

STANDARD COLLOCATION AND PERTURBED COLLOCATION METHODS FOR SOLVING LINEAR VOLTERRA INTEGRO DIFFERENTIAL EQUATIONS

K. A. Bello¹, O. A. Taiwo¹, A. Abubakar², A. Abdulkareem¹, M.A. Adeyanju¹ and S. K. Ige¹

¹Department of Mathematics, University of Ilorin, Ilorin, Nigeria

²Department of Mathematical Sciences, Ibrahim Badamasi Babangida University, Lapai, Niger State, Nigeria. (bello.ak@unilorin.edu.ng)

Abstract

This project deals with the numerical approximation of linear fourth order integro differential equations. The numerical methods considered are standard collocation and perturbed collocation methods using shifted chebyshev polynomial as basis function. The result obtained shows that perturbed collocation method proved to have a better approximation than that of standard collocation methods in the cases considered. Three examples are considered to illustrate efficiency method.

Keywords: Standard collocation, Perturbed collocation, Integro - Differential equation

1.0 Introduction

The integro differential equations is an equations that involves both integrals and derivatives of unknown functions. The integro differential equations are often involved in the mathematical formulation of physical and engineering phenomena. These equations arise in many fields like physics, astronomy, potential theory, fluid dynamics, biology models and chemical kinetic. Mathematical modeling of real life problems usually results in functional equations like ordinary differential equations, partial differential equations, integral equations, integro differential equations and stochastic equations. Integro differential equations is an important of modern mathematics and have wide applications in Physics, Mechanics, Heat Transfer, Electrical Circuits, Electro Chemistry, Astronomy, Chemistry and some other many scientific areas.

Integro differential equations (IDEs) are usually solve analytically, thus numerical methods are required. Several researchers have worked in this area in recent years. The numerical methods used by numerical analyst are Goswani, J. C., Chan, A. K. and Chui, C. K. solved first kind integral equations by using wavelets on a bounded interval in (1995), Wavelet Galerkin Compression technique used to solved integro differential equations by Chrysatinos, K. in (2007) and in (2010) Aruchuan, E. and Sulaiman, J. solved second order linear fredholm integro differential equations by using Generalised Minimal Residual techniques.

Furthermore, Wazwaz (2001) proposed a reliable algorithm techniques for solving boundary value problems for higher order integro differential equation. Mixed interpolation collocation

methods for first and second order Volterra linear integro differential equations have been suggested by H. Brunner, A. Makroglou, K. Miller, (1997). Fourth order linear Volterra integro differential equations with constants coefficients are usually difficult to solve analytically, Then a pertubed and collocation numerical methods are used to solve volterra linear integro differential equations to the approximate solutions of the form

$$P_0 y(x) + P_1 y^i(x) + P_2 y^{ii}(x) + P_3 y^{iii}(x) + P_4 y^{iv}(x) = g(x) + \int_0^x k(x,t) y(t) dt$$

With the boundary conditions

$$y^n(a) = \alpha_1, y^n(a) = \alpha_2, y^n(b) = \alpha_3, y^n(b) = \alpha_4 \quad \left. \vphantom{y^n(a)} \right\} \quad (2)$$

Where $P_0, P_1, P_2, P_3, P_4, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ are constants, $g(x)$ and $k(x,t)$ are the given functions and $y(x)$ is to be determined.

In this project, standard collocation method and perturbed collocation method is proposed to solve fourth order Volterra integro differential equations taking shifted chebyshev as a basis functions.

2.0 Methodology

In this project, linear fourth order volterra integro differential equations with boundary conditions is solved using the methods :- standard collocation method and perturbed collocation method

2.1 Standard collocation method

A collocation method is a method for the numerical solution of ordinary differential equation(ODE),partial differential equation(PDE),integral equation and integro differential equations. The ideal is to choose a finite dimensional space of candidate solution(polynomial up to a certain degree) and a number of points in the domain(called collocation point) and to select that solution which satisfies the given equation at the collocation point. This method is used to solve equation (1) by assuming an approximate solution of the form,

$$y_N(x) = \sum_{r=0}^N a_r T_r^*(x) \quad (3)$$

Where $T_n^*(x)$ is a shifted chebyshev polynomials and N is the degree of the approximate $a_r; r \geq 0$ are the unknown constants to be determined. Hence substituting equation (3) into equation (1) and (2), then

$$p_0 y_N(x) + p_1 y_N^i(x) + p_2 y_N^{ii}(x) + p_3 y_N^{iii}(x) + p_4 y_N^{iv}(x) = g(x) + \int_0^x k(x,t) y_N(t) dt$$

Together with the boundary conditions

$$\left. \begin{aligned} y_N^n(a) &= \alpha_1 \\ y_N^n(a) &= \alpha_2 \\ y_N^n(b) &= \alpha_3 \\ y_N^n(b) &= \alpha_4 \end{aligned} \right\} \quad (5)$$

Simplifying equation(4), then

$$p_0 \sum_{r=0}^N a_r T_r^*(x) + p_1 \frac{d}{dx} (\sum_{r=0}^N a_r T_r^*(x)) + p_2 \frac{d^2}{dx^2} (\sum_{r=0}^N a_r T_r^*(x)) + p_3 \frac{d^3}{dx^3} (\sum_{r=0}^N a_r T_r^*(x)) + p_4 \frac{d^4}{dx^4} (\sum_{r=0}^N a_r T_r^*(x)) = g(x) + \int_0^x k(x,t) \sum_{r=0}^N a_r T_r^*(t) dt \quad (6)$$

