x^2 - s_x^2}{S_x^2 + (1 - \alpha)s_x^2} \right] \quad (11)$$

The minimum MSE of η_5 up to first order approximation is given as

$$MSE_{\min}(\eta_5) = \gamma S_y^4 \left[(\lambda_{40} - 1) - \frac{(\lambda_{22} - 1)^2}{(\lambda_{04} - 1)} \right] \quad (12)$$

Many other authors like Bandyopadhyay [5], Dubey [6], Goodman [7], Grover, et al. [8], Gupta and Shabbir [9], Ray and Sahai [10], Singh et al. [11], Singh and Solanki [12,13], Solanki and Singh [14], Srivastava [15], Srivastava et al. [16], Subramani and Kumarapandiyam [17], Tracy et al. [18], have suggested both ratio and product exponential estimators in single and two-phase sampling for population mean.

2 Proposed Estimator

After studying various variance estimators in the literature some of which were stated in section 1 and motivated by the work of Yadav and Kadilar [19] estimators for population mean, an extension of Adewara et al. [20], defined as

$$\eta_1^* = k_1 \bar{y}^* \left(\frac{\bar{X}}{\bar{X}^*} \right), \eta_2^* = k_2 \bar{y}^* \left(\frac{\bar{X}^*}{\bar{X}} \right), \eta_3^* = k_3 \bar{y}^* \left(\frac{\bar{X} + C_x}{\bar{X}^* + C_x} \right),$$

$$\eta_4^* = k_4 \bar{y}^* \left(\frac{\bar{X}^* + C_x}{\bar{X} + C_x} \right), \eta_5^* = k_5 \bar{y}^* \left(\frac{\bar{X} + \rho}{\bar{X}^* + \rho} \right), \eta_6^* = k_6 \bar{y}^* \left(\frac{\bar{X}^* + \rho}{\bar{X} + \rho} \right)$$

Where $\bar{X}^*$ and $\bar{Y}^*$ are the respective sample means of the auxiliary and study variables, having the relationship: (1) $\bar{X} = f \bar{x} + (1-f) \bar{X}^*$ (2) $\bar{Y} = f \bar{y} + (1-f) \bar{Y}^*$ where $f = \frac{n}{N}$ is finite population correction, $k_i, i=1,2,3,4,5,6$, are real constants.

We proposed the following class of improved exponential estimators for finite population variance S_y^2 of the study variable Y as follows;

$$\phi_1^* = k_1 s_y^{2*} \exp \left(\frac{S_x^2 - S_x^{2*}}{S_x^2 + S_x^{2*}} \right), \quad \phi_2^* = k_2 s_y^{2*} \exp \left(\frac{S_x^{2*} - S_x^2}{S_x^{2*} + S_x^2} \right),$$

$$\phi_3^* = k_3 s_y^{2*} \exp \left(\frac{\bar{X} - \bar{X}^*}{\bar{X} + \bar{X}^*} \right), \quad \phi_4^* = k_4 s_y^{2*} \exp \left(\frac{\bar{X}^* - \bar{X}}{\bar{X}^* + \bar{X}} \right),$$

Where S_x^{2*} and S_y^{2*} are the respective finite sample variances of the auxiliary and study variables, having the relationship: (1) $S_x^2 = f S_x^2 + (1-f) S_x^{2*}$ (2) $S_y^2 = f S_y^2 + (1-f) S_y^{2*}$.

These relationships are subjected to the condition that $n < \frac{1}{2} N$

In order to obtain the MSE, we defined $e_0 = \frac{S_y^2 - S_y^{2*}}{S_y^2}$, $e_1 = \frac{S_x^2 - S_x^{2*}}{S_x^2}$ and $e_2 = \frac{\bar{X} - \bar{X}^*}{\bar{X}}$ such that

$$\left. \begin{aligned} E(e_0) &= E(e_1) = E(e_2) = 0, E(e_0^2) = \gamma(\lambda_{40} - 1), E(e_1^2) = \gamma(\lambda_{04} - 1) \\ E(e_2^2) &= \gamma C_x^2, E(e_0 e_1) = \gamma(\lambda_{22} - 1), E(e_0 e_2) = \gamma \lambda_{21} C_x \end{aligned} \right\} \quad (13)$$

Expressing $\phi_i^*, i=1,2,3,4$, in terms of e_0 and e_1 or e_2 , we have

$$\phi_i^* = \left. \begin{aligned} &k_i S_y^2 (1 - h e_0) \exp \left[\frac{1}{2} (-1)^{i+1} (1 - h e_1)^{-1} h e_1 \right] & i=1,2 \\ &k_i S_y^2 (1 - h e_0) \exp \left[\frac{1}{2} (-1)^{i+1} (1 - h e_2)^{-1} h e_2 \right] & i=3,4 \end{aligned} \right\} \quad (14)$$

We now assume that $|he_1| < 1$ and $|he_2| < 1$ so that $(1 - he_1)^{-1}$ and $(1 - he_2)^{-1}$ are expandable.

