

COMPARISON OF SOME ESTIMATORS OF BAYESIAN HETEROSCEDASTIC LINEAR MODEL

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ABSTRACT

In order to investigate the asymptotic consistency and efficiency of estimators with normal-gamma double sided Heteroscedastic error structure, the study explored full Bayesian Metropolis Hasting Algorithm experiments, an approach of Markov Chain Monte Carlo simulation. The study contaminated the model with one component of two sided error structure. A Metropolis-Hasting Algorithm adopted to perform simulation on the marginal posterior distribution of heteroscedastic linear econometric model. Absolute Bias and Mean Squares Error criteria were used to evaluate finite properties of the estimators. The study chose the following sample sizes: 25; 50; 100; 200; 500 and 1000. Thus 10,000 simulations with varying degree of heteroscedasticity were carried out. This is subject to the level of convergence. Absolute Bias and Minimum Mean Squares Error criteria revealed that Bayesian generalized least squares outperformed other estimators considered in the study.

Keywords: Markov Chain Monte Carlo Method, Heteroscedasticity, Bayesian Inference, Metropolis-Hasting Algorithm.

1. INTRODUCTION

Economic problems rely on regression model of which in most cases come with errors the most significant of it all is heteroscedasticity of either one or two components which may be additive or multiplicative. Oloyede (2010). Heteroscedasticity arises when the diagonal element of variance-covariance regression disturbances term are not constant across observation such that $\text{var}(\epsilon) = \sigma^2 \Omega$, assuming the disturbances are pairwise uncorrelated.

Thus

$$\sigma^2 \Omega = \sigma^2 \begin{bmatrix} w_1 \sigma_1^2 & 0 & \dots & 0 \\ 0 & w_2 \sigma_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & w_n \sigma_n^2 \end{bmatrix}$$

Such that $\sigma_1^2 \neq \sigma_2^2 \neq \dots \neq \sigma_n^2$

Heteroscedasticity arises from two sources: model and data based heteroscedasticity, the former arises from the construction and modification of the model, the error term may be regarded as disturbances error which may be associated with y-level of dependent variable and/or X-level of explanatory variables which is known as *misspecification*. The example of such model is the celebrated Cobb-Douglas production function, since 1928 when it was formulated a lot of criticism and modification had been placed on it. The error structure is then denoted by u with its variance as σ_u^2 whereas the latter arises from the data gathering and it can be regarded as

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measurement error. In most cases researchers do assumed normality when intend to compute regression. The error is denoted as v with its variance as σ_v^2 . It is impossible to know *a priori* what the variances of both errors structured are. Thus two forms of variances exist-one with known form and the other with unknown form. It should be noted that both errors are present particularly when data generating process (DGP) is not used.

Bayesian inference is an alternative to classical statistical point of view. In Bayesian framework, the knowledge about the parameter estimates of the model is described by probability distribution. The joint posterior distribution is the product of likelihood and prior which is divided by normalizing constant, thus normalizing constant often portend computational intensive. It is therefore usually assumed equal to unity. Meanwhile, Markov Chain Monte Carlo simulation technique that draw correlated sample of parameters from the joint posterior distribution with normalizing constant set to unity would proffer solution to the problem of intensive computation. Gilks *et al* (1996), Boscardin and Gelman (1996) developed a Bayesian approach in which Gibbs and Metropolis-Hasting algorithms were used to the model parameters of heteroscedasticity in the linear model. Ohtani (1982) derived the Bayesian estimator for heteroscedasticity linear model, using Monte Carlo experiment, they found that the Bayesian estimator when compared with some traditional estimators shows the best properties in the mean squares error criterion. Since Ohtani (1982) obtained the Bayesian estimate by numerical integration, it is not easy to extend to the multidimensional cases of both the regression coefficient and heteroscedastic parameters.