Then, evaluating equation (6) and then collocating at the point $x = x_i$ to have

$$p_0 \sum_{r=0}^N a_r T_r^*(x_i) + p_1 \frac{d}{dx_i} (\sum_{r=0}^N a_r T_r^*(x_i)) + p_2 \frac{d^2}{dx_i^2} (\sum_{r=0}^N a_r T_r^*(x_i)) + p_3 \frac{d^3}{dx_i^3} (\sum_{r=0}^N a_r T_r^*(x_i)) + p_4 \frac{d^4}{dx_i^4} (\sum_{r=0}^N a_r T_r^*(x_i)) = g(x_i) + \int_0^{x_i} k(x_i,t) \sum_{r=0}^N a_r T_r^*(t) dt \quad (7)$$

Where

$$x_i = a + \frac{(b-a)i}{N+1}, \quad i = 1, 2, \dots, [N - (n - 1)] \quad (8)$$

Where N is the degree of the approximation and n is the order of the differential equation. Thus, equation (7) gives rise to $[N - (n - 1)]$ system of algebraic linear equations in $N + 1$ unknown constants. The extra equations are obtained using the boundary condition given in equation (5). Altogether we have $N + 1$ by $N + 1$ algebraic system of equations, which are then solved by using Gaussian Elimination Methods to obtain the unknown constants $\sum_{r=0}^N a_r$, which are then substitute back into equation(1) to get the approximation solution.

2.2 Perturbedcollocation method

This method is used to solve (1) by assuming an approximate solution of the form

$$y_N(x) = \sum_{r=0}^N a_r T_r^*(x) \quad (9)$$

Where $T_n^*(x)$ is a shifted chebyshev polynomials, N is the degree of the approximate, τ_1 and τ_2 are tau parameters $a_r: r \geq 0$ are the unknown constants to be determined along with tau parameters. Hence substituting equation(9) into equation (1) and (2), then

$$p_0 y_N(x) + p_1 y_N^I(x) + p_2 y_N^{II}(x) + p_3 y_N^{III}(x) + p_4 y_N^{IV}(x) = g(x) + \int_0^x k(x,t) y_N(t) dt + \tau_1 T_{N+1}^*(x) + \tau_2 T_{N+2}^*(x) \quad (10)$$

Together with the boundary conditions

$$\left. \begin{aligned} y_N^n(a) &= \alpha_1 \\ y_N^n(a) &= \alpha_2 \\ y_N^n(b) &= \alpha_3 \\ y_N^n(b) &= \alpha_4 \end{aligned} \right\} \quad (11)$$

Simplifying equation(10), then

$$\begin{aligned} & p_0 \left(\sum_{r=0}^N a_r T_r^*(x) \right) + p_1 \frac{d}{dx} \left(\sum_{r=0}^N a_r T_r^*(x) \right) + p_2 \frac{d^2}{dx^2} \left(\sum_{r=0}^N a_r T_r^*(x) \right) + \\ & p_3 \frac{d^3}{dx^3} \left(\sum_{r=0}^N a_r T_r^*(x) \right) + p_4 \frac{d^4}{dx^4} \left(\sum_{r=0}^N a_r T_r^*(x) \right) = g(x) + \int_0^x k(x, t) \left(\sum_{r=0}^N a_r T_r^*(t) \right) dt + \\ & \tau_1 T_{N+1}^*(x) + \tau_2 T_{N+2}^*(x) \end{aligned} \quad (12)$$

Then, evaluating equation(12) and then collocating at the point $x = x_i$ to have

$$\begin{aligned} & p_0 \left(\sum_{r=0}^N a_r T_r^*(x_i) \right) + p_1 \frac{d}{dx_i} \left(\sum_{r=0}^N a_r T_r^*(x_i) \right) + p_2 \frac{d^2}{dx_i^2} \left(\sum_{r=0}^N a_r T_r^*(x_i) \right) + \\ & p_3 \frac{d^3}{dx_i^3} \left(\sum_{r=0}^N a_r T_r^*(x_i) \right) + p_4 \frac{d^4}{dx_i^4} \left(\sum_{r=0}^N a_r T_r^*(x_i) \right) = g(x_i) + \int_0^{x_i} k(x_i, t) \left(\sum_{r=0}^N a_r T_r^*(t) \right) dt + \\ & \tau_1 T_{N+1}^*(x_i) + \tau_2 T_{N+2}^*(x_i) \end{aligned} \quad (13)$$

Where

$$x_i = a + \frac{(b-a)i}{N}, \quad i = 1, 2, \dots, [N-1] \quad (14)$$

Where N is the degree of the approximation and n is the order of the differential equation. Thus, equation (13) gives rise to $N-1$ system of algebraic linear equations in $N+3$ unknown constants. The extra equations are obtained using the boundary condition given in equation (11). Altogether we have $N+1$ by $N+1$ algebraic system of equations, which are then solved by using Gaussian Elimination Methods to obtain the unknown constants $\sum_{r=0}^N a_r$, which are then substitute back into equation(9) to get the approximation solution.