Expanding the right hand side of (14) up to second degree approximation, subtract S_y^2 from its both sides, square the both sides and taking expectation using the results in equation (13), we obtain the MSEs of the proposed estimators as:

$$MSE(\phi_i^*) = \begin{cases} S_y^4 \left\{ \gamma h^2 \left[\begin{aligned} &k_i^2 (\lambda_{40} - 1) + \left(k_i^2 - \frac{3}{4}k_i\right)(\lambda_{04} - 1) \\ &- (-1)^{i+1} (2k_i^2 - k_i)(\lambda_{22} - 1) \end{aligned} \right] + (k_i - 1)^2 \right\} & i = 1, 2 \\ S_y^4 \left\{ \gamma h^2 \left[\begin{aligned} &k_i^2 (\lambda_{40} - 1) + \left(k_i^2 - \frac{3}{4}k_i\right)C_x^2 \\ &- (-1)^{i+1} (2k_i^2 - k_i)\lambda_{21}C_x \end{aligned} \right] + (k_i - 1)^2 \right\} & i = 3, 4 \end{cases} \quad (15)$$

The $MSE(\phi_i^*)$, $i = 1, 2, 3, 4$, expressions are minimized for the optimum values of k given by

$$k_i^{opt} = \frac{\frac{1}{8}h^2\gamma[(\lambda_{04} - 1) - 4(\lambda_{22} - 1)] + 1}{h^2\gamma[(\lambda_{04} - 1) - 2(-1)^{i+1}(\lambda_{22} - 1) + (\lambda_{40} - 1)] + 1} = \frac{P_1}{Q_1}, \quad i = 1, 2 \quad (16)$$

$$k_i^{opt} = \frac{\frac{3}{8}h^2\gamma[C_x^2 - 4\lambda_{21}C_x] + 1}{h^2\gamma[C_x^2 - 2(-1)^{i+1}\lambda_{21}C_x + (\lambda_{40} - 1)] + 1} = \frac{P_2}{Q_2}, \quad i = 3, 4 \quad (17)$$

where $h = \frac{n}{N - n}$

Replacing k_i by k_i^{opt} , $i = 1, 2, 3, 4$, in equation (15), we obtain the minimum MSE of the proposed estimators as

$$MSE_{\min}(\phi_i^*) = \begin{cases} S_y^4 \left(1 - \frac{P_1}{Q_1}\right), & i = 1, 2 \\ S_y^4 \left(1 - \frac{P_2}{Q_2}\right), & i = 3, 4 \end{cases} \quad (18)$$

3 Efficiency Comparisons

In this section efficiencies of the proposed estimators are compared with efficiencies of some estimators in the literature.

The ϕ_i^* – family of proposed exponential estimators of the population variance is more efficient than η_0 if,

$$MSE_{\min}(\phi_i^*) < Var(\eta_0) \quad i = 1, 2, 3, 4,$$

$$\left\{ \begin{array}{l} \left(1 - \frac{P_1^2}{Q_1}\right) < \gamma[(\lambda_{40} - 1)] \quad i = 1, 2 \\ \left(1 - \frac{P_2^2}{Q_2}\right) < \gamma[(\lambda_{40} - 1)] \quad i = 3, 4 \end{array} \right\} \quad (19)$$

The ϕ_i^* – family of proposed exponential estimators of the population variance is more efficient than η_1 and η_2 , if,

$$MSE_{\min}(\phi_i^*) < MSE(\eta_j) \quad i = 1, 2, 3, 4, \quad j = 1, 2$$

$$\left\{ \begin{array}{l} \left(1 - \frac{P_1^2}{Q_1}\right) < \gamma\left[(\lambda_{40} - 1) + \frac{1}{4}(\lambda_{04} - 1) - (-1)^{j+1}(\lambda_{22} - 1)\right] \quad i = 1, 2 \quad j = 1, 2 \\ \left(1 - \frac{P_2^2}{Q_2}\right) < \gamma\left[(\lambda_{40} - 1) + \frac{1}{4}(\lambda_{04} - 1) - (-1)^{j+1}(\lambda_{22} - 1)\right] \quad i = 3, 4 \quad j = 1, 2 \end{array} \right\} \quad (20)$$

The ϕ_i^* – family of proposed exponential estimators of the population variance is more efficient than η_3 and η_4 , if,

$$MSE_{\min}(\phi_i^*) < MSE(\eta_j) \quad i = 1, 2, 3, 4, \quad j = 3, 4$$

$$\left\{ \begin{array}{l} \left(1 - \frac{P_1^2}{Q_1}\right) < \gamma\left[(\lambda_{40} - 1) + \frac{1}{4}C_x^2 - (-1)^{j+1}\lambda_{22}C_x\right] \quad i = 1, 2 \quad j = 3, 4 \\ \left(1 - \frac{P_2^2}{Q_2}\right) < \gamma\left[(\lambda_{40} - 1) + \frac{1}{4}C_x^2 - (-1)^{j+1}\lambda_{22}C_x\right] \quad i = 3, 4 \quad j = 3, 4 \end{array} \right\} \quad (21)$$