Guermat & Hadri (1999) examined Heteroscedasticity in stochastic frontier models using a Monte Carlo approach, they affirmed that correcting heteroscedasticity when present improve efficiency, but when not present, correcting it would bring about adverse effect. Carlos *et al* (2012): worked on the Multiplicative Heteroscedastic Von-Bertalanffy Model and propose heteroscedastic sigmoidal growth model. They opined that small and moderate number of observation do not harm empirical converge of probabilities of the credible interval and they examined asymptotic properties of the their estimator. Arto & Jani (2012) worked on Bayesian two-stage regression with parametric heteroscedasticity where they opined that data lend strong support to the use of the Heteroscedasticity Bayesian 2Stage (HB2S) in lieu of Homoscedasticity Bayesian 2Stage (homoB2S). Reis & *et al* (2003) examined Bayesian Generalized least squares regression with application to log pearson type 3 regional skew estimation where profile likelihood and quasi analytic solution were used to marginalizing the posterior density. Daniel and Enes (2012, 2013) examined Minimum Message Length Ridge Regression for Generalized Linear Models where they established superiority of minimum message length to other estimators, likewise Enes & Daniel (2006, 2009) looked into Efficient linear regression by minimum message length & MML Invariant Linear Regression and opined that mml87 outperformed other estimators.

MODEL DESIGNS

Let $y = X\beta + u$ with $u \sim N(0, \sigma_u^2 \Omega)$ where Ω is a positive definite matrix of order n . A case where $u \sim N(0, \sigma^2 I)$ is a homoscedastic model with constant variance, but $u \sim N(0, \sigma_u^2 \Omega)$ indicates unequal variances of the diagonal element of $n \times n$ matrix of $E(uu')$ which is regarded as heteroscedastic error structure.

$$y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + u \quad (1)$$

The paper adopted Harvey (1976) heteroscedastic error structure $\sigma_i^2 = \sigma^2(\beta_1 X_{i1} + \beta_2 X_{i2})^2$ [doubled sided]. We represent $(\beta_1 X_{i1} + \beta_2 X_{i2})^2$ by w . The likelihood function of θ , where $\theta = (\beta_0, \beta_1, \beta_2, \sigma, \lambda)$ given the sample vector $X_1, X_2 = (1, 2, \dots, n)'$ and $y = (y_1, y_2, \dots, y_n)'$ is expressed as $L(\theta, \sigma | X, y) = (2\pi\sigma^2)^{-n/2} \prod_{i=1}^n \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^n [y_i - x\beta]^2\right\}$

(2)

Incorporating $\sigma_i^2 \Omega$ into (2) produces MLE, GLS and MML87 likelihoods

$$L(\theta, \sigma | X, y) = (2\pi\sigma^2)^{-n/2} \prod_{i=1}^n w^{-\lambda/2} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^n w^{-\lambda} (y_i - X\beta)^2\right\} \Leftarrow MLE_{het} \quad (3)$$

$$L(\theta, \sigma | X, y) = (2\pi\sigma^2)^{-n/2} \prod_{i=1}^n |w|^{-\lambda/2} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - X\beta) w^{-\lambda} (y_i - X\beta)\right\} \Leftarrow GLS_{het} \quad (4)$$

$$MsgLen(\theta, \sigma | X, y)_{het} = -\left[h(\theta) \frac{(2\pi\sigma^2)^{-n/2} \prod_{i=1}^n |w|^{-\lambda/2} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^n w^{-\lambda} (y_i - X\beta)^2\right\}}{\sqrt{\left(\frac{n}{\sigma^2}\right) \left(\frac{|X'X|}{\sigma^{2k}}\right) \left(\frac{1}{2\pi\exp(1)}\right)}} \right] \Leftarrow MML87_{het} \quad (5)$$

To optimize full Bayesian density we conjugate heteroscedastic truncated likelihood (3), (4) and (5) with conjugate priors define below:

Prior Density

Multivariate normal Prior:

$$\pi(\beta) \propto (2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2} (\beta - \mu)^2\right\}, \beta > 0 \quad (6)$$

Inverse Gamma Prior:

$$\pi(\sigma^2) \propto (\sigma^2)^{-a_1+1} \exp(-b_1/\sigma^2), \sigma^2 > 0 \quad (7)$$

Uniform Prior:

$$\pi(\lambda) \propto c \quad (8)$$

Thus the Joint Posterior Density $\pi(\beta_0, \beta_1, \beta_2, \lambda, \sigma | X, y)$ are:

$$(2\pi\sigma^2)^{-n/2} (\sigma^2)^{-(a_1-1-n/2)} \exp\left\{-\frac{1}{2\sigma^2} (\beta - \mu)^2\right\} \prod_{i=1}^n w^{-\lambda/2} \exp\left\{-\frac{1}{\sigma^4} \left(b_1 + \frac{1}{2} \sum_{i=1}^n w^{-\lambda} (y_i - X\beta)^2\right)\right\} \Leftarrow \text{joint P.D MLE}_{het} \quad (9)$$