3.0 ChebyshevPolynomials of the First Kind in the Interval [0,1]

3.1 Shifted Chebyshev polynomial of the first kind

Shifted Chebyshev polynomial are orthogonal on [0,1].we have four different kind of shifted Chebyshev polynomial on [0,1] but we are going to deal the first kind only and it has rodrigue formula

$$T_n^*(x) = \cos(ncos^{-1}(2x-1)) \quad (15)$$

Recurrence relation

Using co-ordinate transformation from $[-1,1]$ to $[0,1]$ and substitute the value into the rodrigue formula we have

$$T_n^*(x) = \cos(ncos^{-1}(2x - 1))$$

let $\theta = \cos^{-1}(2x - 1)$ and $(2x-1) = \theta$ therefore $T_n^*(x) = \cos(n\theta)$

$$T_{n+1}^*(x) = \cos(n+1)\theta \quad (16)$$

$$T_{n+1}^*(x) = \cos n\theta \cos \theta - \sin n\theta \sin \theta$$

$$T_{n-1}^*(x) = \cos(n-1)\theta \quad (17)$$

$$T_{n-1}^*(x) = \cos n\theta \cos \theta + \sin n\theta \sin \theta \quad (18)$$

Adding (16) and (18) we have

$$T_{n+1}^*(x) + T_{n-1}^*(x) = 2\cos n\theta \cos \theta$$

$$T_{n+1}^*(x) + T_{n-1}^*(x) = 2(2x - 1)T_n^*(x)$$

We obtained the recurrence relation formula

$$T_{n+1}^*(x) = 2(2x - 1)T_n^*(x) - T_{n-1}^*(x) \quad (19)$$

These are the terms obtain using the rodrigue and the recurrence relation respectively.

$$T_0^*(x) = 1$$

$$T_1^*(x) = 2x - 1$$

$$T_2^*(x) = 8x^2 - 8x + 1$$

$$T_3^*(x) = 32x^3 - 48x^2 + 18x - 1$$

$$T_4^*(x) = 128x^4 - 256x^3 + 160x^2 - 32x + 1$$

$$T_5^*(x) = 512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1$$

$$T_6^*(x) = 2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1 \text{ and so on.}$$

4.0 Demonstration of the Numerical Method

In this section, we demonstrated the proposed method discussed above on some examples.

Example 1

Consider the fourth order linear volterra integro differential equations

$$y^{iv} = x(1 + e^x) + 3e^x + y(x) - \int_0^x y(t)dtx \in [0,1] \quad (20)$$

Subject to the boundary conditions

$$\left. \begin{aligned} y(0) &= 1 \\ y^i(0) &= 1 \\ y(1) &= 1 + e \\ y^i(1) &= 2e \end{aligned} \right\} \quad (21)$$

The exact solution of the problem is given as

$$y(x) = 1 + xe^x \quad (22)$$

Solution

• Standard Collocation Method

Let the trial solution be

$$y_N(x) = \sum_{r=0}^N a_r T_r^*(x) \quad (23)$$

Taking $N = 4$, therefore equation (23)

$$y_4(x) = \sum_{r=0}^4 a_r T_r^*(x) \cong a_0 T_0^*(x) + a_1 T_1^*(x) + a_2 T_2^*(x) + a_3 T_3^*(x) + a_4 T_4^*(x)$$

Then

$$y_4(x) = a_0 + (2x - 1)a_1 + (8x^2 - 8x + 1)a_2 + (32x^3 - 48x^2 + 18x - 1)a_3 + (128x^4 - 256x^3 + 160x^2 - 32x + 1)a_4 \quad (24)$$

Differentiating (24) four times with respect to independent variable x to have

$$y_4^{iv}(x) = 3072a_4 \quad (25)$$

Substituting equation(25) and (24) into equation(20), then

$$3072a_4 = (1 + e^x) + 3e^x + a_0 + (2x - 1)a_1 + (8x^2 - 8x + 1)a_2 + (32x^3 - 48x^2 + 18x - 1)a_3 + (128x^4 - 256x^3 + 160x^2 - 32x + 1)a_4 - \int_0^x (a_0 + (2t - 1)a_1 + (8t^2 - 8t + 1)a_2 + (32t^3 - 48t^2 + 18t - 1)a_3 + (128t^4 - 256t^3 + 160t^2 - 32t + 1)a_4)dt \quad (26)$$

Then

$$3072a_4 = (1 + e^x) + 3e^x + a_0 + (2x - 1)a_1 + (8x^2 - 8x + 1)a_2 + (32x^3 - 48x^2 + 18x - 1)a_3 + (128x^4 - 256x^3 + 160x^2 - 32x + 1)a_4 - \frac{128}{5}a_4x^5 - \frac{1}{4}(32a_3 - 256a_4)x^4 - \frac{1}{3}(8a_2 - 48a_3 + 160a_4)x^3 - \frac{1}{2}(2a_1 - 8a_2 + 18a_3 - 32a_4)x^2 - a_0x + a_1x - a_2x + a_3x - a_4$$

Using the boundary conditions given we have

$$\begin{aligned} y_4(0) &= 1; \Rightarrow a_0 - a_1 + a_2 - a_3 + a_4 = 1 \\ y_4'(0) &= 1; \Rightarrow 2a_1 - 8a_2 + 18a_3 - 32a_4 = 1 \\ y_4(1) &= 1 + e; \Rightarrow a_0 + a_1 + a_2 + a_3 + a_4 = 1 + e \\ y_4'(1) &= 2e; \Rightarrow 2a_1 + 8a_2 + 18a_3 + 32a_4 = 2e \end{aligned} \quad (28)$$

Now collocating equation (27) at the point

$$x_i = \left(\frac{1}{4}\right) i, \quad i = 1 \quad (29)$$

Putting $i = 1$ in equation (29) and then substituting the value of x_1 into equation (27) to have

$$3072.0a_4 = 4.1020 + 0.75000a_0 - 0.31250a_1 - 0.54167a_2 + 0.90625a_3 - 0.35833a_4 \quad (30)$$

