

COLLOCATION TECHNIQUES FOR SOLVING MULTI-
ORDER FRACTIONAL DIFFERENTIAL AND
INTEGRO-DIFFERENTIAL EQUATIONS.

BY

UWAHEREN, Ohigweren Airenoni
MATRIC NO: 86/10613

BEING A THESIS SUBMITTED TO THE DEPARTMENT OF
MATHEMATICS, FACULTY OF PHYSICAL SCIENCES, UNIVERSITY
OF ILORIN, ILORIN, NIGERIA,

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE
AWARD OF DOCTOR OF PHILOSOPHY (Ph.D.) DEGREE IN
MATHEMATICS.

JUNE, 2018.

Certi cation

This is to certify that the research work reported in this thesis was done by UWAHEREN, Ohigweren Airenoni with Matriculation number 86/10613 in the Department of Mathematics, University of Ilorin, Ilorin, Nigeria, for the award of a Doctor of Philosophy Degree in Mathematics.

Prof. O. A Taiwo
Supervisor

Date

Dr. Olubunmi. A. Fadipe-Joseph
Departmental Postgraduate Coordinator

Date

Prof. R. B. Adeniyi
Head of Department

Date

Internal Examiner

Date

External Examiner

Date

Dedication

This research work is dedicated to Almighty God who has been there for me. He sustained me and provided me the needed wisdom, knowledge and understanding throughout the period of the study.

I also dedicate it to the memories of my late parents Mr. Amedu Uwaheren Ahonsi and Mrs. Grace Amedu Uwaheren Ahonsi.

Acknowledgments

I wish to register my profound thanks and gratitude to Almighty God for his mercies, favour and grace that were sufficient for me all through the programme. He gave me endurance and patience to withstand all the challenges faced to this level.

My sincere gratitude and appreciation go to my wonderful supervisor, Prof. O. A. Taiwo, for his words of advice and for sacrificing his time to supervise me. His words of encouragement greatly propelled me. I wholeheartedly appreciate his contribution to the successful completion of this study. I pray that God will continue to bless and strengthen him and his entire family.

My profound gratitude goes to the Head of Department, Prof. R. B. Adeniyi, for his useful contributions toward the success of this thesis. I also sincerely thank the departmental post-graduate coordinator, Dr. (Mrs) O. A. Fadipe-Joseph for her efforts that made the thesis successful.

I wish to appreciate the entire staff of the department of Mathematics, University of Ilorin very profoundly. They are Prof. J. A. Gbadeyan, Prof. T. O. Opoola, Prof. O. M. Bamigbola, Prof. M. O. Ibrahim, Prof. O. A. Taiwo, Prof. R. B. Adeniyi, Dr. K. Rauf, Dr. A. S. Idowu, Dr. M. S. Dada, Dr. K. O Babalola, Dr. S. O. Makanjuola, Dr. E. O. Titiloye, Dr. (Mrs) O. A. Fadipe-Joseph, Dr. (Mrs) Y. O. Aderinto, Dr. (Mrs) C. N. Ejieji, Dr. B. M. Yisa, Dr. J. U. Abubakar, Dr. K. A. Bello, Dr. (Mrs) G. N. Bakare, Dr. (Mrs) H. O. Ibraheem, Dr. O. T. Olotu, Dr. B. M. Ahmed, Dr. (Mrs) I. F. Usamot, Mr. O. Odetunde, Mr. Y. A Ayinla and the following non-Academic staff: Mrs. F. Y. Aderibigbe, Mrs. T. A. Oyedepo, Mrs. F. Otonekwu, Mr. J. Olonka and Mrs. A. Abdulsalam for all their efforts to bring out the best in me.

I am immensely grateful to my parents late Mr. Ahonsi Uwaheren Amedu and late Mrs. Grace Amedu Uwaheren Ahonsi for not only giving birth to me but for also setting my feet on the path of greatness.

The wonderful understanding, support and prayers of my darling wife, Mrs Roseline Opeyemi Uwaheren and my children; Mr. Precious Ovbare Uwaheren, Miss. Comfort Omole Uwaheren and Master Emmanuel Oyaregba Uwaheren cannot go unappreciated. I cannot thank them enough for standing by me during the 'dark and thin' times we have gone through. I say big thanks to them and pray for brighter moments ahead.

I wish to express my profound appreciation to my big brother and his family, Mr and Mrs. Ahonsi and family for the positive role they played in my life. My gratitude also goes to my other siblings and their families; Mrs Abibetu Musa, Mrs Iroegwan Eric, Mr and Mrs Peter .E. Uwaheren, Mr and Mrs Abel .O. Uwaheren, Mrs Blessing (Osen) Irogho and Mr and Mrs Sunday .I. Uwaheren. I say thanks and pray for long life and prosperity. I must express my gratitude to my friends and their families among whom are Dr (Mrs). A. F. Adebisi, Elder and Dns. S.O. Adewoye, Elder and Dns. Oladigbo, Elder and Dns. Odeyemi, Elder and Mrs. Jolayemi, Mr and Mrs Olaogun, Pastor and Mrs Adedeji and others too numerous to mention, who have contributed in one way or the other to the success of this course. May Almighty God bless them all (Amen).

ABSTRACT

Multi-order fractional differential and integro-differential equations are increasingly being used to model many physical phenomena which occur in fluid mechanics, engineering and other areas of science. Unfortunately, exact solutions do not exist for most of this class of problems in closed form. The necessity to find approximate solutions to such problems thus arises. Therefore, an approximate solutions through numerical treatment is desired. The Multi-order fractional integro-differential equations considered in this work are generally in the form:

$$D_{x_1}^{\alpha} (y(x)) + \sum_{i=0}^n p_i y^{(i)}(x) + J^{\alpha} (y(x)) = f(x)$$

subject to the conditions; $y^{(k)}(0) = \kappa_k$; $k = 0; 1; 2; \dots; n-1$
 where D and J denote the fractional order derivative and integral of the function $y(x)$ respectively. The objectives of the study were to: (i) construct new orthogonal polynomials for the general class of Multi-order Fractional Differential Equations (MFDEs) and Multi-order Fractional Integro Differential Equations (MFIDEs); (ii) use Standard and Perturbed Collocation methods to solve the class of problems in (i); (iii) examine the absolute estimated errors; and (iv) examine the accuracy of the proposed methods by comparing the results obtained with exact solutions where such exist. The use of Standard Collocation Method (SCM) and Perturbed Collocation Method (PCM) to find the numerical solution of multi-order fractional differential and integro-differential equations using the constructed orthogonal polynomials as basis function in the approximation used were tested. Some numerical examples were considered to illustrate the accuracy of the methods. From the computational view, the proposed methods are reliable and in total agreement with some known results in the literature. The results obtained include:

- (i) new orthogonal polynomials basis functions were constructed;
- (ii) Standard Collocation Method (SCM) successfully solved linear and nonlinear multi-order fractional differential and integro-differential equations;
- (iii) Perturbed Collocation Method (PCM) successfully solved linear and nonlinear multi-order fractional differential and integro-differential equations;
- (iv) the absolute estimated errors closely approximate the exact solution. The study concluded that Standard and Perturbed Collocation Methods serve as alternative and additional ways of finding numerical solutions of multi-order fractional differential and integro-differential

equations. The use of newly constructed orthogonal polynomial as basis function for solving multi-order fractional differential and integro-differential equations is recommended, and also, the proposed methods can be extended to solve integral and partial fractional order differential equations.

Contents

Title page	i
Certi cation	ii
Dedication	iii
Acknowledgments	iv
TABLE OF CONTENTS	x
1 General Introduction	1
1.1 Background of the Study	1
1.2 De nition of Relevant Terms	3
1.2.1 Orthogonality	3
1.2.2 Weight function	3
1.2.3 Fractional Calculus	4
1.2.4 Fractional Derivative	5
1.2.5 Left and Right Fractional Derivatives	6
1.2.6 Fractional Di erential Equation	6
1.2.7 Multi-order Fractional Di erential Equations	6
1.2.8 Fractional Integrals	7
1.2.9 Fractional Integro-di erential Equations	9
1.2.10 Multi-order Fractional Integro-di erential Equations	9
1.2.11 Initial Value Problems	10
1.2.12 Solution Integral Equations	10
1.2.13 Exact Solution	10
1.2.14 Approximate Solution	11
1.2.15 Absolute, Relative and Percentage Errors	11
1.2.16 Error Estimation	12
1.2.17 Gamma and Beta Functions	12

1.2.18	Gamma Function	12
1.2.19	Beta Function	12
1.3	Aim and Objectives of the Study	13
1.4	Significance of the Study	13
1.5	Outline of the Thesis	14
2	Literature Review	15
2.1	Preamble	15
2.2	Methods of Solution	16
2.2.1	Collocation method (CM)	16
2.2.2	Successive /Picard Approximation Method	17
2.2.3	Perturbed Collocation Method (PCM)	17
2.2.4	Differential Transform Method (DTM)	17
2.2.5	Iterative Decomposition Method (IDM)	18
2.2.6	Variational Iterative Method (VIM)	18
2.2.7	Homotopy Perturbation Method (HPM)	19
2.2.8	Homotopy Analysis Method (HAM)	19
2.2.9	Adomian Decomposition Method (ADM)	19
2.2.10	Laplace Decomposition Method (LDM)	20
2.2.11	Least Squares Method (LSM)	20
2.3	Orthogonal functions	22
2.3.1	Chebyshev polynomial	22
2.3.2	Legendre polynomial	24
2.3.3	Hermite Polynomial	25
2.3.4	Bernstein Polynomial	26
3	Standard Collocation Method for Solving Multi-order Fractional Differential and Integro-differential Equations	27
3.1	Introduction	27
3.2	Construction of Orthogonal Polynomials	27
3.3	Demonstration of the Standard Collocation Method	33
3.4	Numerical Examples	35

4	Perturbed Collocation Method for Solving Multi-order Fractional Differential and Integro-differential Equations	59
4.1	Preamble	59
4.2	The Perturbed Collocation Method	59
4.3	Numerical Examples	61
4.4	Table of Results	86
5	Conclusion, Contribution to Knowledge and Recommendations	107
5.1	Conclusion	107
5.2	Discussion of Results	108
5.3	Contribution to Knowledge	108
5.4	Recommendations for Further Study	109
	References	110

Chapter 1

General Introduction

1.1 Background of the Study

The role of differential equations cannot be over emphasized in the representation and interpretation of real life problems as most of them are formulated using differential equations. They play relevant roles in the field of science and engineering technology. Fractional differential equations (differential equations where the derivatives are of fractional order) also play very important roles in every field of science. Hashim Abdulaziz and Momeni (2009) revealed that fractional differential equations have a lot of applications in many problems in science and engineering technology. Much earlier, He (1999) stated that the Fluid Dynamic Traffic Model (FDTM) was developed based on fractional differential calculus. Besides, fractional calculus is applicable in the modeling of physical occurrences such as polymers and the description of deformation and flow properties of rock materials (Kilbas et al.,(2006)). According to Weilbeer (2005), the field of fractional calculus is as old as calculus itself, but over the last decades the usefulness of this mathematical theory in applications as well as its merits in pure mathematics has become more and more evident. Most formulations of mathematical models of physical problems lead to linear or nonlinear differential equations. Many of these differential equations do not have analytical solution. To solve these types of equations, we have to employ numerical techniques and numerical collocation methods have been found to be very useful.

From Fractional Calculus, which is the branch of mathematical analysis that studies the possibility of taking fractional powers of differential operators and from the fundamental theorem of classical calculus, we deduce that

$$D \frac{d}{dt} f(t) = f^0(t) \quad (1.1)$$

where D represents fractional operator of $f(t)$.

If $f(t) = t^r$, then

$$\frac{d^n}{dt^n} t^r = \frac{r!}{(r-n)!} t^{r-n} \quad (1.2)$$

Rewriting equation (1.2) using gamma function, gives

$$\frac{d^n}{dt^n} t^r = \frac{\Gamma(r+1)}{\Gamma(r-n+1)} t^{r-n}; \quad r \geq 0 \quad (1.3)$$

Hence, if $r = 1$ and $n = \frac{1}{2}$, we obtained the half derivative of the function t as

$$\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} t^1 = \frac{\Gamma(1+1)}{\Gamma(1+1-\frac{1}{2})} t^{1-\frac{1}{2}} = \frac{2^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} t^{\frac{1}{2}} = \frac{1}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} t^{-\frac{1}{2}} \quad (1.4)$$

(see Weilbeer (2005)). Also, denoting J_a as an integration operator that maps a function f onto its primitive and centered at 'a' then:

$$J_a f(t) = \int_a^x f(t) dt \quad a < t < b \quad (1.5)$$

If $n \in \mathbb{N}$, we use the symbols D^n and J^n to denote the n -fold iterates of D and J_a respectively. Fractional differential equation, also known as extraordinary differential equation is a generalization of differential equations with the application of fractional calculus. According to Podlubny (1999), fractional calculus does not mean the calculus of fractions, neither does it mean the fractions of any calculus (differential or integral) but the theory of integrals and derivatives of arbitrary order, which unify and generalize the notions of integer-order differentiation and n -fold integration. Generally, it is denoted by $D f(t)$. Simply put, a fractional differential equation of order α , is an equation of the form:

$$D y(x) = f(x; y(x)) \quad a < x < b \quad (1.6)$$

The concept of fractional differential equations has been defined by many researchers, but two commonly used are that of Riemann liouville and Caputo definitions. The Riemann-Liouville differential operator is denoted as D while the Caputo differential operator is given as D_c . Fractional Calculus is a generalization of ordinary differentiation and integration to an arbitrary order. This class of calculus has gained tremendous attention in recent years as researchers utilize it in solving problems in the field of Physics, Chemistry, Engineering and Technology. It is worth noting that most problems of fractional integro differential equation

can not be solved analytically. Hence good approximate solutions using numerical methods are usually employed to solve such problems. To this end, many numerical techniques have been advanced to include Homotopy Analysis Method (HAM), Adomian Decomposition Method (ADM), Perturbation Collocation Method (PCM), Variational Iterative Method (VIM), Laplace Decomposition Method (LDM), Iterative Decomposition Method (IDM), Galerkin Method (GM), Collocation Method (CM), Successive Substitution Method (SSM), and so on. According to Gradimir (1991), orthogonal polynomial functions are very useful in many problems of approximation theory, mathematical and numerical analysis and their applications. Orthogonal polynomials are special functions of mathematical physics which are important research subject for pure and applied mathematics, and their applications.