$$(2\pi\sigma^2)^{-n/2} (\sigma^2)^{-(a_1-1-n/2)} \exp\left\{-\frac{1}{2\sigma^2} (\beta - \mu)^2\right\} \prod_{i=1}^n |w|^{-\lambda/2} \exp\left\{-\frac{1}{\sigma^4} \left(b_1 + \frac{1}{2} \sum_{i=1}^n (y_i - X\beta) w^{-\lambda} (y_i - X\beta)\right)\right\} \Leftarrow \text{joint P.D GLS}_{het} \quad (10)$$

$$\text{joint P.D MML87}_{het} = \left[\frac{(2\pi\sigma^2)^{-n/2} (\sigma^2)^{-(n_1-1-n/2)} \exp\left\{-\frac{1}{2\sigma^2}(\beta - \mu u)^2\right\} \prod_{i=1}^n w^{-\lambda/2} \exp\left\{-\frac{1}{2\sigma^2}\left(b_i + \frac{1}{2} \sum_{j=1}^n w^{-\lambda} (y_i - x_j \beta)^2\right)\right\}}{\left(\frac{n}{2}\right)! \left(\frac{K^2 \lambda^2}{\sigma^2 2\pi}\right) \left(\frac{1}{2\pi \exp(1)}\right)} \right] \quad (11)$$

Integrating the joint densities (9), (10), & (11) w.r.t σ^2 produce marginal density for β -parameters that is updated with M.H algorithm. Ditto for σ^2 . For full detail of joint and marginal posteriors density of Bayesian Linear Heteroscedasticity Linear Model, reader should see Oloyede *et al* (2013a and 2013b).

3. DATA GENERATION PROCESSES

In an attempt to investigate the asymptotic consistency, and efficiency of estimator of econometric model in the presence of heteroscedastic error structure, we adopted Markov Chain Monte Carlo Experiment. The sample sizes are specified with 6 sets as follow: 25, 50, 100, 200, 500 and 1000. Harvey (1976) multiplicative heteroscedastic error structure is adopted to truncate linear econometric model.

The scale of λ -heteroscedastic error structure is selected as 0.0 homoscedastic, 0.3 less heteroscedastic, 0.5 & 0.6-moderately heteroscedastic, 0.9-mildly heteroscedastic and 1 & 2-severely heteroscedastic. Following Germa and Saez, (2000), the distribution of the main regression is assumed to be moderately heteroscedastic when variance is proportional to x_1 where x_1 ranged from 11 to 15-mesokurtos) and strongly heteroscedastic where x_2 ranged from 4 to 8-platokurtos. The error term U is generated based on $E(U) = 0$ and $E(UU') = (\lambda_0 + \lambda_1 x_1 + \lambda_2 x_2)^2$. $\delta_1 = 0.0; 0.3; 0.5, 0.6; 0.9, 1$ and 2 and λ_0, λ_1 and λ_2 are set at -2, 0.25 and 1 respectively. Thereafter, we incorporated U into the model to generate variable y . The parameters β_0, β_1 and β_2 are set at $\beta_0=10, \beta_1 = 1$ and $\beta_2 = 1$ respectively to generate variable y . For the Bayesian experiment, a Metropolis Hasting Algorithm was developed to simulate our MML87 heteroscedastic based models. This was invoked in [R 3.0.0]-a Statistical software. The data set class contained the posterior sample for the model parameters. The *n.sim* option used to specify the number of posterior simulation iteration was set to 10,000. The BURN-IN is set at 1000 which specified the draws that were discarded to remove the effect of the initial values. The THINING is set at 5 to ensure the removal of the effect of autocorrelation.

4. RESULTS AND DISCUSSION

Table 1: Comparison of the Performances of Estimators [BMLE, BGLS, BMML87] with Absolute Bias Criterion $[\hat{\beta}_0]$

Samples	Bias Criterion $[\hat{\beta}_0]$		
	BMLE	BGLS	BMML87
25	6(86%)	7(100%)	6(86%)
50	6(86%)	7(100%)	6(86%)
100	6(86%)	7(100%)	6(86%)
200	7(100%)	6(86%)	6(86%)
500	2(29%)	6(86%)	1(14%)
1000	6(86%)	2(29%)	6(86%)

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The Absolute bias for β_0 at various degree of heteroscedasticity indicates that BGLS outperformed all other estimators at sample sizes 25, 50, 100 and 500 with minimum biases, the BMLE outperformed other estimators at sample size 200 while BMLE and BMML87 outperformed other estimators at sample size 1000. Thus performance at β_0 shows that BGLS has overall best performance compare with all other estimators.