Solving equation (28) and (30) to obtained the unknown constant

$$\left. \begin{aligned} a_0 &= 2.086868704 \\ a_1 &= 1.327890914 \\ a_2 &= 0.2706012043 \\ a_3 &= 0.03125000000 \\ a_4 &= 0.001671006056 \end{aligned} \right\} \quad (31)$$

Substitute (31) into equation (23) to have approximate solution

$$y_4(x) = 1.0 + 1.0x + 0.932170603x^2 + 0.5722224497x^3 + 0.2138887752x^4 \quad (32)$$

• Perturbed Collocation Method

Let the trial solution be

$$y_N(x) = \sum_{r=0}^N a_r T_r^*(x) \quad (33)$$

Taking $N = 4$, therefore equation (33)

$$y_4(x) = \sum_{r=0}^4 a_r T_r^*(x) \cong a_0 T_0^*(x) + a_1 T_1^*(x) + a_2 T_2^*(x) + a_3 T_3^*(x) + a_4 T_4^*(x)$$

Then

$$y_4(x) = a_0 + (2x - 1)a_1 + (8x^2 - 8x + 1)a_2 + (32x^3 - 48x^2 + 18x - 1)a_3 + (128x^4 - 256x^3 + 160x^2 - 32x + 1)a_4 \quad (34)$$

Differentiating (34) four times with respect to independent variable x to have

$$y_4^{iv}(x) = 3072a_4 \quad (35)$$

Substituting equation(34) and (35) into equation(20), then

$$3072a_4 = x(1 + e^x) + 3e^x + a_0 + (2x - 1)a_1 + (8x^2 - 8x + 1)a_2 + (32x^3 - 48x^2 + 18x - 1)a_3 + (128x^4 - 256x^3 + 160x^2 - 32x + 1)a_4 - \int_0^x (a_0 + (2t - 1)a_1 + (8t^2 - 8t + 1)a_2 + (32t^3 - 48t^2 + 18t - 1)a_3 + (128t^4 - 256t^3 + 160t^2 - 32t + 1)a_4)dt + (512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1)\tau_1 + (2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1)\tau_2 \quad (36)$$

Then

$$3072a_4 = x(1 + e^x) + 3e^x + a_0 + (2x - 1)a_1 + (8x^2 - 8x + 1)a_2 + (32x^3 - 48x^2 + 18x - 1)a_3 + (128x^4 - 256x^3 + 160x^2 - 32x + 1)a_4 - \frac{128}{5}a_4x^5 - \frac{1}{4}(32a_3 - 256a_4)x^4 - \frac{1}{3}(8a_2 - 48a_3 + 160a_4)x^3 - \frac{1}{2}(2a_1 - 8a_2 + 18a_3 - 32a_4)x^2 - a_0x + a_1x - a_2x + a_3x - a_4x + (512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1)\tau_1 + (2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1)\tau_2 \quad (37)$$

Using the boundary conditions given we have

$$\begin{aligned} y_4(0) &= 1; \Rightarrow a_0 - a_1 + a_2 - a_3 + a_4 = 1 \\ y_4^i(0) &= 1; \Rightarrow 2a_1 - 8a_2 + 18a_3 - 32a_4 = 1 \\ y_4(1) &= 1 + e; \Rightarrow a_0 + a_1 + a_2 + a_3 + a_4 = 1 + e \\ y_4^i(1) &= 2e; \Rightarrow 2a_1 + 8a_2 + 18a_3 + 32a_4 = 2e \end{aligned} \quad (38)$$

Now collocating equation (37) at the point

$$x_i = \left(\frac{1}{4}\right)i, \quad i = 1, 2, 3 \quad (39)$$

Putting $i = 1$ in equation (39) and then substituting the value of x_1 into equation(37) to have

$$3072.0a_4 = 4.1020 + 0.75000a_0 - 0.31250a_1 - 0.54167a_2 + 0.90625a_3 - 0.35833a_4 - 0.50000\tau_1 + \tau_2 \quad (40)$$

Putting $i = 2$ in equation (39) and then substituting the value of x_2 into equation(37) to have

$$3072.0a_4 = 5.4461 + 0.50000a_0 - 0.83333a_2 + 1.0333a_4 - 0.25000a_3 + 0.25000a_1 - 1.0\tau_2 \quad (41)$$

Putting $i = 3$ in equation (39) and then substituting the value of x_3 into equation(37) to have

$$3072.0a_4 = 7.1010 + 0.25000a_0 + 0.68750a_1 - 0.12500a_2 - 1.0938a_3 - 0.57500a_4 + 0.50000\tau_1 + \tau_2 \quad (42)$$

Solving equation (38),(40),(41) and (42) to obtained the unknown constant

$$\left. \begin{aligned} a_0 &= 2.088395781 \\ a_1 &= 1.327890914 \\ a_2 &= 0.2685651017 \\ a_3 &= 0.03125000000 \\ a_4 &= 0.002180031716 \end{aligned} \right\} \quad (43)$$

Substitute (43) into equation (33) to have approximate solution

$$y_4(x) = 1.000000001 + 0.999999999x + 0.997325889x^2 + 0.4419118807x^3 + 0.2790440596x^4 \quad (44)$$

Example 2

Consider the fourth order linear volterra integro differential equations

$$y^{iv} = g(x) + \int_0^x y(t)dt \quad x \in [0,1] \quad (45)$$

Where $g(x) = -x + 5e^x - 1$ and Subject to the boundary conditions

$$\left. \begin{aligned} y(0) &= 1 \\ y^i(0) &= 1 \\ y(1) &= 1 + e \\ y^i(1) &= 2e \end{aligned} \right\} \quad (46)$$