1.2 Definition of Relevant Terms

Here, we present some relevant definitions which are very useful in this work.

1.2.1 Orthogonality

Two functions, say $y_n(x)$ and $y_m(x)$ defined on the interval $a \leq x \leq b$ are orthogonal if; the inner product of the functions is zero i.e

$$\langle y_n(x); y_m(x) \rangle = \int_a^b y_n(x)y_m(x)dx = 0 \quad (1.7)$$

If, on the other hand, a third function $w(x) > 0$ exists, then we have:

$$\langle y_n(x); y_m(x) \rangle = \int_a^b w(x)y_n(x)y_m(x)dx = 0 \quad (1.8)$$

where $y_n(x)$ and $y_m(x)$ are mutually orthogonal with respect to the weight function $w(x)$ and in the interval $[0,1]$, we have:

$$\int_0^1 w(x)y_n(x)y_m(x)dx = \begin{cases} 0 & \text{if } n \neq m \\ \delta_{nm} & \text{if } n = m \end{cases} \quad (1.9)$$

1.2.2 Weight function

a weight function is a mathematical device used when performing a sum, integral or average to give some elements more influence on the same set. The result of this application of weight function is a weighted sum or weighted average. (Wikipedia)

1.2.3 Fractional Calculus

According to Weilbeer (2005), fractional calculus is almost as old as calculus itself. The fact remains that until recently, the eld was not well known and used by many researchers. Until 1823, fractional calculus operations was not used to solve speci c physical problem when Abel applied it to solve an integral equation arising in the Tautochrone problem (Weilbeer, (2005)). The problem has to do with the physical law that the Potential energy lost during the descent of a particle is equal to the Kinetic energy it gains.

$$\frac{1}{2} m \frac{d^2}{dt^2} = mg(y_0 - y)$$

where m is mass, is distant of particle from starting point, g is gravitational acceleration and t is the time parameter. By changing some variables and further simplifications, the above equation is transformed to

$$k = \int_0^x f(t) dt \quad (1.10)$$

where f(t) is the function to be determined. Multiplying both sides by $\frac{1}{\Gamma(\frac{1}{2})}$, we obtained

$$\frac{k}{\Gamma(\frac{1}{2})} = \frac{1}{\Gamma(\frac{1}{2})} \int_0^x f(t) dt = \frac{1}{\Gamma(\frac{1}{2})} \int_0^x f(t) dt \quad (1.11)$$

multiplying by the left-inverse of the fractional derivative, gives

$$f(t) = \frac{1}{\Gamma(\frac{1}{2})} \frac{d^{\frac{1}{2}} k}{dx^{\frac{1}{2}}} = \frac{k}{x^{\frac{1}{2}}} \quad (1.12)$$

Lacroix (1819) used the generalized factorial given by Euler's Gamma function, to formulate non-integer order derivatives and gets

$$\frac{d^n}{dx^n} y(x) = \frac{(m+1)}{\Gamma(n-m)} x^{n-m} \quad (1.13)$$

when $y(x) = x^m$, $m = \frac{1}{2}$ and $n = 1$ then equation (2.6) is represented in a closed form as

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} y(x) = \frac{(2)}{\Gamma(\frac{3}{2})} x^{\frac{1}{2}} \quad (1.14)$$

Weilbeer (2005) noted that the mathematical relevance of Lacroix's work lies in the fact that the special result obtained then is the same as today's representation of the fractional derivatives of Riemann-liouville type. While working to generalize the Taylor's series, Riemann in 1847 (see Weilbeer(2005)) ended up deducing a definition for fractional integral of order for a function f(x).

$$D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t) dt \quad (1.15)$$

and Liouville, much later improved on Riemann's work and came up with the relation

$${}_c D_x^n f(x) = {}_c D_x^{n-1} f(x) = \dots = {}_c D_x f(x) = \frac{1}{\Gamma(n)} \int_c^x (x-t)^{n-1} f(t) dt \quad (1.16)$$

which is known as the Riemann-Liouville fractional derivative to date. Additional results of fractional calculus were presented in the twentieth century by Caputo (1967) and fractional differential equation of a function $f(x)$ with $(m-1)$ absolute continuous derivatives was there defined as

$$D f(x) = \frac{1}{\Gamma(m)} \int_0^x (x-t)^{m-1} \frac{d^m f(t)}{dt^m} dt \quad (1.17)$$

Caputo (1967) concluded that the advantage of the Caputo's fractional derivative is that one can specify the initial conditions of the fractional differential equation in classical form. i.e.

$$D^{(k)} y(k) = b_k; \quad k = 0; 1; 2; \dots; m-1 \quad (1.18)$$

We can however write the Caputo's fractional derivative of order $\alpha > 0$ as:

$$D f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m f(\tau)}{d\tau^m} d\tau \quad m-1 < \alpha < m \quad (1.19)$$

1.2.4 Fractional Derivative

Simply put, fractional derivative is non-integer derivative of a function. It involves computing the differentiation of a function to any non-integer order. Podlubny (1999) while discussing the usefulness and advantages of fractional derivatives said that it provides a very good instrument for the description of memory and hereditary properties of many materials and processes. Furthermore noticed that it is an interdisciplinary concept with advantages that are more noticeable in the modeling of mechanical and electrical properties of real materials. Weibull (2005) corroborated and said that the most important properties of any fractional derivative in applications is its non-local character and correspondingly its memory effect on materials.

Now, let $\alpha \in \mathbb{R}_+$ and $n = \lceil \alpha \rceil$, the operator, D_a^α which is defined as

$$(D_a^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t) dt \quad (1.20)$$

for $a \leq x \leq b$, is called Riemann-Liouville's differential operator of order α .

and similarly if $\alpha \in \mathbb{R}_+$ and $n = \lceil \alpha \rceil$ then, the operator D_a^α is defined by

$$(D f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n f(t)}{dt^n} dt \quad (1.21)$$

for $a < b$, is called Caputo differential operator of order α . It is noted therefore that Caputo differential operator also uses the Riemann-Liouville's integral operator, but the order of application of the differentiation and integration that are interchanged. For Riemann-Liouville, we differentiate before integration while we integrate before differentiation in Caputo.

1.2.5 Left and Right Fractional Derivatives

Suppose the function $f(x)$ is defined in the interval $[a, b]$, then we say that the fractional derivatives of the function with fixed lower and upper limits are:

$${}_a D_t^\alpha f(x) \quad \text{and} \quad {}_t D_b^\alpha f(x)$$

where $a < x < b$ in the interval $[a, b]$. They are referred to as past and future derivatives of $f(x)$ respectively. They could be defined similarly as:

$${}_a D_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \frac{d^\alpha}{dx^\alpha} \int_a^x (x-t)^{\alpha-1} f(t) dt \quad (1.22)$$

and,

$${}_x D_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \frac{d^\alpha}{dx^\alpha} \int_x^b (t-x)^{\alpha-1} f(t) dt \quad (1.23)$$

1.2.6 Fractional Differential Equation

A differential equation is called a fractional differential equation if it contains at least one fractional order derivative, D^α of the unknown function $y(x)$. The general form of a fractional differential equation is given as:

$$D^\alpha y(x) = f(x; y(x))$$

subject to the conditions: $y(0) = y_k; \quad k = 0; 1; \dots; n$

where D^α is the fractional order derivative in the Caputo sense and α is a non-integer value and $\alpha > 0$

1.2.7 Multi-order Fractional Differential Equations

It is a differential equation with more than one power of different fractional order. The general form of multi-order fractional differential equation is given as:

$$D^\alpha (y(t)) + \sum_{i=0}^n c_i y^{(i)}(t) = f(t) \quad (1.24)$$

subject to conditions:

$$y^k(0) = k; \quad k = 1; 2:::n \quad (1.25)$$

When the assumed approximate solution given as

$$y_N(x) = \sum_{i=1}^N c_i Q_i(x) \quad (1.26)$$

is substituted in (1.14), it takes the form:

$$D^{\alpha} \sum_{i=1}^N c_i Q_i(t) y^{(i)}(t) = f(t) \quad (1.27)$$

$$Q_j(x), \quad j = 0, 1, \dots, N$$

where $Q_j(x)$, $j = 0, 1, \dots, N$ are the functions or the basis polynomial functions to be determined, D and J are differential and integral operators respectively given in Caputo sense. Generally speaking, a multi-order fractional differential equation can be obtained from a multi-order fractional integro-differential equation when we set $\alpha = 0$.

1.2.8 Fractional Integrals

Weibeer (2005) defined fractional integral in lemma form: Let f be Riemann integrable on $[a, b]$, then, for $a \leq x \leq b$ and $n \in \mathbb{N}$, we have

$$J_n^{\alpha} f(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t) dt \quad (1.28)$$

where J_a^{α} is the integral operator of order α . He stated that the identity relationship between derivatives and integration of non-integer orders is

$$D^n f(x) = D^m J^{m-n} f(x)$$

where $m, n \in \mathbb{N}$ (natural numbers) and $m > n$.

Some general important properties of $D f(t)$ and $J f(t)$ in Gamma functions can be shown as follows:

suppose

$$f(t) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t) dt$$

Let

$$f(t) = t^n$$

and applying the fractional derivative

$$(D^{\alpha}) (t^n) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} \frac{d}{dt} (t^n) dt$$

Therefore, one very important property of D is given as

following the same process, for fractional integral of a function $f(t) = t^n$ with an arbitrary parameter is given as

and Riemann fractional derivative for a function $f(t) = t^\alpha$ with an arbitrary parameter α is given by

For functions of the type $f(t) = t^{-1}$, with an arbitrary parameter , Liouville developed the definition:

also for integration, Riemann-liouville fractional integral operator J^{α} is defined as:

However, when solving multi-order fractional differential equations.

$$D(C) = 0$$

where C is a constant. Therefore

8

where d is the ceiling function known as the smallest integer greater than or equal to α , $\alpha > 0$ and $N_0 = \{0; 1; 2; \dots\}$ set of natural numbers and C is a constant.

Other properties of derivatives and integral operator of fractional includes:

$$(i) J J^\alpha = J^{\alpha+1}$$

$$(ii) J D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0^+)}{k!} \frac{x^k}{\Gamma(k+1)}$$

$$(iii) J D^\alpha f(x) = J^\alpha f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} \frac{x^k}{\Gamma(k+1)}$$

1.2.9 Fractional Integro-differential Equations

A differential equation is called a fractional integro-differential equation when the unknown function $y(x)$ appears under the integral sign and the equation also contains a fractional derivative D^α . The general form of fractional integro-differential equation is:

$$D^\alpha y(x) = f(x) + \int_{l(x)}^{p(x)} K(x; t)y(t)dt \quad (1.35)$$

subject to the conditions: $D^\alpha y(0) = k$

Where $k(x; t)$ and $y(t)$ are given smooth functions

Fractional integro-differential equation is classified into two major types; Fredholm fractional integro-differential and Volterra fractional integro-differential equations.

A fractional integro-differential equation is called a Fredholm fractional integro-differential equation if the upper and the lower limits of the equation, $l(x)$ and $p(x)$ are both constants.

$$y^n(x) = f(x) + \int_a^b K(x; t)y(t)dt \quad (1.36)$$

Where $k(x; t)$ is a given smooth function known as the kernel and $y(t)$ is to be determined. When the limits of such equation are not both constants but one a constant and the other a variable as in:

$$y^n(x) = f(x) + \int_a^{p(x)} K(x; t)y(t)dt \quad (1.37)$$

then the equation is called Volterra fractional integro-differential equation. Where $k(x; t)$ and $y(t)$ are given smooth functions

1.2.10 Multi-order Fractional Integro-differential Equations

As the name implies, it is an integral equation with more than one power of different fractional order. The general form of multi-order fractional integro-differential equation is given

as:

$$D(y(t)) + \sum_{i=0}^n c_i y^{(i)}(t) + J(y(t)) = f(t) \quad (1.38)$$

subject to conditions:

$$y^{(k)}(0) = k; \quad k = 1; 2:::n \quad (1.39)$$

However, when and occur more than once in the equation, we have

$$\begin{aligned} D^1 c_1 y(t) + D^2 c_2 y(t) + \dots + D^N c_N y(t) + \sum_{i=0}^n c_i(t) y^{(i)}(t) \\ + {}_1J^1 c_1 y(t) + {}_2J^2 c_2 y(t) + \dots + {}_3J^3 c_3(t) + {}_N J^N c_N y(t) \\ = f(t) \end{aligned} \quad (1.40)$$

where and are non-integer. and k are integers. D and J are denoting the fractional order derivative and fractional integral of the functions respectively. If = 0 equations becomes Multi-order Fractional Differential Equations

1.2.11 Initial Value Problems

An initial value problem is a differential equation together with some specified conditions called the initial conditions of the unknown function at the initial point in the domain of the problem. Initial value problem is a type of problem involving some predetermined values at the initial point that are required to determine the solution of the problem at the initial point.