Table 2: Comparison of the Performances of Estimators [BMLE, BGLS, BMML87] with Absolute Bias Criterion $|\hat{\beta}_1|$

Samples	Bias Criterion $ \hat{\beta}_1 $		
	BMLE	BGLS	BMML87
25	6(86%)	7(100%)	6(86%)
50	6(86%)	7(100%)	6(86%)
100	7(100%)	7(100%)	6(86%)
200	7(100%)	6(86%)	6(86%)
500	2(29%)	6(86%)	1(14%)
1000	7(100%)	0(0%)	6(86%)

The Absolute bias for β_1 at various sample sizes indicates that BGLS outperformed all other estimators at sample sizes 25, 50 and 500 with minimum biases, the BMLE and BGLS outperformed other estimators at sample size 100, the BMLE outperformed other estimators at sample size 200 while BMLE outperformed other estimators at sample size 1000. Thus performance at β_1 shows that BGLS has overall best performance compare with all other estimators.

Table 3: Comparison of the Performances of Estimators [BMLE, BGLS, BMML87] with Absolute Bias Criterion $|\hat{\beta}_2|$

Samples	Bias Criterion $ \hat{\beta}_2 $		
	BMLE	BGLS	BMML87
25	6(86%)	6(86%)	7(100%)
50	7(100%)	6(86%)	6(86%)
100	6(86%)	6(86%)	7(100%)
200	7(100%)	6(86%)	6(86%)
500	1(14%)	7(100%)	2 (29%)
1000	6(86%)	1 (14%)	6(86%)

The Absolute bias for β_2 at various sample sizes indicates that BMML87 outperformed all other estimators at sample sizes 25 and 100 with minimum biases, the BGLS outperformed other estimators at sample size 50, the BMLE outperformed other estimators at sample size 200, BGLS outperformed other estimators at sample size 500 whereas BMLE and BMML87 outperformed other estimators at sample size 1000. Thus performance at β_2 shows that BMML87 has overall best performance compare with all other estimators.

Table 4: Comparison of the Performances of Estimators [BMLE, BGLS, BMML87] With Mean Squares Error Criterion [β_0]

Samples	Mean Squares Error Criterion [β_0]		
	BMLE	BGLS	BMML87
25	6(86%)	7(100%)	6(86%)
50	6(86%)	7(100%)	6(86%)
100	6(86%)	7(100%)	6(86%)
200	7(100%)	6(86%)	6(86%)
500	3(43%)	5(71%)	2(29%)
1000	1(14%)	7(100%)	1(14%)

The mean squares error for β_0 at various sample sizes indicates that BGLS outperformed all other estimators at sample sizes 25, 50 and 100 with minimum mean squares error, the BMLE outperformed other estimators at sample size 200, the BGLS outperformed other estimators at sample size 500 and 1000. Thus performance at β_0 shows that BGLS has overall best performance compare with all other estimators.

Table 5: Comparison of the Performances of Estimators [BMLE, BGLS, BMML87] with Mean Squares Error Criterion [β_1]

Samples	Mean Squares Error Criterion% [β_1]		
	BMLE	BGLS	BMML87
25	6(86%)	7(100%)	6(86%)
50	6(86%)	7(100%)	6(86%)
100	6(86%)	7(100%)	6(86%)
200	7(100%)	6(86%)	6(86%)
500	3(43%)	4(57%)	2(29%)
1000	3(43%)	4(57%)	2(29%)

The mean squares error for β_1 at various sample sizes indicates that BGLS outperformed all other estimators at sample sizes 25, 50 and 100 with minimum mean squares error, the BMLE outperformed other estimators at sample size 200, the BGLS outperformed other estimators at sample size 500 and 1000. Thus performance at β_1 shows that BGLS has overall best performance compare with all other estimators.

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Table 6: Comparison of the Performances of Estimators [BMLE, BGLS, BMML87] With Mean Squares Error Criterion [$\hat{\beta}_2$]

Samples	Mean Squares Error Criterion [$\hat{\beta}_2$]		
	BMLE	BGLS	BMML87
25	6(86%)	7(100%)	6(86%)
50	6(86%)	7(100%)	6(86%)
100	7(100%)	6(86%)	6(86%)
200	6(86%)	6(86%)	7(100%)
500	2(29%)	6(86%)	3(43%)
1000	1(14%)	6(86%)	1(14%)

The mean squares error for β_2 at various sample sizes indicates that BGLS outperformed all other estimators at sample sizes 25 and 50, the BMLE outperformed other estimators at sample size 100, the BMML87 outperformed other estimators at sample size 200 whereas the BGLS outperformed other estimators at sample size 500 and 1000. Thus performance at β_2 shows that BGLS has overall best performance compare with all other estimators.