The exact solution of the problem is given as

$$y(x) = 1 + xe^x \quad (47)$$

Solution

• Standard Collocation Method

Let the trial solution be

$$y_N(x) = \sum_{r=0}^N a_r T_r^*(x) \quad (48)$$

Taking $N = 4$, therefore equation (48)

$$y_4(x) = \sum_{r=0}^4 a_r T_r^*(x) \cong a_0 T_0^*(x) + a_1 T_1^*(x) + a_2 T_2^*(x) + a_3 T_3^*(x) + a_4 T_4^*(x)$$

Then

$$y_4(x) = a_0 + (2x - 1)a_1 + (8x^2 - 8x + 1)a_2 + (32x^3 - 48x^2 + 18x - 1)a_3 + (128x^4 - 256x^3 + 160x^2 - 32x + 1)a_4 \quad (49)$$

Differentiating (49) four times with respect to independent variable x to have

$$y_4^{iv}(x) = 3072a_4 \quad (50)$$

Substituting equation(50) and (49) into equation(45), then

$$3072a_4 = -x + 5e^x - 1 + \int_0^x (a_0 + (2x - 1)a_1 + (8x^2 - 8x + 1)a_2 + (32x^3 - 48x^2 + 18x - 1)a_3 + (128x^4 - 256x^3 + 160x^2 - 32x + 1)a_4)dt$$

Then

$$3072a_4 = -x + 5e^x - 1 + \frac{128}{5}a_4x^5 + \frac{1}{4}(32a_3 - 256a_4)x^4 + \frac{1}{3}(8a_2 - 48a_3 + 160a_4)x^3 + \frac{1}{2}(2a_1 - 8a_2 + 18a_3 - 32a_4)x^2 + a_0x - a_1x + a_2x - a_3x + a_4x \quad (52)$$

Using the boundary conditions given we have

$$\begin{aligned} y_4(0) &= 1; \Rightarrow a_0 - a_1 + a_2 - a_3 + a_4 = 1 \\ y_4^i(0) &= 1; \Rightarrow 2a_1 - 8a_2 + 18a_3 - 32a_4 = 1 \\ y_4(1) &= 1 + e; \Rightarrow a_0 + a_1 + a_2 + a_3 + a_4 = 1 + e \\ y_4^i(1) &= 2e; \Rightarrow 2a_1 + 8a_2 + 18a_3 + 32a_4 = 2e \end{aligned} \quad (53)$$

Now collocating equation (52) at the point

$$x_i = \left(\frac{1}{4}\right)i, \quad i = 1 \quad (54)$$

Putting $i = 1$ in equation (54) and then substituting the value of x_1 into equation(52) to have

$$3072.0a_4 = 5.1700 - 0.14167a_4 + 0.093750a_3 + 0.041667a_2 - 0.18750a_1 + 0.25000a_0 \quad (55)$$

Solving equation (55) and (53) to obtained the unknown constant

$$\left. \begin{aligned} a_0 &= 2.087184545 \\ a_1 &= 1.327890914 \\ a_2 &= 0.2701800824 \\ a_3 &= 0.0312500000 \\ a_4 &= 0.001776286531 \end{aligned} \right\} \quad (56)$$

Substitute (56) into equation (48) to have approximate solution

$$y_4(x) = 0.9999999995 + 1.0x + 0.945646504x^2 + 0.5452706481x^3 + 0.2273646760x^4$$

• Perturbed Collocation Method

Let the trial solution be

$$y_N(x) = \sum_{r=0}^N a_r T_r^*(x) \quad (58)$$

Taking $N = 4$, therefore equation (58)

$$y_4(x) = \sum_{r=0}^4 a_r T_r^*(x) \cong a_0 T_0^*(x) + a_1 T_1^*(x) + a_2 T_2^*(x) + a_3 T_3^*(x) + a_4 T_4^*(x)$$

Then

$$y_4(x) = a_0 + (2x - 1)a_1 + (8x^2 - 8x + 1)a_2 + (32x^3 - 48x^2 + 18x - 1)a_3 + (128x^4 - 256x^3 + 160x^2 - 32x + 1)a_4 \quad (59)$$

Differentiating (59) four times with respect to independent variable x to have

$$y_4^{iv}(x) = 3072a_4 \quad (60)$$

Substituting equation(60) and (59) into equation(45), then

$$3072a_4 = -x + 5e^x - 1 + \int_0^x (a_0 + (2t - 1)a_1 + (8t^2 - 8t + 1)a_2 + (32t^3 - 48t^2 + 18t - 1)a_3 + (128t^4 - 256t^3 + 160t^2 - 32t + 1)a_4)dt + (512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1)\tau_1 + (2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1)\tau_2 \quad (61)$$

Then

$$3072a_4 = -x + 5e^x - 1 + \frac{128}{5}a_4x^5 + \frac{1}{4}(32a_3 - 256a_4)x^4 + \frac{1}{3}(8a_2 - 48a_3 + 160a_4)x^3 + \frac{1}{2}(2a_1 - 8a_2 + 18a_3 - 32a_4)x^2 + a_0x - a_1x + a_2x - a_3x + a_4x + (512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1)\tau_1 + (2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1)\tau_2 \quad (62)$$

Using the boundary conditions given we have

$$\begin{aligned} y_4(0) &= 1; \Rightarrow a_0 - a_1 + a_2 - a_3 + a_4 = 1 \\ y_4^i(0) &= 1; \Rightarrow 2a_1 - 8a_2 + 18a_3 - 32a_4 = 1 \\ y_4(1) &= 1 + e; \Rightarrow a_0 + a_1 + a_2 + a_3 + a_4 = 1 + e \\ y_4^i(1) &= 2e; \Rightarrow 2a_1 + 8a_2 + 18a_3 + 32a_4 = 2e \end{aligned} \quad (63)$$