1.2.12 Solution Integral Equations

The solution of a differential or an integral equation is obtained or expressed in two ways; namely exact solution or as an approximate form of the exact solution.

1.2.13 Exact Solution

A solution is called exact solution if it is expressed in a closed form, such as a polynomial, exponential function, trigonometric function or the combination of two or more of these elementary functions (Wazwaz (2011)).

1.2.14 Approximate Solution

For many problems, sometimes we cannot obtain exact solutions. So we seek to get the solution in a series or polynomial form that may converge to exact solution that can be used for numerical purposes. Such a solution, also referred to as inexact representation of the exact solution and can be obtained through different numerical methods is referred to as approximate solutions. The more coefficient terms of the problem that we determine, the higher accuracy level that we can achieve. A solution of an integral equation or integro-differential equation is a function $y(x)$ that must, to some level, satisfy the given equation. In other words, the obtained solution $y_n(x)$ must satisfy both sides of the examined equation. A simple example is, suppose $y(x) = \sin x$ is the solution of the integro-differential equation:

$$y^0(x) = 1 - \int_0^x y(t) dt \quad (1.41)$$

then,

$$\cos(x) = 1 - \int_0^x \sin(t) dt$$

$$\cos(x) = 1 - [\cos(x)]^x_0$$

$$\cos(x) = 1 - [\cos(x) + \cos(0)]$$

$$\cos(x) = 1 - [\cos(x) + 1]$$

$$\cos(x) = 1 + \cos(x) - 1$$

$$\cos(x) = \cos(x)$$

$$\text{LHS} = \text{RHS}$$

1.2.15 Absolute, Relative and Percentage Errors

The numerical difference between the true value of a quantity and its approximate value is

referred to as error. Denoted by E_A , $E_A = x - x^1 = x$:

The relative error E_R , is defined as

$$E_R = \frac{E_A}{x} = \frac{-x}{xx}$$

and the percentage error (E_P) by $E_P = 100E_R$

Where x is the true value of the quantity and x^1 is the approximate value. However, the truncation error committed as a result of using series of approximation is evaluated by the Taylor's series as stated:

$$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i)f^0(x_i) + \dots + \frac{(x_{i+1} - x_i)^n}{n!} f^n(x_i) + R_n(x)$$

$$\frac{(x_{i+1} - x_i)^{n+1}}{(n+1)!}$$

$R_n(x)$ is referred to as the remainder term which, for a convergence series, tends to zero as $n \rightarrow \infty$

1.2.16 Error Estimation

In this thesis, the formula used for the estimation of errors when the exact solution is given is

$$\text{Error} = |Y_{\text{(exact)}}(x) - Y_{\text{(appr)}}(x)| \quad (1.42)$$

In the case where the exact solution is not known in closed form, we estimate the error using the formula

$$\text{Error} = |Y_{(N+1)}(x) - Y_N(x)| \quad (1.43)$$

1.2.17 Gamma and Beta Functions

These two functions arise from the solution of physical problems and they play very important role in various branches of mathematical analysis. They are one of the convenient functions been used in solving fractional differential and integro-differential equations.

1.2.18 Gamma Function

The Gamma function, denoted by Γ is a generalization of the positive factorial numbers is defined.

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

1.2.19 Beta Function

The Beta function, denoted by B is a definite integral is defined as:

$$B(m; n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx; \quad m, n > 0$$

is called the Beta function of m and n (see Dass (2000)) Weibeeer (2005) said that Beta function is a special function, that is, directly connected to Eulers Gamma function. The author therefore defined it in terms of two variables; $(m; n) \in \mathbb{R}^+$ as

$$B(m; n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

where (m, n) are the $B(m+1, n+1)$ is noted to be the solution of the Beta integral

$$\int_0^1 t^m (1-t)^n dt = B(m+1; n+1)$$

. Three identities of Beta function are as follows

$$B(m; n) = B(n; m)$$

$$B(m; n) = B(m+1; n) + B(m; n+1)$$

$$B(m; n+1) = \frac{m}{m+n} B(m+1; n) = \frac{m}{m+n} + n B(m; n)$$

1.3 Aim and Objectives of the Study

The aim of this study is to construct orthogonal polynomials and use them as basis functions for numerical solution of multi-order fractional differential and integro-differential equations. The objectives of the work includes:

- (i) constructing new orthogonal polynomials for the general class of multi-order fractional differential and integro-differential equations that are valid in the interval under consideration;
- (ii) using the newly constructed orthogonal polynomial as basis functions to obtain numerical solutions of multi-order fractional differential and integro-differential equations by Standard and Perturbed Collocation Methods;
- (iii) computing the absolute estimated errors; and
- (iv) examine the performance of the proposed methods in the determination of approximate solutions to multi-order fractional differential and integro-differential equations.

1.4 Significance of the Study

This study is significant based on the following reasons:

- (i) solutions of problems that cannot be solved analytically or that are difficult to solve in a closed form are easily obtained by applying the proposed methods.
- (ii) the proposed methods produce solution to a high level of accuracy even at lower values of N (the degree of the approximant basis function) and at few iterations.

- (iii) the proposed numerical methods can reduce problems to matrices form which then become easier to solved using the well known Gaussian elimination method or computers based softwares like maple 18 with low cost of computation.
- (iv) the proposed numerical methods produce comparatively better approximate solutions to both linear and nonlinear problems.

1.5 Outline of the Thesis

Chapter one contains general introduction to the research work, relevant de nitions, aim and objectives, and signi cance of the study. Chapter two deals with the literature review while in chapter three, derivation of orthogonal polynomial, methodology and numerical examples using the Standard Collocation Method. Chapter four, methodology and numerical examples using the Perturbed Collocation Method. In chapter ve, general conclusion, contribution to knowledge and recommendation for further studies were presented.

Chapter 2

Literature Review

2.1 Preamble

Most Scientific and Engineering problems are better presented and solved using fractional calculus. Khader (2011) stated that owing to the growing applications of fractional calculus in the formulation and representation of physical life problems, it may soon become the calculus of the twenty-first century. This statement speaks volume of the fact that the applications of fractional calculus in the representation and presentation of life problems can not be over emphasized. The field of fractional calculus has become a very attractive field to many researchers for recent decades because of its applications in many fields of Engineering, Mathematical modeling, Physics, Banking and Finance, etc. Fractional integro-differential equations on the other hand has grown tremendously in applications so much so that a lot of researchers have picked interest in it. Omar (2014) considered fractional integro-differential equation as a very important integro-differential equations where the differentiation and the integration appearing in the equation is of non-integer order. Omar (2014) solved some multi fractional order integro differential equations using Variational Iterative Method (VIM). Mittal and Nigam (2014) solved multi order fractional integro differential equations using Adomian Decomposition Method (ADM) and got satisfactory results. Odetunde (2013) used Iterative Decomposition Method (IDM) for the approximation of multi order fractional differential equations. The study presented solutions of rapidly convergent series of computable terms. Ibtisam (2011) applied the Homotopy Analysis Method (HAM) to solve multi order fractional integro differential equations involving Initial value problems. Khader (2011) carried out implementation of operational matrix of fractional derivative to solve nonlinear multi order fractional differential equations using Legendre polynomial. Atabakzadeh et al., (2012) used Chebyshev

operational matrix method to solve multi order fractional ordinary differential equations. By using shifted Chebyshev polynomial, they were able to obtain satisfactory results without linearization, discretization or perturbation. Davood et al., (2013) solved multi term fractional differential equations by operational matrices of boundary value problems with convergence analysis using Bernstein polynomials. The method involves using matrices to reduce linear multi-term order fractional differential equations to a system of algebraic equations which then make it easier to solve.

The solution of multi order fractional differential and or integro differential equations have been approached by many researchers using various methods like as enumerated above and by different basis functions including Chebyshev polynomials, Legendre polynomials, Hermite polynomials, Bernstein polynomials etc. The contribution of researchers using these basis functions are discussed later.

2.2 Methods of Solution

2.2.1 Collocation method (CM)

Collocation method is one of the methods of solving differential equations. The procedure involves first assuming an approximate solution for the problem. The assumed solution is substituted into the problem and the resulting equation is collocated at various points within the interval of consideration. The most commonly used method for solving differential and integro-differential equations by many researchers is the collocation method. It has been described as a very effective method to handle ordinary, fractional and partial differential equations (Taiwo and Odetunde, (2013)). Collocation methods for ordinary differential equations with Initial Value Problems (IVP) was first studied systematically in the late 1960s. The method is known to be the simplest way of making differential and integral equations of both linear and non linear to be in discrete form, and it involves the use of an assumed or trial solution (Brunner, (2004)). The derivative and integral of the problem are approximated by a finite smooth functions that are linearly independent, then the coefficients which are associated with the basis functions are determined by fitting the approximate solution at certain points known as the collocation points to satisfy the differential, integral equations as well as the initial or boundary conditions as the case may be (Gegele, (2016)). It is common to see many researchers applying collocation method to solve differential or integro-differential equations by using polynomials as basis functions. Such polynomials as Bernstein, Chebyshev,

Hermite, Legendre etc.

2.2.2 Successive /Picard Approximation Method

This is another method to solve Fredholm and Volterra fractional differential equations and in particular multi-term fractional differential equations. Wazwaz (2011) stated that the method of successive approximation admits the use of the iteration formula

$$u_{n+1}(x) = f(x) + \int_0^x K(x; t) F(u_n(t)) dt \quad n = 0 \quad (2.1)$$

Where

$$u_n(x) = f(x) + \int_0^x K(x; t) F(u(t)) dt \quad (2.2)$$

is the previous iteration. He described the method as very efficient and accurate. The zeroth approximation $u_0(x)$ was assumed as $U_0(x) = 0$ or any selected real-valued function that is used in a recurrence relation to determine the other approximations. Taiwo and Uwaheren (2015) applied the method to solve multi-order fractional integro-differential equations

2.2.3 Perturbed Collocation Method (PCM)

This method involves the addition of an extra term, called the perturbation term, to problem with the aim of improving the results. Abbasbandy (2006), Abbasbandy (2007) and Taiwo and Adebisi(2013) are some of the researchers that used perturbation method to solve various problems involving Fredholm integral and Volterra integral equations.

2.2.4 Differential Transform Method (DTM)

According to Arikogu and Ozkol (2009), the Differential Transform Method of a function $u(t)$ is given as:

$$U(t) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k u(t)}{dt^k} \bigg|_{t=t_0} \quad (2.3)$$

where the differential inverse of $U(t)$ is given as:

$$u(t) = \sum_{k=0}^{\infty} \frac{1}{k!} U(t - t_0)^k \quad (2.4)$$

and by neglecting the terms from $n + 1$ to ∞ , the infinite series is reduced to a finite series and we use:

$$u(t) = \sum_{k=0}^n \frac{1}{k!} U(t - t_0)^k \quad (2.5)$$

Some properties of DTM are given in Gon-Chang (2011) and include:

(i) if $u(t)=p(t) + \text{ or } -q(t)$, then $U(t)=P(t)+ \text{ or } -Q(t)$

(ii) if $u(t)= kp(t)$, then $U(t)= kP(t)$

(iii) if $u(t)= \frac{d^k u(t)}{dt^k}$, then $U(t)= \frac{(k+m)!}{k!} U(k + m)$

(iv) if $u(t)= p(t)q(t)$, then $U(k)= \sum_{k=0}^1 P(c)Q(k - c)$

(v) if $u(t)= t^n$, then $U(k)= (k - n)^P$

2.2.5 Iterative Decomposition Method (IDM)

Taiwo and Odetunde (2013) applied Iterative Decomposition Method to find approximate solution of multi-term fractional differential equations. The work clearly shows that multi-order fractional differential equations are actually the same as multi-term fractional differential equations as they used two words interchangeably. The paper noted that in spite of the merits of multi-term fractional differential equations in many models, the constraints include the existence of limited numerical schemes for their solutions. Other researchers that have worked with Iterative Decomposition Method include Taiwo et al., (2009).