Table 7: Comparison of the Performances of Estimators [BMLE, BGLS, BMML87]

With Mean Squares Error Criterion [$\hat{\beta}_0$] based on Degree of Heteroscedasticity.

Scales	Mean Squares Error Criterion [$\hat{\beta}_0$]		
	BMLE	BGLS	BMML87
0	2(29%)	4(57%)	0
0.3	4(57%)	6(86%)	4(57%)
0.5	4(57%)	6(86%)	4(57%)
0.6	4(57%)	6(86%)	4(57%)
0.9	4(57%)	6(86%)	4(57%)
1	5(71%)	6(86%)	5(71%)
2	6(86%)	4(57%)	6(86%)

The mean squares error for β_0 at various degree of heteroscedasticity indicates that BGLS outperformed all other estimators at 0.0, 0.3, 0.5, 0.6, 0.9 and 1 degree of heteroscedasticity with minimum mean squares error while the BMLE and BMML87 outperformed other estimators at 2 degree of heteroscedasticity with minimum mean squares error. Thus performance at β_0 shows that BGLS has overall best performance compare with all other estimators.

Table 8: Comparison of the Performances of Estimators [BMLE, BGLS, BMML87] With Mean Squares Error Criterion [β_1] based on Degree of Heteroscedasticity.

Scales	Mean Squares Error Criterion [β_1] %		
	BMLE	BGLS	BMML87
0	3(43%)	3(43%)	0
0.3	4(57%)	6(86%)	4(57%)
0.5	4(57%)	6(86%)	4(57%)
0.6	4(57%)	6(86%)	4(57%)
0.9	4(57%)	6(86%)	4(57%)
1	6(86%)	4(57%)	6(86%)
2	6(86%)	4(57%)	6(86%)

The mean squares error for β_1 at various degree of heteroscedasticity indicates that BMLE and BGLS outperformed all other estimators at 0.0 degree of heteroscedasticity with minimum mean squares error, the BGLS outperformed all other estimators at 0.3, 0.5, 0.6 and 0.9 degree of heteroscedasticity with minimum mean squares error while the BMLE and BMML87 outperformed other estimators at 1 and 2 degree of heteroscedasticity. Thus performance at β_1 shows that BGLS has overall best performance compare with all other estimators.

Table 9: Comparison of the Performances of Estimators [BMLE, BGLS, BMML87]

with Mean Squares Error Criterion [β_2] based on Degree of Heteroscedasticity.

Scales	Mean Squares Error Criterion % [β_2]		
	BMLE	BGLS	BMML87
0	1(14%)	4(57%)	2(29%)
0.3	4(57%)	6(86%)	4(57%)
0.5	4(57%)	6(86%)	4(57%)
0.6	4(57%)	6(86%)	4(57%)
0.9	4(57%)	6(86%)	4(57%)
1	5(71%)	6(86%)	5(71%)
2	6(86%)	4(57%)	6(86%)

The mean squares error for β_2 at various degree of heteroscedasticity indicates that BGLS outperformed other estimators at 0.0 degree of heteroscedasticity, the BGLS outperformed all other estimators at 0.3, 0.5, 0.6, 0.9 and 1 degree of heteroscedasticity with minimum mean squares error while the BMLE and BMML87 outperformed other estimators at 2 degree of heteroscedasticity with minimum mean squares error. Thus performance at β_2 shows that BGLS has overall best performance compare with all other estimators.

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CONCLUSION

In this paper I have presented a simple way of modeling and estimating heteroscedastic linear econometric model under simulation technique (MCMC). The paper observed that modeling heteroscedasticity in a full Bayesian improve the precision of the inferences of the parameter estimates. The papers concluded that asymptotically there exist consistency and efficiency in the estimation. The paper compared three different estimators and reported that in the overall Bayesian Generalized Least Squares has robust performance over other estimators, the Bayesian Minimum Message Heteroscedasticity also had robust performance after BGLS. The approach can be applied to other econometrics and other field of Statistics where high dimensional data is the major concern.

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