Now collocating equation (62) at the point

$$x_i = \left(\frac{1}{4}\right)i, \quad i = 1, 2, 3 \quad (64)$$

Putting $i = 1$ in equation (64) and then substituting the value of x_1 into equation(62) to have

$$3072.0a_4 = 5.1700 - 0.14167a_4 + 0.093750a_3 + 0.041667a_2 - 0.18750a_1 + 0.25000a_0 - 0.50000\tau_1 + \tau_2 \quad (65)$$

Putting $i = 2$ in equation (64) and then substituting the value of x_2 into equation(62) to have

$$3072.0a_4 = 6.7435 - 0.033333a_4 + 0.25000a_3 - 0.16667a_2 - 0.25000a_1 + 0.50000a_0 - 1.0\tau_2 \quad (66)$$

Putting $i = 3$ in equation (64) and then substituting the value of x_3 into equation(62) to have

$$3072.0a_4 = 8.8350 + 0.075000a_4 + 0.093750a_3 - 0.37500a_2 - 0.18750a_1 + 0.75000a_0 + 0.50000\tau_1 + \tau_2 \quad (67)$$

Solving equation (63),(65),(66) and (67) to obtained the unknown constant

$$\left. \begin{aligned} a_0 &= 2.089265722 \\ a_1 &= 1.327890914 \\ a_2 &= 0.2674051797 \\ a_3 &= 0.03125000000 \\ a_4 &= 0.002470012216 \end{aligned} \right\} \quad (68)$$

Substitute (68) into equation (58) to have approximate solution

$$y_4(x) = 1.0 + 0.999999999x + 1.034443393x^2 + 0.3676768727x^3 + 0.3161615636x^4 \quad (69)$$

Example 3

Consider the fourth order linear volterra integro differential equations

$$y^{iv} = x(1 + e^x) + 3e^x + y(x) - \int_0^x y(t)dt, x \in [0,1] \quad (70)$$

Subject to the boundary conditions

$$\left. \begin{aligned} y(0) &= 1 \\ y^{ii}(0) &= 2 \\ y(1) &= 1 + e \\ y^i(1) &= 3e \end{aligned} \right\} \quad (71)$$

The exact solution of the problem is given as

$$y(x) = 1 + xe^x \quad (72)$$

Solution

• Standard Collocation Method

Let the trial solution be

$$y_N(x) = \sum_{r=0}^N a_r T_r^*(x) \quad (73)$$

Taking $N = 4$, therefore equation (73)

$$y_4(x) = \sum_{r=0}^4 a_r T_r^*(x) \cong a_0 T_0^*(x) + a_1 T_1^*(x) + a_2 T_2^*(x) + a_3 T_3^*(x) + a_4 T_4^*(x)$$

Then

$$y_4(x) = a_0 + (2x - 1)a_1 + (8x^2 - 8x + 1)a_2 + (32x^3 - 48x^2 + 18x - 1)a_3 + (128x^4 - 256x^3 + 160x^2 - 32x + 1)a_4 \quad (74)$$

Differentiating (74) four times with respect to independent variable x to have

$$y_4^{iv}(x) = 3072a_4 \quad (75)$$

Substituting equation(75) and (74) into equation(70), then

$$3072a_4 = (1 + e^x) + 3e^x + a_0 + (2x - 1)a_1 + (8x^2 - 8x + 1)a_2 + (32x^3 - 48x^2 + 18x - 1)a_3 + (128x^4 - 256x^3 + 160x^2 - 32x + 1)a_4 - \int_0^x (a_0 + (2x - 1)a_1 + (8x^2 - 8x + 1)a_2 + (32x^3 - 48x^2 + 18x - 1)a_3 + (128x^4 - 256x^3 + 160x^2 - 32x + 1)a_4)dt \quad (76)$$

Then

$$3072a_4 = (1 + e^x) + 3e^x + a_0 + (2x - 1)a_1 + (8x^2 - 8x + 1)a_2 + (32x^3 - 48x^2 + 18x - 1)a_3 + (128x^4 - 256x^3 + 160x^2 - 32x + 1)a_4 - \frac{128}{5}a_4x^5 - \frac{1}{4}(32a_3 - 256a_4)x^4 - \frac{1}{3}(8a_2 - 48a_3 + 160a_4)x^3 - \frac{1}{2}(2a_1 - 8a_2 + 18a_3 - 32a_4)x^2 - a_0x + a_1x - a_2x + a_3x - a_4 \quad (77)$$

Using the boundary conditions given we have

$$\begin{aligned} y_4(0) &= 1; \Rightarrow a_0 - a_1 + a_2 - a_3 + a_4 = 1 \\ y_4^{ii}(0) &= 2; \Rightarrow 16a_2 - 96a_3 + 320a_4 = 2 \\ y_4(1) &= 1 + e; \Rightarrow a_0 + a_1 + a_2 + a_3 + a_4 = 1 + e \\ y_4^{ii}(1) &= 3e; \Rightarrow 16a_2 + 96a_3 + 320a_4 = 3e \end{aligned} \quad (78)$$

Now collocating equation (??) at the point

$$x_i = \left(\frac{1}{4}\right)i, \quad i = 1 \quad (79)$$

Putting $i = 1$ in equation (79) and then substituting the value of x_1 into equation(77) to have

$$3072.0a_4 = 4.4230 + 0.75000a_0 - 0.31250a_1 - 0.54167a_2 + 0.90625a_3 - 0.35833a_4 \quad (80)$$