2.2.6 Variational Iterative Method (VIM)

Some of the researchers that used Variational Iterative Method include He (2008), Chen and Wang (2010), Omar (2014), to mention but a few. Omar (2014) admitted that Variational Iterative Method was originally proposed by Ji-Huan and that the method have been successfully applied to solving various kinds of nonlinear problems, parabolic partial differential equations, heat-like, wave-like equations with variable coefficient etc. The author stated that to solve a potential nonlinear equation, the problem is written in the form

$$L(u(x)) + N(u(x)) = g(x)$$

where L is the linear operator, N is the nonlinear operator and $g(x)$ is any given function which is called the non-homogeneous term. The n th approximate solution U_n is substituted into the equation and then the correction functional is of the form:

$$U_{n+1}(x) = U_n(x) + \int_{x_0}^x \lambda(s)(L(U_n(s)) + N(U_n(s)) - g(s))ds \quad (2.6)$$

where λ is the general Lagrange multiplier. Neda et al (2012) used the method to solve higher order integro differential equations. He (2003) used VIM coupled with perturbation method

to solve the similar equations. He (2006) also solved nonlinear integro differential equations. Wazwaz (2011) described the method as one that is easily used to convert Volterra integral equations into initial value problem or to an equivalent integro differential equations. Besides, the method leads to rapidly convergence approximate solution to exact solution. It is observed that all the researchers agreed that the method is good and efficient in handling both linear and nonlinear differential and integro differential equations. They also generally agreed that the method gives the exact solution after a few steps. Omar (2014) however added that in some cases, the method will require more calculations which may pose some difficulties.

2.2.7 Homotopy Perturbation Method (HPM)

Fesojaye (2016) reported that He (1999) proposed the Homotopy Perturbation Method for solving differential equations with the idea of introducing what is called homotopy perturbation parameter, say q . Jafari and Momani (2007) employed the method to solve fractional Diffusion and Wave equations but with some modifications and got very good results. Abdulaziz et al., (2008) also applied Homotopy Perturbation Method to solve Fractional Initial Valued Problems and their remark shows that the method is efficient in handling fractional order differential and integro differential equations.

2.2.8 Homotopy Analysis Method (HAM)

The method have been known to produce very good results over the years. Ibtisam (2011) used HAM to solve multi order fractional integro differential equations with satisfactory results. Then after, remarked that the method provides simple way to adjust and control the convergence of solution series. Jafari and Sei (2004), Hashim et al., (2009) and Mohammed (2010) are among researchers that have successfully applied HAM to solve various classes of linear and nonlinear differential and integro differential equations.

2.2.9 Adomian Decomposition Method (ADM)

Mittal and Nigam (2014) solved fractional integro differential equations by Adomian Decomposition Method. It was noted in the paper that ADM provides analytical approximation to linear and nonlinear problems. It was noted in the method also that the solution is considered as the sum of an infinite series, that rapidly converge to an accurate solution. Besides, ADM provide solutions without any need for linearization or discretizations. The method

provides a systematic computational algorithm for solving equations of physical significance. The authors concluded that though other methods exist but ADM justifies its applicability and efficiency in solving integro-differential equations. Abdasbandy (2006) sought the solution of the Homotopy integral equations by Homotopy perturbation and Adomian Decomposition method. Wazwaz (2011) stated that Adomian Decomposition method provides solutions in an infinite series of components that is recurrently determined.

2.2.10 Laplace Decomposition Method (LDM)

Changqing and Jianhua (2013) described LDM as a numerical algorithm to solve nonlinear ordinary, partial differential equations. In the paper, LDM was used to solve special kind of nonlinear fractional integro-differential equations of the form:

$$D^\alpha y(t) = p(t)y(t) + g(t) + \int_0^t k(t;\tau)F(y(\tau))d\tau \quad (2.7)$$

and

$$D^\alpha y(t) = p(t)y(t) + g(t) + \sum_{i=1}^n \int_0^t k_i(t;\tau)F(y(\tau))d\tau \quad (2.8)$$

with initial conditions, $y(0) = y_i$ $i = 0; 1; 2; \dots; n-1$ and $t \in [0; 1]$. In their conclusion, after successful application of the method, was that the method is a powerful and efficient method for finding analytical as well as numerical solutions for wide classes linear and nonlinear fractional integro-differential equations. Wazwaz (2011) however noted that the method is developed to handle the generalized Abel's form of integral equations. Khuri (2001) had earlier used LDM to solve some classes of nonlinear differential equations in which he illustrated a basic algorithm for manipulating the decomposition method. Other researchers that have worked using LDM include Yusufoglu (2006), Wazwaz (2010) etc and they also agree that the method is efficient in solving linear and nonlinear differential and integro-differential equations. Details of the Laplace convolution product of two functions can be found in Wazwaz (2011)

2.2.11 Least Squares Method (LSM)

Mohammed (2014) and Mashdy and Mohammed (2016) determined the numerical solution of fractional integro-differential equations by least squares method and shifted Chebyshev polynomial. They described the method as very effective as it converges fast as the number of terms increases. The method involves getting the residual expression $R(a_0; a_2; \dots; a_n)$ from

the general class of the fractional integro-differential equation:

$$R(a_0; a_2; \dots; a_n) = D^\alpha y(x) - f(x) - \int_0^1 k(x; t)y(t)dt$$

and substituting the approximate solution of the form

$$y_n(x) = \sum_{i=0}^n a_i T_i(x)$$

into the above equation, where T_i is the Chebyshev polynomial, we get

$$R(a_0; a_2; \dots; a_n) = \sum_{i=0}^n D^\alpha T_i(x) - f(x) - \int_0^1 k(x; t) \left[\sum_{i=0}^n T_i(t) \right] dt$$

If we let

$$S(a_0; a_2; \dots; a_n) = \int_0^1 [R(a_0; a_2; \dots; a_n)]^2 w(t) dt$$

Then,

$$S(a_0; a_2; \dots; a_n) = \int_0^1 \left[\sum_{i=0}^n D^\alpha T_i(x) - f(x) - \int_0^1 k(x; t) \left[\sum_{i=0}^n T_i(t) \right] dt \right]^2 dx = 0$$

and that can be converted into matrices of $n+1$ terms:

$$\begin{bmatrix} R(x; a_0)h_0 dx & R(x; a_1)h_1 dx & \dots & R(x; a_n)h_n dx \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = \begin{bmatrix} C_0 & C_1 & \dots & C_n \end{bmatrix} \begin{bmatrix} f(x)h_0 dx \\ f(x)h_1 dx \\ \vdots \\ f(x)h_n dx \end{bmatrix}$$

where,

$$R(x; a) = \sum_{i=0}^n D^\alpha T_i(x) - f(x) - \int_0^1 k(x; t) \left[\sum_{i=0}^n T_i(t) \right] dt$$

The system of equation is solved to obtain the constant coefficients. Taiwo and Fesojaye (2016) did not agree less. Their least square method was with some perturbation which also produced satisfactory results. Other researchers that have applied the method with success includes; Zuppa (2003), Masdy and Shwayye (2016) and Shehab et al. (2010) just to mention a few.

2.3 Orthogonal functions

Zarowsky (2004) observed that Orthogonal polynomials arise in highly diverse settings and they can be used to define solutions to special classes of differential equations that arise in mathematical physics problems and are also very vital in the design of analog and digital separations. Zarowsky (2004) also agreed that they arise in numerical integration methods, and also have a considerable role to play in solving least-squares and uniform approximation problems.

Two functions $h(x)$ and $g(x)$ are referred to as orthogonal functions in a defined interval say $[a, b]$ if

$$\int_a^b h(x)g(x)dx = 0$$

Some examples of orthogonal functions includes $\sin x$ and $\cos x$. These two functions are orthogonal to each other in the interval $[-\pi; \pi]$ or any other interval of width 2π . So we say that the trigonometric functions $\sin mx$ and $\cos nx$ where $m, n = 0; 1; 2; \dots$ form an infinite collection of periodic functions that are mutually orthogonal in the interval x and hence we write that

$\int$

$$\begin{aligned} \int \sin nx \cos mx dx &= 0 & n \neq m \\ \int \sin nx \sin mx dx &= 0 & n \neq m \\ \int \cos nx \cos mx dx &= 0 & n \neq m \end{aligned}$$

Zarowsky (2004), while discussing orthogonal polynomials noted that some important terminologies related to the concept are "inner product, domain D (i.e interval of consideration) and the weighting function, $w(x)$. Zarowsky (2004) noticed that changing the domain D and the weighting function, $w(x)$ we generate various forms and types of orthogonal polynomials. This leads us to what is called special orthogonal polynomials like Chebyshev, Hermite, Legendre etc

2.3.1 Chebyshev polynomial

The Chebyshev polynomials of the first kind, denoted by $T_m(t)$ and of order m is defined in interval $[-1, 1]$ as

$$T_m(t) = \cos^{-1}(\cos mt)$$

where,

$$T_0(t) = 1; \quad T_1(t) = t; \quad T_2(t) = 2t^2 - 1$$

The recurrence relation is given as:

$$T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t); \quad m \geq 1$$

The shifted Chebyshev polynomial denoted by $T(t)$ of order m is defined on interval $[0,1]$ as:

$$T_m(t) = T_m(2t-1)$$

and the recurrence function is given as:

$$T_{m+1}(t) = 2(2t-1)T_m(t) - T_{m-1}(t) \quad \text{for } m = 1; 2; 3; \dots$$

Few terms of the shifted Chebyshev polynomial are given as:

$$T_0(t) = 1$$

$$T_1(t) = 2t - 1$$

$$T_2(t) = 8t^2 - 8t + 1 \quad (2.9)$$

$$T_3(t) = 32t^3 - 48t^2 + 18t - 1$$

$$T_4(t) = 128t^4 - 256t^3 + 160t^2 - 32t + 1$$

The orthogonality condition of Chebyshev polynomials is given by

$$\int_0^1 \frac{T_i(t)T_j(t)}{w(t)} dt = \begin{cases} 0 & \text{for } i \neq j; \\ \frac{\pi}{2} & \text{for } i = j = 0; \\ \frac{\pi}{4} & \text{for } i = j \neq 0; \end{cases} \quad (2.10)$$

where $w(t)$ is known as the weight function, is given as $w(t) = \frac{1}{\sqrt{1-t^2}}$.

Doha et al., (2011) worked on the solution of linear and nonlinear multi-term fractional differential equations (MFDEs) using Chebyshev spectral method based on operational matrix method for initial and boundary value problems. Their approach was based on shifted Chebyshev tau and collocation methods. They proposed an algorithm which they applied to solve two types of fractional differential equations; linear and nonlinear. It was noted in the work that, for the fact that most problems of fractional differential equations do not have exact analytical solutions, finding accurate and efficient methods for solving them have become an active research interest for many researchers in the last few decades. In the discussion, it was noted that the main advantage of spectral method lies in the degree of accuracy of the results guaranteed by the method and that the method achieves exponential rates of convergence/spectral accuracy. Their works were described in Caputo sense which they claimed

allows traditional initial and boundary conditions to be included in the formulation of the problem.

Mahnaz and Ali (2013) also employed Chebyshev polynomial (shifted Chebyshev polynomial) to find the numerical solution of nonlinear multi order fractional differential equations by operational matrix method. The paper discussed some important merits of the method to include that it could convert any given nonlinear problem into a system of linear equations and that makes it easier to be solved.

2.3.2 Legendre polynomial

Zarowsky (2004) said that "the Legendre polynomials arise in potential problems in electro-magnetics. Modeling the scattering of electromagnetic radiation by particles involves working with these polynomials. Legendre polynomials appear in quantum mechanics as part of the solution to Schrodingers equation for the hydrogen atom".

The well known Legendre polynomials $P(s)$ is defined in the interval $[-1,1]$ by the Rodrigues' formula:

$$P_n(s) = \frac{1}{2^n n!} \frac{d^n}{ds^n} [(s^2 - 1)^n] \quad (2.11)$$

where, $P_0(s) = 1$ and $P_1(s) = s$ for $n = 0; 1$. It will be observed that an n^{th} derivative of the formula has to be carried out before a polynomial of degree n is obtained at all times. That process is however time consuming and so a recurrence formula to obtain the polynomial for $n > 2$ is given as:

$$P_{n+1}(s) = \frac{(2n+1)}{n+1} s P_n(s) - \frac{n}{n+1} P_{n-1}(s) \quad (2.12)$$

To transform (2.16) to shifted Legendre polynomial in the interval $[0,1]$ we let $s = 2t - 1$ where $t \in [0; 1]$, then $P_n(s) = L_n(2t - 1)$ and $L_{n+1}(t)$ will be defined as:

$$L_{n+1}(t) = \frac{(2n+1)(2t-1)}{n+1} L_n(t) - \frac{n}{n+1} L_{n-1}(t); \quad n = 1; 2; \dots; \quad (2.13)$$

A few numbers of the Legendre polynomials are given as follows:

$$L_0(t) = 1$$

$$L_1(t) = (2t - 1)$$

$$L_2(t) = (6t^2 - 6t + 1) \quad (2.14)$$

$$L_3(t) = (20t^3 - 30t^2 + 12t - 1)$$

$$L_4(t) = (70t^4 - 140t^3 + 90t^2 - 20t + 1)$$

The orthogonality conditions of the Legendre polynomials is stated as follows:

$$\int_{-1}^1 L_i(t)L_j(t)dt = \begin{cases} \frac{2}{2i+1} & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases} \quad (2.15)$$

Khadar (2011) used implementation of operational matrix of fractional derivative to solve nonlinear multi order fractional differential equations using truncated Legendre series together with the Legendre operational matrix of fractional derivatives. Khadar (2011) stated that the main characteristics in using the method is that it can reduce a given problem to that of a system of algebraic linear equations and thus greatly simplifying the problem. Yin Yang (2012) also used Legendre Pseudo Spectral method for solving a nonlinear multi order fractional differential equation. In the paper, spectral-collocation method was used to obtain approximate solutions of nonlinear classes of multi order fractional equations.