The ϕ_i^* – family of proposed exponential estimators of the population variance is more efficient than η_5 , if,

$$MSE_{\min}(\phi_i^*) < MSE_{\min}(\eta_5) \quad i = 1, 2, 3, 4,$$

$$\left. \begin{aligned} \left(1 - \frac{P_1^2}{Q_1}\right) &< \gamma \left[(\lambda_{40} - 1) - \frac{(\lambda_{22} - 1)^2}{(\lambda_{04} - 1)} \right] & i = 1, 2 \\ \left(1 - \frac{P_2^2}{Q_2}\right) &< \gamma \left[(\lambda_{40} - 1) - \frac{(\lambda_{22} - 1)^2}{(\lambda_{04} - 1)} \right] & i = 3, 4 \end{aligned} \right\} \quad (22)$$

4 Numerical Illustration

In order to investigate the merits of the proposed estimators, we have considered the following two real populations given as:

Data 1: Murthy [21]

$$N = 25, \bar{Y} = 33.8465, \bar{X} = 283.875, \rho = 0.9136, C_y = 0.352$$

$$C_x = 0.746, \lambda_{40} = 2.2667, \lambda_{04} = 3.65, \lambda_{21} = 0.5475, \lambda_{22} = 4.6977$$

Data 2: Gujarati [22]

$$N = 81, \bar{Y} = 2137.086, \bar{X} = 112.4568, \rho = -0.691135, C_y = 0.1248$$

$$C_x = 0.48, \lambda_{40} = 3.59, \lambda_{04} = 6.82, \lambda_{21} = 0.05137, \lambda_{22} = 2.110$$

The numerical demonstration to justify the merit of the suggested class of variance estimator has been conducted using the two data sets. Tables 1 and 2 show the mean square errors of proposed estimators and that of some existing estimators in respect to Data 1 and Data 2 respectively.

Table 1. MSEs of the proposed estimators and some existing estimators for data 1

Estimators	Different sample sizes				
	$n = 2$	$n = 5$	$n = 7$	$n = 10$	$n = 12$
η_0	12760.55	5104.221	3645.872	2552.111	2126.759
η_1	5958.686	2383.474	1702.482	1191.737	993.1143
η_2	32910.28	13164.11	9402.937	6582.056	5485.046
η_3	10047.6	4019.041	2870.744	2009.521	1674.601
η_4	18276.63	7310.653	5221.895	3655.327	3046.106
η_5	5958.072	2383.229	1702.306	1191.614	993.0121
ϕ_1^*	145.2706	474.9356	811.4012	1622.111	2509.035
ϕ_2^*	540.0776	1690.579	2768.077	5029.455	7073.922
ϕ_3^*	96.79651	317.2208	543.2958	1092.694	1701.529
ϕ_4^*	219.4212	709.2566	1197.93	2330.993	3503.569

Table 2. MSEs of the proposed estimators and some existing estimators for data 2

Estimators	Different sample sizes				
	$n = 5$	$n = 10$	$n = 13$	$n = 20$	$n = 40$
η_0	26.21×10^8	13.11×10^8	10.08×10^8	6.55×10^8	3.28×10^8
η_1	29.70×10^8	14.85×10^8	11.42×10^8	7.43×10^8	3.71×10^8
η_2	52.17×10^8	26.08×10^8	20.06×10^8	13.04×10^8	6.52×10^8
η_3	26.54×10^8	13.27×10^8	10.21×10^8	6.64×10^8	3.32×10^8
η_4	27.04×10^8	13.52×10^8	10.40×10^8	6.76×10^8	3.38×10^8
η_5	24.07×10^8	12.03×10^8	9.26×10^8	6.02×10^8	3.01×10^8
ϕ_1^*	0.25×10^8	0.58×10^8	0.82×10^8	1.54×10^8	6.13×10^8
ϕ_3^*	0.45×10^8	1.01×10^8	1.42×10^8	2.65×10^8	9.88×10^8
ϕ_3^*	0.12×10^8	0.27×10^8	0.39×10^8	0.73×10^8	3.09×10^8
ϕ_4^*	0.12×10^8	0.28×10^8	0.40×10^8	0.76×10^8	3.20×10^8

5 Conclusion

From results of Tables 1 and 2, we infer that the suggested variance estimators are more efficient than the existing estimators in the sense of having lesser mean square error and therefore recommended for use in estimating finite population variance.

Competing Interests

Authors have declared that no competing interests exist.

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