Solving equation (79) and (80) to obtained the unknown constant

$$\left. \begin{aligned} a_0 &= 2.075451441 \\ a_1 &= 1.327084427 \\ a_2 &= 0.2819184493 \\ a_3 &= 0.03205648690 \\ a_4 &= 0.001771023604 \end{aligned} \right\} \quad (81)$$

Substitute (81) into equation (73) to have approximate solution

$$y_4(x) = 0.9999999997 + 0.919165269x + 1.0x^2 + 0.5724255384x^3 + 0.2266910213x^4$$

• Perturbed Collocation Method

Let the trial solution be

$$y_N(x) = \sum_{r=0}^N a_r T_r^*(x) \quad (83)$$

Taking $N = 4$, therefore equation (83)

$$y_4(x) = \sum_{r=0}^4 a_r T_r^*(x) \cong a_0 T_0^*(x) + a_1 T_1^*(x) + a_2 T_2^*(x) + a_3 T_3^*(x) + a_4 T_4^*(x)$$

Then

$$y_4(x) = a_0 + (2x - 1)a_1 + (8x^2 - 8x + 1)a_2 + (32x^3 - 48x^2 + 18x - 1)a_3 + (128x^4 - 256x^3 + 160x^2 - 32x + 1)a_4 \quad (84)$$

Differentiating (84) four times with respect to independent variable x to have

$$y_4^{iv}(x) = 3072a_4 \quad (85)$$

Substituting equation(84) and (85) into equation(70), then

$$\begin{aligned} 3072a_4 = & x(1 + e^x) + 3e^x + a_0 + (2x - 1)a_1 + (8x^2 - 8x + 1)a_2 + \\ & (32x^3 - 48x^2 + 18x - 1)a_3 + (128x^4 - 256x^3 + 160x^2 - 32x + 1)a_4 - \int_0^x (a_0 + \\ & (2t - 1)a_1 + (8t^2 - 8t + 1)a_2 + (32t^3 - 48t^2 + 18t - 1)a_3 + (128t^4 - 256t^3 + 160t^2 - \\ & 32t + 1)a_4)dt + (512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1)\tau_1 + (2048x^5 - \\ & 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1)\tau_2 \end{aligned} \quad (86)$$

Then

$$3072a_4 = x(1 + e^x) + 3e^x + a_0 + (2x - 1)a_1 + (8x^2 - 8x + 1)a_2 + (32x^3 - 48x^2 + 18x - 1)a_3 + (128x^4 - 256x^3 + 160x^2 - 32x + 1)a_4 - \frac{128}{5}a_4x^5 -$$

$$\frac{1}{4}(32a_3 - 256a_4)x^4 - \frac{1}{3}(8a_2 - 48a_3 + 160a_4)x^3 - \frac{1}{2}(2a_1 - 8a_2 + 18a_3 - 32a_4)x^2 - a_0x + a_1x - a_2x + a_3x - a_4x + (512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1)\tau_1 + (2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1)\tau_2 \quad (87)$$

Using the boundary conditions given we have

$$\begin{aligned} y_4(0) &= 1; \Rightarrow a_0 - a_1 + a_2 - a_3 + a_4 = 1 \\ y_4^{ii}(0) &= 1; \Rightarrow 16a_2 - 96a_3 + 320a_4 = 2 \\ y_4(1) &= 1 + e; \Rightarrow a_0 + a_1 + a_2 + a_3 + a_4 = 1 + e \\ y_4^{ii}(1) &= 2e; \Rightarrow 16a_2 + 96a_3 + 320a_4 = 3e \end{aligned} \quad (88)$$

Now collocating equation (87) at the point

$$x_i = \left(\frac{1}{4}\right)i, \quad i = 1, 2, 3 \quad (89)$$

Putting $i = 1$ in equation (89) and then substituting the value of x_1 into equation(87) to have

$$3072.0a_4 = 4.4230 + 0.75000a_0 - 0.31250a_1 - 0.54167a_2 + 0.90625a_3 - 0.35833a_4 - 0.50000\tau_1 + \tau_2 \quad (90)$$

Putting $i = 2$ in equation (89) and then substituting the value of x_2 into equation(87) to have

$$3072.0a_4 = 6.2704 + 0.50000a_0 - 0.83333a_2 + 1.0333a_4 - 0.25000a_3 + 0.25000a_1 - 1.0\tau_2 \quad (91)$$

Putting $i = 3$ in equation (89) and then substituting the value of x_3 into equation(87) to have

$$3072.0a_4 = 8.6888 + 0.25000a_0 + 0.68750a_1 - 0.12500a_2 - 1.0938a_3 - 0.57500a_4 + 0.50000\tau_1 + \tau_2 \quad (92)$$

Solving equation (88),(90),(91) and (92) to obtained the unknown constant

$$\left. \begin{aligned} a_0 &= 2.088724942 \\ a_1 &= 1.327084427 \\ a_2 &= 0.2679463436 \\ a_3 &= 0.03205648690 \\ a_4 &= 0.002469628893 \end{aligned} \right\} \quad (93)$$

Substitute (93) into equation (83) to have approximate solution

$$y_4(x) = 1.000000001 + 1.008586744x + 1.000000001x^2 + 0.3935825844x^3 + 0.3161124983x^4 \quad (94)$$

5.0 Table of Results

In this section, the results obtained in the Numerical examples are represented in tabular form.