2.3.3 Hermite Polynomial

Shaqul and Azizur (2013) noted that the Hermite polynomial is named after a French mathematician, Charles Hermite (1822 - 1901) and that the polynomial is a family of polynomial functions which is the solution of special case of differential equation in the treatment of harmonic oscillator in quantum mechanics.

The Hermite polynomial denoted $H_n(x)$, is given as

$$H_r(x) = \sum_{i=0}^{\frac{r}{2}} \frac{r!}{i!(r-i)!} (-1)^i (2x)^{r-2i}$$

where,

$$\frac{r}{2} = \begin{cases} \frac{r-1}{2} & \text{for } r \text{ is even;} \\ \frac{r}{2} < \frac{r-1}{2} & \text{for } r \text{ is odd;} \end{cases}$$

The first few terms of Hermite polynomial are:

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2 \quad (2.16)$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

2.3.4 Bernstein Polynomial

A Bernstein polynomial, though not orthogonal but a space polynomial is also a very useful polynomial. Bernstein polynomial or a polynomial in the Bernstein form, named after Sergei Natanovich Bernstein, is a linear combination of Bernstein basis polynomials. Davaeifar, et,al (2016) sees the Bernstein polynomials as very important tools usually applied to determine the approximate solutions of differential, linear and nonlinear Fredholm integral and integro-differential equations and nonlinear Volterra-Fredholm-Hammerstein integral equations. The polynomial is defined as:

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}; \quad i = 0(1)n \quad (2.17)$$

X

where $\binom{n}{i}$ is a binomial coefficient, and it is given as $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ which as usual form Pascal's triangle arrangement. Few numbers of the polynomials using the definition above are

$$B_{(0;0)}(t) = 1$$

$$B_{(0;1)}(t) = 1-t; \quad B_{(1;1)}(t) = t$$

$$B_{(0;2)}(t) = (1-t)^2; \quad B_{(1;2)}(t) = 2t(1-t); \quad B_{(2;2)}(t) = t^2 \quad (2.18)$$

$$B_{(0;3)}(t) = (1-t)^3; \quad B_{(1;3)}(t) = 3t(1-t)^2; \quad B_{(2;3)}(t) = 3t^2(1-t); \quad B_{(3;3)}(t) = t^3$$

$$B_{(0;4)}(t) = (1-t)^4; \quad B_{(1;4)}(t) = 4t(1-t)^3; \quad B_{(2;4)}(t) = 6t^2(1-t)^2; \quad B_{(3;4)}(t) = 4t^3(1-t);$$

$$B_{(4;4)}(t) = t^4$$

There are $(n+1)$, n^{th} - degree Bernstein polynomials. However, we do for convenience sake, set $B_{i;n} = 0$, if $i < 0$ or $i > n$

Jianping et al., (2012) published a paper titled, solution of fractional differential equations by Bernstein polynomials method. The paper deals with the application of Bernstein polynomial method for the solution of fractional differential equations.

In this work, the constructed Orthogonal Polynomial are used as basis functions in the assumed approximate solution to solve multi-order fractional integro-differential equations.

Chapter 3

Standard Collocation Method for Solving Multi-order Fractional Differential and Integro-differential Equations

3.1 Introduction

In this chapter, we present the construction of the orthogonal polynomial used, the methodology of the proposed methods and some numerical examples.

3.2 Construction of Orthogonal Polynomials

Recall the conditions of orthogonality of two functions $y_p(x)$ and $y_q(x)$. Following the conditions of orthogonality of two functions $v_p(x)$ and $v_q(x)$, we construct our orthogonal polynomial based on the use of weight function, $w_m(z) = (1-z)^m$, for $m = 1, 2$ and 3 . This corresponds to linear, quadratic and cubic functions respectively and each satisfying the orthogonality conditions in the interval $[a, b]$ under consideration.

According to Gram-Schmidt orthogonalization process, if we have orthogonal polynomial $u_j(z)$ valid in the interval $[a, b]$ with the leading terms z^j , then

$$u_j(z) = z^j + v_j(z) \tag{3.1}$$

where

$$v_j(z) = \sum_{i=0}^j c_{j,i} v_i(z) \quad i = 0; 1; 2; \dots; j-1 \quad (3.2)$$

and $v_j(z)$ is an increasing polynomial of degree j and $v_i(z)$ are the corresponding values of the approximating functions in z .

Now, to determine the values of $c_{j,i}$ in (3.2), we say that, let v_j be some functions of z and $v_0(z); v_1(z); v_2(z); v_{j-1}(z)$ be the corresponding values of the approximating functions, where $u_i(z)$ satisfies (1.9). Let us write

$$v_j(z) = c_{j,0}v_0(z) + c_{j,1}v_1(z) + c_{j,2}v_2(z) + c_{j,3}v_3(z) + \dots + c_{j,j-1}v_{j-1}(z) \quad (3.3)$$

Now applying the least squares method for continuous functions in $[a,b]$, we set:

$$S(c_{j,0}; c_{j,1}; c_{j,2}; \dots; c_{j,j-1}) = \int_a^b w(z) [v(z) - (c_{j,0}v_0(z) + c_{j,1}v_1(z) + c_{j,2}v_2(z) + \dots + c_{j,j-1}v_{j-1}(z))]^2 dz \quad (3.4)$$

where S is the sum of the squares of the errors; and $[a,b]$ is the interval of consideration, $w(z)$ is a non negative weight function and $c_i; i = 0; 1; 2; \dots; j-1$ are constants to be determined.

Differentiating equation (3.4) partially with respect to $c_{j,0}; c_{j,1}; c_{j,2}; \dots; c_{j,j-1}$, we have

$$\begin{aligned} \frac{\partial S}{\partial c_{j,0}} &= 2 \int_a^b w(z) [v(z) - (c_{j,0}v_0(z) + c_{j,1}v_1(z) + c_{j,2}v_2(z) + \dots + c_{j,j-1}v_{j-1}(z))] v_0(z) dz = 0 \\ \frac{\partial S}{\partial c_{j,1}} &= 2 \int_a^b w(z) [v(z) - (c_{j,0}v_0(z) + c_{j,1}v_1(z) + c_{j,2}v_2(z) + \dots + c_{j,j-1}v_{j-1}(z))] v_1(z) dz = 0 \\ &\vdots \\ \frac{\partial S}{\partial c_{j,j-1}} &= 2 \int_a^b w(z) [v(z) - (c_{j,0}v_0(z) + c_{j,1}v_1(z) + c_{j,2}v_2(z) + \dots + c_{j,j-1}v_{j-1}(z))] v_{j-1}(z) dz = 0 \end{aligned}$$

Then, the resulting equations will now be:

$$\int_a^b w(z) v(z) v_0(z) dz = \int_a^b w(z) v_0^2(z) dz + c_{j,0} \int_a^b w(z) v_1(z) v_0(z) dz + \dots + c_{j,j-1} \int_a^b w(z) v_{j-1}(z) v_0(z) dz$$

(3.6)

(3.5)

Thus (3.6) is re-written as one equation:

That is

$$c_{j;0} \int_a^b w(z) v_0(z) v_j(z) dz + c_{j;1} \int_a^b w(z) v_1(z) v_j(z) dz + \dots + c_{j;j-1} \int_a^b w(z) v_{j-1}(z) v_j(z) dz = \int_a^b w(z) v(z) v_j(z) dz; \quad i=0, 1, 2, \dots, j-1 \quad (3.7)$$

which satisfies equation (1.9).

Therefore, from equation (3.7), we have the general relation of $c_{j;i}$ as

$$c_{(j;i)} = \frac{\int_a^b w(z) v_i(z) v_j(z) dz}{\int_a^b w(z) v_i(z) v_i(z) dz} \quad (3.8)$$

R

We apply the combination of equations (3.8) and (3.1) to generate the polynomials as follows:

CASE 1: $m = 1$, so $w_1(z) = 1 - z$, in the interval $[-1, 1]$.

$$u_0(z) = 1 \quad (3.9)$$

$$u_1(z) = z^j + c_{(j;i)} v_i(z) = z + c_{(1;0)} v_0(z) \quad (3.10)$$

$$c_{(1;0)} = \frac{\int_{-1}^1 z^1 (1-z) dz}{\int_{-1}^1 (1-z) dz} = \frac{1}{3} \quad (3.11)$$

$$u_1(z) = z + \frac{1}{3} v_0(z) = z + \frac{1}{3} = \frac{1}{3} (3z + 1) \quad (3.12)$$

$$u_2(z) = z^j + c_{(j;0)} v_0(z) + c_{(j;1)} v_1(z) = z^2 + c_{(2;0)} v_0(z) + c_{(2;1)} v_1(z) \quad (3.13)$$

$$c_{(2;0)} = \frac{\int_{-1}^1 z^2 (1-z) dz}{\int_{-1}^1 (1-z) dz} = \frac{1}{5} \quad (3.14)$$

$$c_{(2;1)} = \frac{\int_{-1}^1 z^2 (1-z)(z + \frac{1}{3}) dz}{\int_{-1}^1 (1-z)(z + \frac{1}{3}) dz} = \frac{2}{5} \quad (3.15)$$

$$u_2(z) = z^2 + \frac{1}{5} + \frac{2}{5} (z + \frac{1}{3}) = z^2 + \frac{2}{5} z + \frac{1}{5}$$

$$u_3(z) = z^j + c_{(j;0)} v_0(z) + c_{(j;1)} v_1(z) + c_{(j;2)} v_2(z) = z^3 + c_{(3;0)} v_0(z) + c_{(3;1)} v_1(z) + c_{(3;2)} v_2(z) \quad (3.16)$$

$$c_{(3;0)} = \frac{\int_{-1}^1 z^3 (1-z) dz}{\int_{-1}^1 (1-z) dz} = \frac{1}{5} \quad (3.17)$$

$$c_{(3;1)} = \frac{\int_{-1}^1 z^3 (1-z)(z + \frac{1}{3}) dz}{\int_{-1}^1 (1-z)(z + \frac{1}{3}) dz} = \frac{3}{5} \quad (3.18)$$

$$c_{(3;2)} = \frac{\int_{-1}^1 z^3 (1-z)(z^2 + \frac{2z}{5} + \frac{1}{5}) dz}{\int_{-1}^1 (1-z)(z^2 + \frac{2z}{5} + \frac{1}{5}) dz} = \frac{3}{7} \quad (3.19)$$

R
1

5 5

$$u_3(z) = z^3 + c_{(3;0)}v_0(z) + c_{(3;1)}v_1(z) + c_{(3;2)}v_2(z) = z^3 + \frac{1}{5} + \frac{3}{5}z + \frac{1}{5}z^2 + \frac{1}{7}(z^2 + \frac{2z}{5}) = z^3 + \frac{1}{7} + \frac{3z}{7} + \frac{3}{35} \quad (3.20)$$

So a few number of the polynomial on the interval [-1; 1] are:

$$\begin{aligned} u_0(z) &= 1 \\ u_1(z) &= \frac{1}{3}(3z + 1) \\ u_2(z) &= \frac{1}{5}(5z^2 + 2z - 1) \\ u_3(z) &= \frac{1}{35}(35z^3 + 15z^2 - 15z - 3) \\ u_4(z) &= \frac{1}{63}(63z^4 + 28z^3 - 42z^2 - 12z + 3) \\ u_5(z) &= \frac{1}{231}(231z^5 + 105z^4 - 210z^3 - 70z^2 + 35z + 5) \end{aligned} \quad (3.21)$$

The shifted equivalent of (3.21) that are valid in [0,1], are given as:

$$\begin{aligned} L_0(t) &= 1 \\ L_1(t) &= \frac{1}{3}(6t - 2) \\ L_2(t) &= \frac{1}{5}(20t^2 - 16t + 2) \\ L_3(t) &= \frac{1}{35}(280t^3 - 360t^2 + 120t - 8) \\ L_4(t) &= \frac{1}{63}(1008t^4 - 1792t^3 + 1008t^2 - 192t + 8) \\ L_5(t) &= \frac{1}{231}(7392t^5 - 16800t^4 + 13440t^3 - 4480t^2 + 560t - 16) \end{aligned} \quad (3.22)$$

CASE 2: $m = 2$, so $w_2(z) = (1 - z)^2$, in the interval [-1,1].

$$u_0(z) = 1 \quad (3.23)$$

$$u_1(z) = z + c_{(1;0)}v_0(z) = z + c_{(1;0)}v_0(z) \quad (3.24)$$

$$c_{(1;0)} = \frac{\int_{-1}^1 z(1 - 2z + z^2)dz}{\int_{-1}^1 (1 - 2z + z^2)dz} = \frac{1}{2} \quad (3.25)$$

$$u_1(z) = z + \frac{1}{2}v_0(z) = z + \frac{1}{2} = \frac{1}{2}(2z + 1) \quad (3.26)$$

$$u_2(z) = z^2 + c_{(2;0)}v_0(z) + c_{(2;1)}v_1(z) = z^2 + c_{(2;0)}v_0(z) + c_{(2;1)}v_1(z) \quad (3.27)$$

$$c_{(2;0)} = \frac{\int_{-1}^1 z^2(1 - 2z + z^2)dz}{\int_{-1}^1 (1 - 2z + z^2)dz} = \frac{2}{5} \quad (3.28)$$

$$c_{(2;1)} = \frac{\int_{-1}^1 z(1 - 2z + z^2)(z + \frac{1}{2})dz}{\int_{-1}^1 (1 - 2z + z^2)(z + \frac{1}{2})dz} = \frac{2}{5} \quad (3.29)$$

$$u_2(z) = z^2 \left(\frac{2}{5} + \frac{2}{3}z + \frac{1}{2} \right) = z^2 + \frac{2}{3}z + \frac{1}{2} = \frac{1}{15}(15z^2 + 10z + 3) \quad (3.30)$$

$$u_3(z) = z^3 + c_{(3;0)}v_0(z) + c_{(3;1)}v_1(z) + c_{(3;2)}v_2(z) = z^3 + c_{(3;0)}v_0(z) + c_{(3;1)}v_1(z) + c_{(3;2)}v_2(z) \quad (3.31)$$

$$c_{(3;0)} = \frac{\int_0^1 z^3(1-2z+z^2)dz}{\int_0^1 (1-2z+z^2)dz} = \frac{3}{10} \quad (3.32)$$

$$c_{(3;1)} = \frac{\int_0^1 \frac{1}{z}(1-2z+z^2)(z-2z^2+\frac{1}{2}z^3)dz}{\int_0^1 \frac{1}{z}(1-2z+z^2)dz} = \frac{7}{5} \quad (3.33)$$

$$c_{(3;2)} = \frac{\int_0^1 \frac{1}{z^2}(1-2z+z^2)(z-2z^2+\frac{1}{2}z^3)dz}{\int_0^1 \frac{1}{z^2}(1-2z+z^2)dz} = \frac{3}{4} \quad (3.34)$$

$$u_3(z) = z^3 + c_{(3;0)}v_0(z) + c_{(3;1)}v_1(z) + c_{(3;2)}v_2(z) = z^3 + \frac{3}{10} \left(\frac{5}{7}z + \frac{3}{4} \right) + \frac{7}{5} \left(\frac{3}{14}z + \frac{28}{3} \right) = z^3 + \frac{3}{4}z^2 + \frac{3}{14}z + \frac{28}{3} \quad (3.35)$$

So a few number of the polynomial on the interval [-1; 1] are:

$$\begin{aligned} u_0(z) &= 1 \\ u_1(z) &= \frac{1}{2}(2z + 1) \\ u_2(z) &= \frac{1}{15}(15z^2 + 10z + 3) \\ u_3(z) &= \frac{1}{28}(28z^3 + 12z^2 + 6z + 1) \\ u_4(z) &= \frac{1}{105}(105z^4 + 84z^3 + 42z^2 + 28z + 1) \\ u_5(z) &= \frac{1}{198}(198z^5 + 165z^4 + 100z^3 + 505z^2 + 25z + 5) \end{aligned} \quad (3.36)$$

$$\begin{aligned} Q_0(t) &= 1 \\ Q_1(t) &= \frac{1}{2}(4t + 1) \\ Q_2(t) &= \frac{1}{15}(60t^2 + 40t + 4) \\ Q_3(t) &= \frac{1}{7}(56t^3 + 63t^2 + 18t + 1) \\ Q_4(t) &= \frac{1}{105}(1680t^4 + 2688t^3 + 1344t^2 + 224t + 8) \\ Q_5(t) &= \frac{1}{99}(3168t^5 + 6600t^4 + 4800t^3 + 1440t^2 + 160t + 4) \end{aligned} \quad (3.37)$$

CASE 3: $m = 3$, so $w_3(z) = (1 - z)^3$, in the interval $[-1, 1]$

$$u_0(z) = 1 \quad (3.38)$$

$$u_1(z) = z^1 + c_{(1;0)}v_0(z) = z + c_{(1;0)}v_0(z) \quad (3.39)$$

$$c_{(1;0)} = \frac{\int_{-1}^1 z^1 (1 - 3z + 3z^2 - z^3) dz}{\int_{-1}^1 (1 - 3z + 3z^2 - z^3) dz} = \frac{3}{5} \quad (3.40)$$

$$u_1(z) = z + \frac{3}{5}v_0(z) = z + \frac{3}{5} = \frac{1}{5}(5z + 3) \quad (3.41)$$

$$u_2(z) = z^2 + c_{(2;0)}v_0(z) + c_{(2;1)}v_1(z) = z^2 + c_{(2;0)}v_0(z) + c_{(2;1)}v_1(z) \quad (3.42)$$

$$c_{(2;0)} = \frac{\int_{-1}^1 z^2 (1 - 3z + 3z^2 - z^3) dz}{\int_{-1}^1 (1 - 3z + 3z^2 - z^3) dz} = \frac{7}{15} \quad (3.43)$$

$$c_{(2;1)} = \frac{\int_{-1}^1 z^2 (1 - 3z + 3z^2 - z^3) dz}{\int_{-1}^1 (1 - 3z + 3z^2 - z^3) dz} = \frac{6}{15} \quad (3.44)$$

$$u_2(z) = z^2 + \frac{7}{15} + \frac{6}{15}(z + \frac{3}{5}) = z^2 + \frac{6z}{15} + \frac{1}{5} \quad (3.45)$$

$$u_3(z) = z^3 + c_{(3;0)}v_0(z) + c_{(3;1)}v_1(z) + c_{(3;2)}v_2(z) = z^3 + c_{(3;0)}v_0(z) + c_{(3;1)}v_1(z) + c_{(3;2)}v_2(z) \quad (3.46)$$

$$c_{(3;0)} = \frac{\int_{-1}^1 z^3 (1 - 3z + 3z^2 - z^3) dz}{\int_{-1}^1 (1 - 3z + 3z^2 - z^3) dz} = \frac{13}{35} \quad (3.47)$$

$$c_{(3;1)} = \frac{\int_{-1}^1 z^3 (1 - 3z + 3z^2 - z^3) dz}{\int_{-1}^1 (1 - 3z + 3z^2 - z^3) dz} = \frac{7}{15} \quad (3.48)$$

$$c_{(3;2)} = \frac{\int_{-1}^1 z^3 (1 - 3z + 3z^2 - z^3) dz}{\int_{-1}^1 (1 - 3z + 3z^2 - z^3) dz} = \frac{4}{21} \quad (3.49)$$

$$u_3(z) = z^3 + c_{(3;0)}v_0(z) + c_{(3;1)}v_1(z) + c_{(3;2)}v_2(z) = z^3 + \frac{13}{35} + \frac{7}{15}(z + \frac{3}{5}) + \frac{4}{21}(z^2 + \frac{6z}{15} + \frac{1}{5}) = z^3 + z^2 + \frac{2}{21} \quad (3.50)$$

So a few number of the polynomial on the interval $[-1, 1]$ are:

$$\begin{aligned} u_0(z) &= 1 \\ u_1(z) &= \frac{1}{5}(5z + 3) \\ u_2(z) &= \frac{1}{21}(21z^2 - 18z + 1) \\ u_3(z) &= \frac{1}{21}(21z^3 - 21z^2 - 2) \\ u_4(z) &= \frac{1}{165}(165z^4 + 180z^3 - 30z^2 - 44z - 5) \\ u_5(z) &= \frac{1}{1287}(z^5 + 1485z^4 - 3300z^3 - 630z^2 - 45z + 25) \end{aligned} \quad (3.51)$$

The shifted equivalent of (3.51) that are valid in [0,1], are given as:

$$\begin{aligned}
 C_0(t) &= 1 \\
 C_1(t) &= \frac{1}{5}(10t - 2) \\
 C_2(t) &= \frac{1}{21}(84t^2 - 48t + 4) \\
 C_3(t) &= \frac{1}{21}(168t^3 - 168t^2 + 42t - 2) \\
 C_4(t) &= \frac{1}{165}(2640t^4 - 3840t^3 + 31728t^2 - 256t + 8) \\
 C_5(t) &= \frac{1}{1287}(41184t^5 - 79200t^4 + 52800t^3 - 14400t^2 + 1440t - 32)
 \end{aligned} \tag{3.52}$$

For convenience, in this work we refer to the polynomials constructed above on the interval [0,1] as Linear Weighted Polynomial (LWP), Quadratic Weighted Polynomial (QWP) and Cubic Weighted Polynomial (CWP) respectively.

Some properties of the polynomial

Some of the properties of our orthogonal polynomials are presented below

- i, the constructed polynomials are polynomials of degree j and are defined on interval; [0,1]
- ii, the weight function used is non-negative on the interval; [0,1].
- iii, the polynomials are orthogonal in the interval [0,1] with respect to the inner product;

$$\langle P_n(t); P_m(t) \rangle = \int_0^1 w(t) P_n(t) P_m(t) dt = \begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{8} \int_0^1 w(t) P_n^2(t) dt & \text{if } n = m \end{cases}$$

3.3 Demonstration of the Standard Collocation Method

Here, the methodology of the Standard Collocation Method is presented.

In this method, we consider the general class of multi-order fractional differential and integro-differential equation:

$$D^\alpha(y(t)) + \sum_{i=0}^n c_i y^{(i)}(t) + \int_0^1 J(y(t)) dt = f(t) \tag{3.53}$$

subject to conditions:

$$y^{(k)}(0) = \gamma_k; \quad k = 1; 2; \dots; n \tag{3.54}$$

We wish to determine a single polynomial approximation $y_N(x)$ of the unique solution $y(t)$ of equations (3.53) and (3.54) in the form:

$$y_N(t) = \sum_{j=0}^N c_j Q_j(t) \tag{3.55}$$

where

$$Q_j(t) = \sum_{r=0}^j q_r^{(j)} t^r$$

c_j ; $j = 0(1)N$ are unknown constants to be determined and $q_r^{(j)} t^r$ are orthogonal polynomials functions.

On substituting (3.55) into (3.53), we have

$$\sum_{j=0}^N c_j Q_j(t) + \sum_{j=0}^N c_j Q_j(t) = f(t) \quad (3.56)$$

which becomes

$$\sum_{j=0}^N c_j \sum_{r=0}^j q_r^{(j)} t^r + \sum_{j=0}^N c_j \sum_{r=0}^j q_r^{(j)} t^r = f(t) \quad (3.57)$$

with the initial conditions

$$\sum_{j=0}^N c_j q_j^{(k)}(0) = k \quad k = 1; 2, \dots, n \quad (3.58)$$

Applying equation (6) and (7) on (3.57) gives

$$\sum_{j=0}^N c_j \frac{(j+1)}{(j+1)} t^j + \sum_{j=0}^N c_j \sum_{r=0}^j q_r^{(j)} t^r + \sum_{j=0}^N c_j \frac{(j+1)}{(j+1)} t^j = f(t) \quad (3.59)$$

Thus, equation (3.58), is collocated at equally spaced interior points $t = t_e$, we obtain:

where $t_e = a + \frac{b-a}{N+1} e$, ($e = 1, 2, \dots, N+1$).

Thus equation (3.38) resulted to $(N+1)$ linear system of algebraic equations with $N+1$ unknown constants. Hence, additional equation is obtained with the boundary conditions (3.54). Altogether, we have $(N+n+1)$ linear system algebraic equations for the determination of c_j ; ($j = 0; 1; 2; \dots; N$). The $(N+n+1)$ algebraic equations are put in matrix form,:

$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & \dots & A_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} c_0 Q^{(1)}(0) \\ c_1 Q^{(1)}(0) \\ \vdots \\ c_n Q^{(1)}(0) \end{bmatrix}$$

where

$$3 \quad 2 \quad B_1 \quad 3$$

34

$A_{13} = C_2 D Q_2(t) + Q_2(t)y^{(2)}(t)$ etc and B_j are the values $f(t)$.

These (+ 1) linear

or any suitable computer package like Maple 18 to obtain the unknown constants, and these values are then substituted back into the approximate solution to give the required approximate solution.

3.4 Numerical Examples

Here we present some examples to illustrate the Standard Collocation method. Initial value problems that consist of the multi-order fractional integro-differential equations were considered.