Table 1: Result of example 1 and errors obtained, $N=4$

x	Exact	Standard Collocation	Error	Perturbed collocation	Error
0.0	1.000000000	1.000000000	0.0000	1.000000001	1×10^{-9}
0.1	1.110517092	1.109915317	6.01×10^{-4}	1.110443076	7.40×10^{-5}
0.2	1.244280552	1.242206826	2.07×10^{-3}	1.243874802	4.05×10^{-4}
0.3	1.404957642	1.401077859	3.87×10^{-3}	1.403951209	1.01×10^{-3}
0.4	1.596729879	1.591245086	5.48×10^{-3}	1.594998031	1.73×10^{-3}
0.5	1.824360636	1.817938505	6.42×10^{-3}	1.822010711	2.34×10^{-3}
0.6	2.093271280	2.086901451	6.37×10^{-3}	2.090654396	2.62×10^{-3}
0.7	2.409626895	2.404390591	5.24×10^{-3}	2.407263940	2.36×10^{-3}
0.8	2.780432742	2.777175922	3.26×10^{-3}	2.778843899	1.59×10^{-3}
0.9	3.213642800	3.212540779	1.10×10^{-3}	3.213068538	5.74×10^{-4}
1.0	3.718281828	3.718281828	0.0000	3.718281830	2×10^{-9}

Table 2: Result of example 2 and errors obtained, $N=4$

x	Exact	Standard Collocation	Error	Perturbed collocation	Error
0.0	1.000000000	0.999999995	1×10^{-9}	1.000000000	0.0000
0.1	1.110517092	1.110024472	4.92×10^{-4}	1.110743727	2.26×10^{-4}
0.2	1.244280552	1.242551808	1.72×10^{-3}	1.244825010	5.44×10^{-4}
0.3	1.404957642	1.401672146	3.29×10^{-3}	1.405588090	6.30×10^{-4}
0.4	1.596729879	1.592021298	4.71×10^{-3}	1.597135999	4.06×10^{-4}
0.5	1.824360636	1.818780749	5.58×10^{-3}	1.824330555	3.01×10^{-5}
0.6	2.093271280	2.087677663	5.59×10^{-3}	2.092792363	4.79×10^{-4}
0.7	2.409626895	2.404984878	4.64×10^{-3}	2.408900820	7.26×10^{-4}
0.8	2.780432742	2.777520906	2.91×10^{-3}	2.779794106	6.39×10^{-4}
0.9	3.213642800	3.212649934	9.93×10^{-4}	3.213369189	2.74×10^{-4}
1.0	3.718281828	3.718281828	0.0000	3.718281829	1×10^{-9}

Table 3: Result of example 3 and errors obtained, $N=4$

x	Exact	Standard Collocation	Error	Perturbed collocation	Error
0.0	1.000000000	0.999999997	3×10^{-9}	1.000000001	1×10^{-9}
0.1	1.110517092	1.102511622	8.01×10^{-3}	1.111283869	7.67×10^{-4}
0.2	1.244280552	1.228775164	1.54×10^{-2}	1.245371791	1.09×10^{-3}
0.3	1.404957642	1.383041267	2.19×10^{-2}	1.405763265	8.06×10^{-4}
0.4	1.596729879	1.570104631	2.66×10^{-2}	1.596716464	1.34×10^{-5}
0.5	1.824360636	1.795304015	2.91×10^{-2}	1.823248227	1.11×10^{-3}
0.6	2.093271280	2.064522233	2.88×10^{-2}	2.091134065	2.14×10^{-3}
0.7	2.409626895	2.384186162	2.54×10^{-2}	2.406908159	2.72×10^{-3}
0.8	2.780432742	2.761266733	1.92×10^{-2}	2.777863359	2.57×10^{-3}
0.9	3.213642800	3.203278939	1.04×10^{-2}	3.212051186	1.59×10^{-3}
1.0	3.718281828	3.718281828	0.0000	3.718281828	0.0000

6.0 Conclusions

In this project, the Standard collocation Method and Perturbed collocation method are used to solve some numerical examples on fourth order linear integro-differential equations using the shifted Chebyshev polynomials as basis function. From the table of results presented, we observe that Perturbed collocation method gives a better approximation compared to Standard collocation Method. This shows that the Perturbed collocation method is more suitable for solving the classes of fourth order linear integro-differential problems discussed in this project.

References

- Abbasbandy, S., & Shivanian, E. (2009). *Application of Variational Iteration Method for nth-Order Integro-Differential Equations*. Journal of Applied Mathematics Computation. 64(1), 439-444.
- Hashim, I. (2006). *Adomian decomposition method for solving BVPs for fourth-order integro-differential equations*. Journal of Computational and Applied Mathematics. 193(2), 658-664.
- Kythe, P. K., & Puri, P. (2002). *Computational methods for linear integroequations*. University of New Orleans, New Orleans, USA.
- Laeli, D. H., & Maalek-Ghaini, F. M. (2012). *Numerical solution of Volterra-Fredholm integral equations by moving least square method and Chebyshev polynomials*. Applied Mathematics Modeling. 3(6), 3283-3288.

Sweilam, N. H., Khader M. M., & Kota, Y. W. (2013). *Numerical and Analytical Study for Fourth-Order Integro-Differential Equations Using a Pseudospectral Method*. Journal of Mathematical Problems in Engineering. 7(2), 434-753.

Wazwaz, A. M. (2011). *A reliable algorithm for solving boundary value problems for higher-order integro-differential equations*. Journal of Applied Mathematics Computation. 11(8), 327-342.