Example 1

Consider an Initial value multi-order fractional integro-differential equation

$$D y(t) + x y^0(t) + y(t) = \frac{6}{(3.5)t^{2.5}} + \frac{(4)}{(5.5)t^{4.5}} + J^{1.5}y(t) \quad t \in [0; 1] \quad (3.61)$$

Subject to $y(0) = 0$. Exact solution is $y(t) = t^3$ at $x = 1$. (Mittal and Nigam (2014))

Solution of example 1 by Linear Weighted Polynomial (LWP) at $x = 0.5$ for $N = 4$

Here, we assumed an approximate solution of the form:

$$y_N(t) = \sum_{j=0}^4 c_j L_j(t) \quad (3.62)$$

where c_j ; $j = 0(1)4$ are unknown constants to be determined and $L_j(t)$ are the Linear Weighted Polynomials defined in equation (3.22). Equation (3.61) is expanded in conjunction with equation (3.22) to get

$$y_4(t) = c_0 + \frac{1}{3}c_1(6t - 2) + \frac{1}{5}c_2(20t^2 - 16t + 2) + \frac{1}{35}c_3(280t^3 - 360t^2 + 120t - 8) + \frac{1}{63}c_4(1008t^4 - 1792t^3 + 1008t^2 - 192t + 8) \quad (3.63)$$

Putting (3.62) in (3.60), and applying (1.15) and (1.18), gives

$$\begin{aligned}
 & \frac{512t_5c_3}{3} + \frac{128t_3c_2}{7} + \frac{16t_3c_1}{15} + \frac{2048t_6c_4}{3} + \frac{256t_4c_3}{1925} + \frac{16384t_5c_4}{2475} + \frac{128t_4c_2}{495} + \frac{p}{165} p_{tc_0+4c_1t+} \\
 & \frac{3151051534651053465105}{3} + \frac{32t^2c_2}{7} + \frac{32tc_2}{15} + \frac{368t^3c_3}{3} + \frac{64t^2c_3}{1024} + \frac{4tc_3}{1925} + \frac{106496t^4c_4}{2475} + \frac{182272t^3c_4}{495} + \frac{13792t^2c_4}{165} + \frac{512tc_4}{165} + \frac{8c_4}{165} \\
 & \frac{p}{2=5} - \frac{p}{tc_1+4} - \frac{p}{t} - \frac{p}{3=2} - \frac{p}{1024} - \frac{p}{t^{7=2}c_4} - \frac{p}{tc_3+} - \frac{p}{1728} - \frac{p}{t^{5=2}c_4} - \frac{p}{8} - \frac{p}{tc_4} - \frac{p}{165} + \frac{12}{165} \\
 & \frac{11}{2=21} - \frac{t^{3=2}c_4}{p} - \frac{p}{p} - \frac{p}{p} - \frac{32}{p} - \frac{t^{3=2}c_2}{7} + \\
 & \frac{t^{5=2}c_2+80}{t} - \frac{p}{t} - \frac{p}{9=2} - \frac{c_4}{512p} - \frac{165}{165} + 4 - \frac{p}{t} - \frac{p}{3=2} - \frac{c_1}{32} - \frac{p}{t} - \frac{p}{7=2} - \frac{c_3}{24} - \frac{p}{t} - \frac{p}{5=2} - \frac{c_3}{16} - \frac{t^3}{64} - \frac{t^5}{64} + \\
 & \frac{2}{1} = 5 \quad 315 \quad (3.64)
 \end{aligned}$$

Thus equation (3.63) is collocated at equally spaced interior points $t = t_i$ on $[0,1]$, where $t_i = \frac{i}{4}$. Hence 4 algebraic linear equations are obtained with 5 unknown constants. One extra equation is obtained using the initial condition. Solving these 5 linear algebraic equations together with the use of some computer software (maple 18), the values of the constants, c_i ; $i = 0; 1; \dots; 4$ are obtained as:

$$\begin{aligned}
 c_0 &= 0.9999999988; \quad c_1 = :3000000002; \quad c_2 = :3214285712; \quad c_3 = :1250000004; \\
 c_4 &= 5.435819021 \cdot 10^{-10}
 \end{aligned}$$

Substituting these values into the assumed approximate solution given in equation (3.62), after simplification, gives

$$y_4(t) = :8999999983 + 4 \cdot 10^{-19}t + 1 \cdot 10^{-8}t^2 + 1.000000018t^3 + 8.697310433 \cdot 10^{-9}t^4$$

Solution of example 1 by Quadratic Weighted Polynomial (QWP)

Here, we assumed an approximate solution of the form:

$$y_4(t) = \sum_{j=0}^4 c_j Q_j(t) \quad (3.65)$$

where c_j ; $j = 0(1)4$ are unknown constants to be determined and $Q_j(t)$ are the Quadratic Weighted Polynomials defined in equation (3.37). Equation (3.64) is expanded in conjunction with equation (3.36) to get

$$\begin{aligned}
 y_4(t) &= c_0 + \frac{1}{2}c_1(4t-1) + \frac{1}{15}c_2(60t^2-40t+4) + \frac{1}{7}c_3(56t^3-63t^2+18t-1) + \frac{1}{105}c_4(16 \\
 & \quad 80t^4-2688t^3+1344t^2-224t+8) \quad (3.66)
 \end{aligned}$$

Putting (3.62) into (3.60), and applying (1.15) and (1.18), gives

$$\begin{aligned}
 & 4=3 t_2 c_0 \quad \frac{128 t_4 c_2}{105} + \frac{96 t_4 c_3}{35} + 2=3 t_2 c_1 \quad \frac{512 t_5 c_3}{151} + \frac{16 t_3 c_1}{15} + \frac{8192 t_5 c_4}{157} + \frac{8192 t_6 c_4}{53} + \frac{64 t_3 c_2}{46} \\
 & + \frac{464 t^2 c_2}{45} + \frac{848 t^3 c_3}{35} + \frac{500 t^2 c_3}{21} + \frac{36 t c_3}{7} + \frac{4096 t^4 c_4}{75} + \frac{18176 t^3 c_4}{225} + \frac{2144 t^2 c_4}{63} + \frac{64 t c_4}{15} + 4 c_1 t \\
 & + c_0 \quad 1=2 c_1 + \frac{4 c_2}{15} + \frac{8 c_4}{105} = \frac{16 t^3}{5} + \frac{64 t^5}{315} \quad (3.67)
 \end{aligned}$$

Thus equation (3.66) is collocated at equally spaced interior points $t = t_i$ on $[0,1]$, where $t_i = \frac{i}{4}$. Hence 4 algebraic linear equations are obtained with 5 unknown constants. One extra equation is obtained using the initial condition. Solving these 5 linear algebraic equations together with the use of some computer software (maple 18), the values of the constants, c_i ; $i = 0; 1; \dots; 4$ are obtained as:

$$\begin{aligned}
 c_0 &= 0.5000000127; \quad c_1 = :2142857131; \quad c_2 = :2812500017; \quad c_3 = :1249999986; \\
 c_4 &= 6.852717963 \times 10^{-10}
 \end{aligned}$$

Substituting these values of into assumed approximate solution given in equation (3.65), after simplification, gives

$$y_4(t) = :4500000043 \times 10^{-8} t + 1.42 \times 10^{-7} t^2 + :9999998707 t^3 + 1.096434874 \times 10^{-8} t^4$$

Solution of example 1 by Cubic Weighted Polynomial (CWP)

Here, we assumed an approximate solution of the form:

$$y_4(t) = \sum_{j=0}^4 c_j C_j(t) \quad (3.68)$$

where c_j ; $j = 0(1)4$ are unknown constants to be determined and $C_j(t)$ are the Cubic Weighted Polynomials defined in equation (3.52). Equation (3.64) is expanded in conjunction with equation (3.52) to get

$$\begin{aligned}
 y_4(t) &= c_0 + \frac{1}{5} c_1 (10t - 2) + \frac{1}{21} c_2 (84t^2 - 48t + 4) + \frac{1}{21} c_3 (168t^3 - 168t^2 + 42t - 2) + \frac{1}{165} c_4 (2640t^4 \\
 & \quad - 3840t^3 + 1728t^2 - 256t + 8) \quad (3.69)
 \end{aligned}$$

Putting (3.67) in (3.60), and applying (1.15) and (1.18), gives

$$\begin{aligned}
 & \frac{656 t^2 c_2}{165} + \frac{32 t c_2}{151} + \frac{368 t^3 c_3}{105} + \frac{1336 t^2 c_3}{53} + \frac{106496 t^4 c_4}{157} + \frac{182272 t^3 c_4}{46} + \frac{13792 t^2 c_4}{512} \\
 & + \frac{4 c_1 t}{165} + \frac{63}{151} + \frac{7}{4} + \frac{15}{2} = \frac{63}{8} + \frac{4 t c_3}{16} + \frac{1925}{819} + \frac{2475}{4} + \frac{495}{512} \\
 & + \frac{128 t^3 c_2}{105} + \frac{16384 t^5 c_4}{3465} + \frac{256 t^4 c_3}{105} = \frac{16 t^3}{5} + \frac{64 t^5}{315} \quad (3.70)
 \end{aligned}$$

Thus equation (3.69) is collocated at equally spaced interior points $t = t_i$ on $[0,1]$, where $t_i = \frac{i}{4}$. Hence 4 algebraic linear equations are obtained with 5 unknown constants. One extra equation is obtained using the initial condition. Solving these 5 linear algebraic equations together with the use of some computer software (maple 18), the values of the constants, c_i ; $i = 0; 1; \dots; 4$ are obtained as:

$$c_0 = 0.2857142964; c_1 = 0.1607142845; c_2 = .2500000017; c_3 = .1409999983;$$

$$c_4 = 1.16321965 \times 10^{-9}$$

Substituting these values of into assumed approximate solution given in equation (3.68), after simplification, gives

$$y_4(t) = .2559047713 + 0.30571430 \times 10^{-6}t + .127999979t^2 + 1.127999986t^3 + 1.86115144 \times 10^{-10}t^4$$

Solution of example 1 by Linear Weighted Polynomial (LWP)

Solving equation (3.60) using the LWP as basis function at $t = 0.5$ for $N = 5$, following the same procedure as above, we obtain the constant values as:

$$c_0 = 0.999998568; c_1 = 0.2142857131; c_2 = 0.18125000;$$

$$c_3 = 0.129999986; c_4 = 6.852717963 \times 10^{-9};$$

$$c_5 = 1.79563263 \times 10^{-9}$$

Substituting these values of into assumed approximate solution given in equation (3.61) after simplification, gives:

$$y_5(t) = 2.602214184 \times 10^{-6} + 1.516 \times 10^{-6}t + 2.1876 \times 10^{-6}t^2 + .9999999875t^3$$

$$+ 1.096434874 \times 10^{-8}t^4 + 3.0071796 \times 10^{-8}t^5$$

Solution of example 1 by Quadratic Weighted Polynomial (QWP)

Solving equation (3.60) using the QWP as basis function at $t = 0.5$ for $N = 5$, following the same procedure as above, we obtain the constant values as:

$$c_0 = 0.333333333; c_1 = 1.142857131; c_2 = 0.281550007;$$

$$c_3 = 0.1649999986; c_4 = 5.8527963 \times 10^{-8};$$

$$c_5 = 0.23117963 \times 10^{-8}$$

Substituting these values of into assumed approximate solution given in equation (3.61) after simplification, gives

$$y_5(t) = 2.492215684 \times 10^{-7} + 1.176 \times 10^{-7}t + 2.7585 \times 10^{-8}t^2 + .9999999785t^3$$

$$+ 1.096434874 \times 10^{-8}t^4 + 0.45089631 \times 10^{-8}t^5$$

Solution of example 1 by Cubic Weighted Polynomial (CWP)

Solving equation (3.60) using the CWP as basis function at $\alpha = 0.5$ for $N = 5$, following the same procedure as above, we obtain the constant values as:

$$\begin{aligned} c_0 &= 0.02857142964; c_1 = 0.385072845; c_2 = 0.254390067; \\ c_3 &= 0.1239999983; c_4 = 1.1641960 \times 10^{-7}; \\ c_5 &= 1.032551961 \times 10^{-7} \end{aligned}$$

Substituting these values of into assumed approximate solution given in equation (3.61) after simplification, gives

$$\begin{aligned} y_5(t) &= 2.163848527 \times 10^{-7} + 1.18 \times 10^{-8}t + 3.2475 \times 10^{-6}t^2 + 9.999999592t^3 \\ &\quad + 1.861151384 \times 10^{-6}t^4 + 8.523321961 \times 10^{-7}t^5 \end{aligned}$$

Solution of example 1 by Linear Weighted Polynomial (LWP)

Solving equation (3.60) using the LWP as basis function at $\alpha = 0.5$ for $N = 6$, following the same procedure as above, we obtain the constant values as:

$$\begin{aligned} c_0 &= 0.999999964; c_1 = 0.3521854251; c_2 = 0.25657417; c_3 = 0.1349999983; \\ c_4 &= 1.12321961 \times 10^{-7}; c_5 = 9.00983643 \times 10^{-6}; c_6 = 0.15437 \times 10^{-7} \end{aligned}$$

Substituting these values of into assumed approximate solution given in equation (3.61) after simplification, gives:

$$\begin{aligned} y_6(t) &= 1.6398527 \times 10^{-6} + 1.5167 \times 10^{-8}t + 3.279 \times 10^{-8}t^2 + 9.999999592t^3 \\ &\quad + 1.861151384 \times 10^{-8}t^4 + 1.18 \times 10^{-8}t^5 + 1.38400 \times 10^{-8}t^6 \end{aligned}$$

Solution of example 1 by Quadratic Weighted Polynomial (QWP)

Solving equation (3.60) using the QWP as basis function at $\alpha = 0.5$ for $N = 6$, following the same procedure as above, we obtain the constant values as:

$$\begin{aligned} c_0 &= 0.333333333; c_1 = 0.2342857131; c_2 = 0.281251754; c_3 = 0.1249999986; \\ c_4 &= 6.852717963 \times 10^{-7}; c_5 = 7.010983543 \times 10^{-6}; c_6 = 0.66990017 \times 10^{-7} \end{aligned}$$

Substituting these values of into assumed approximate solution given in equation (3.61) after simplification, gives

$$\begin{aligned} y_6(t) &= 2.302211184 \times 10^{-9} + 0.16990 \times 10^{-8}t + 2.1128 \times 10^{-8}t^2 + 9.999999715t^3 \\ &\quad + 1.096434874 \times 10^{-8}t^4 + 1.10018 \times 10^{-8}t^5 + 1.95138400 \times 10^{-8}t^6 \end{aligned}$$

Solution of example 1 by Cubic Weighted Polynomial (CWP)

Solving equation (3.60) using the CWP as basis function at $\alpha = 0.5$ for $N = 6$, following the same procedure as above, we obtain the constant values as:

$$c_0 = 0.035071358; c_1 = 0.142857131; c_2 = 0.812500017; c_3 = 0.1146999986; \\ c_4 = 6.8527179 \times 10^{-7}; c_5 = 7.04517963 \times 10^{-7}; c_6 = 0.1257179 \times 10^{-6}$$

Substituting these values of into assumed approximate solution, gives:

$$y_6(t) = 2.50502184 \times 10^{-7} + 1.6316 \times 10^{-8}t + 2.8502 \times 10^{-8}t^2 + .9999999715t^3 \\ + 1.096434874 \times 10^{-8}t^4 + 1.951018 \times 10^{-7}t^5 + 1.11238405 \times 10^{-8}t^6$$

Solution of example 1 by Linear Weighted Polynomial (LWP)

Solving equation (3.60) using the LWP as basis function at $\alpha = 0.5$ for $N = 7$, following the same procedure as above, we obtain the constant values as:

$$c_0 = 0.1000000178; c_1 = 0.2999999970; c_2 = 0.3214285716; c_3 = 0.3249999899; \\ c_4 = 1.241103152 \times 10^{-8}; c_5 = 1.382852837 \times 10^{-8}; c_6 = 2.124688474 \times 10^{-8}; \\ c_7 = 9.182353437 \times 10^{-7}$$

Substituting these values of into assumed approximate solution given in equation (3.61) after simplification, gives:

$$y_7(t) = 1.704759008 \times 10^{-8} + 0.4796698377e^{-5}t^6 + 0.7129966938e^{-5}t^4 + 0.117534124 \\ 0e^{-5}t^7 + 0.8029793173 \times 10^{-8}t^5 + .9999963680t^3 + 0.1075507298e^{-5}t^2 \\ 1.775263788 \times 10^{-7}t$$

Solution of example 1 by Quadratic Weighted Polynomial (QWP)

Solving equation (3.60) using the QWP as basis function at $\alpha = 0.5$ for $N = 7$, following the same procedure as above, we obtain the constant values as:

$$c_0 = 0.003571422461; c_1 = 0.2142858388; c_2 = 0.282502336; c_3 = 0.1250201658; \\ c_4 = 8.7449256 \times 10^{-8}; c_5 = 2.7259030 \times 10^{-8}; c_6 = 2.2642756 \times 10^{-8}; \\ c_7 = 1.211991229 \times 10^{-8}$$

Substituting these values of into assumed approximate solution obtained , gives

$$y_7(t) = .9999951289t^3 + 0.1029984644e^{-4}t^5 + 0.9248082086e^{-5}t^4 + 2.747139760 \\ 10^{-8} \\ + 0.6200141731 \times 10^{-8}t^6 + 0.1522838014 \times 10^{-5}t^2 + 2.687548688 \times 10^{-7}t + 0.1551348773e^{-8}t^7$$

Solution of example 1 by Cubic Weighted Polynomial (CWP)

Solving equation (3.60) using the CWP as basis function at $\alpha = 0.5$ for $N = 7$, following the same procedure as above, we obtain the constant values as:

$$\begin{aligned} c_0 &= 0.2857141304 \times 10^{-5}; c_1 = 0.1607142651; c_2 = 0.2499999341; c_3 = 0.1249999762; \\ c_4 &= 6.856099879 \times 10^{-8}; c_5 = 3.938068453 \times 10^{-8}; c_6 = 2.882483665 \times 10^{-8}; \\ c_7 &= 1.180688538 \times 10^{-8} \end{aligned}$$

Substituting these values of into assumed approximate solution given in equation (3.61) after simplification, gives:

$$\begin{aligned} y_7(t) &= 0.99851101 \times 10^{-7} t^4 + 0.1511281 \times 10^{-7} t^7 \\ &+ 530725 \times 10^{-10} t^2 + 0.835729 \times 10^{-1} t^6 \\ &+ 0.531 t^3 + 1.000005031 t^3 + 0.10588227 \times 10^{-1} t^5 \\ &+ 2.61076665 \times 10^{-8} t^5 + 2.586591325 \times 10^{-8} t^7 \end{aligned}$$

Solution of example 1 by Linear Weighted Polynomial (LWP)

Solving equation (3.60) using the LWP as basis function at $\alpha = 0.5$ for $N = 8$, following the same procedure as above, we obtain the constant values as:

$$\begin{aligned} c_0 &= 0.0257142964; c_1 = 0.1657142845; c_2 = 0.2500000017; c_3 = 0.1249999983; \\ c_4 &= 1.16321961 \times 10^{-9}; c_5 = 1.02103152 \times 10^{-8}; c_6 = 1.25610152 \times 10^{-8}; \\ c_7 &= 2.25509720 \times 10^{-8}; c_8 = 0.00751352 \times 10^{-8} \end{aligned}$$

Substituting these values of into assumed approximate solution given in equation (3.61) after simplification, gives:

$$\begin{aligned} y_8(t) &= 0.00850167 \times 10^{-8} t^4 + 0.165128132 \times 10^{-7} t^7 + 0.173072545 \times 10^{-6} t^2 + 0.20080729 \times 10^{-6} t^6 \\ &+ 0.5031 t^3 + 1.10522877 \times 10^{-7} t^5 + 2.110766645 \times 10^{-7} t^7 + 2.08591325 \times 10^{-8} t^8 + 2.105281789 \times 10^{-8} t^8 \end{aligned}$$

Solution of example 1 by Quadratic Weighted Polynomial (QWP)

Solving equation (3.60) using the QWP as basis function at $\alpha = 0.5$ for $N = 8$, following the same procedure as above, we obtain the constant values as:

$$\begin{aligned}
 c_0 &= :10000058; c_1 = :29999970; c_2 = :324285716; c_3 = :1249999899; \\
 c_4 &= 1:21103152 \qquad \qquad \qquad 10^8; c_5 = 1:38285283 \\
 c_7 &= \\
 9:182353 \\
 437 \qquad 10^9; c_8 &= \\
 1:103103152 \qquad 10^{10}
 \end{aligned}$$

Substituting these values of into assumed approximate solution given in equation (3.61) after simplification, gives:

$$y_8(t) = 0.000001704759008 + 0.0004796698377 t^6 + 0.000007129966938 t^4 + 0.000001175341240 t^7 + 0.000008029793173 t^5 + 0.9999963680 t^3 + 0.000001075507298 t^2 + 0.0000001775263788 + 2.0000005603278 t^8$$

Solution of example 1 by Cubic Weighted Polynomial (QWP)

Solving equation (3.60) using the QWP as basis function at $\alpha = 0.5$ for $N = 8$, following the same procedure as above, we obtain the constant values as:

$$\begin{aligned} c_0 &= 0.3256533333; c_1 = 0.42857131; c_2 = 0.21250007; \\ c_3 &= 0.6666666666; c_4 = 0.4531857131; c_5 = 0.275028500017 \\ c_6 &= 0.149999986; c_7 = 6.82352717963 \times 10^{-10}; c_8 \\ &= 5.000082539963 \times 10^{-10} \end{aligned}$$

Substituting these values of into assumed approximate solution given in equation (3.61) after simplification, gives:

$$y_8(t) = 1.562214684 \times 10^{-9} + 0.1762 \times 10^{-8} t + 1.5780 \times 10^{-8} t^2 + 9.999999715 t^3 + 1.096434874 \times 10^{-8} t^4 + 0.000001175341240 t^7 + 0.01280000802 t^5 + 0.99999213680 t^3 + 0.000001075507298 t^2 + 0.000456900012 + 2.00325005603278 t^8$$

SOLUTION OF EXAMPLE 2

Consider initial value multi-order fractional integro differential equation:

$$D^\alpha y(t) + (t-1)y^{(0)}(t) - e^t y^{(0)}(t) = \frac{(3)}{(3.5)} t^{2.5} - \frac{(3)}{(4.5)} t^{3.5} - \frac{(3)}{(9.5)} t^{7.5} + J^{0.5} y(t) + J^{0.75} y(t) \quad t \in [0, 1] \quad (3.71)$$

Subject to $y(0) = 0$, $y(1) = 1$. The exact solution is $y(t) = t^3$ at $\alpha = 2$. (Omar, I.K (2014))

Solution of example 2 by Linear Weighted Polynomial (LWP) at $\alpha = 1.5$ for $N = 4$

Here, we assumed an approximate solution of the form:

$$y_4(t) = \sum_{j=0}^4 c_j L_j(t) \quad (3.72)$$

where c_j ; $j = 0(1)4$ are unknown constants to be determined and $L_j(t)$ are the Linear Weighted Polynomials defined in equation (3.22). Equation (3.71) is expanded in conjunction with

equation (3.21) to get

$$y_4(t) = c_0 + \frac{1}{3}c_1(6t^2) + \frac{1}{5}c_2(20t^2 - 16t + 2) + \frac{1}{35}c_3(280t^3 - 360t^2 + 120t + 8) + \frac{1}{63}c_4(1008t^4 - 1792t^3 + 1008t^2 - 192t + 8) \quad (3.73)$$

Putting (3.71) in (3.70), and applying (1.15) and (1.18), gives

$$\begin{aligned} & 2=5c_2 \frac{8c_3}{35} + \frac{2=3c_1}{224t^2c_2} + \frac{8c_4}{36tc_2} + c_0 + \frac{64t^3c_2}{15} + \frac{256t^4c_3}{35} + \frac{4096t^5c_4}{315} + \frac{264tc_0}{45} + \\ & \frac{16=3c_1t}{2944t^2c_4} + \frac{15}{400tc_4} + \frac{256t^3c_3}{7} + 32t^2c_3 + \frac{256tc_3}{35} + \frac{26624t^4c_4}{315} + \frac{4864t^3c_4}{45} + \\ & \frac{13}{63} - \frac{17}{63} \quad 3:205893728t^4c_2 + 2:204051939t^9c_1 + 18:23797322t^4c_4 + 3:3 \\ & 58555334t^9c_4 - 8:639039941t^4c_4 + 0:4408103875t^5c_3 - 3:778374750t^9c_3 + 3:5264 \\ & 83101t^9c_2 - 12:82357491t^4c_4 + 1:928545447t^5c_0 - 5:129429965t^4c_3 + 1:28569696 \\ & t^5c_1 - 0:7714181783t^5c_2 - 0:2448946598t^5c_4 + 8:243726730t^4c_3 = \frac{13}{5} - \frac{16t^3}{35} - \frac{32t^4}{35} \\ & 0:6411787457t^4 \quad (3.74) \end{aligned}$$

Equation (3.73) is collocated at equally spaced interior points on [0,1], to give 3 equations with 5 unknown constants. Two extra equations were obtained using the initial conditions. Solving these 5 linear algebraic equations together with the use of some computer software (maple 18), the values of the constants, c_i ; $i = 0; 1; \dots; 4$ are obtained as:

$$c_0 = 0:1000000007; c_1 = 0:3000000004; c_2 = 0:3214285720; \\ c_3 = 0:126539862; c_4 = 1:340208933 \cdot 10^{-9}$$

Substituting these values of into assumed approximate solution given in equation (3.72), gives

$$y_4(t) = 0:3428501083 \cdot 10^{-9} + 0:5142856 \cdot 10^{-8}t - 0:15428569 \cdot 10^{-9}t^2 + 1:012000000t^3 + 2:144334293 \cdot 10^{-10}t^4$$

Solution of example 2 by Quadratic Weighted Polynomial (QWP) at $\alpha = 1:5$

Here, we assumed an approximate solution of the form:

$$y_4(t) = \sum_{j=0}^4 c_j Q_j(t) \quad (3.75)$$

where c_j ; $j = 0(1)4$ are unknown constants to be determined and $Q_j(t)$ are the Quadratic Weighted Polynomials defined in equation (3.37). Equation (3.74) is expanded in conjunction

with equation (3.37) to get

$$y_4(t) = c_0 + \frac{1}{2} c_1(4t - 1) + \frac{1}{15} c_2(60t^2 - 40t + 4) + \frac{1}{7} c_3(56t^3 - 63t^2 + 18t - 1) + \frac{1}{105} c_4(1680t^4 - 2688t^3 + 1344t^2 - 224t + 8) \quad (3.76)$$

Putting (3.75) in (3.70), and applying (1.15) and (1.18), gives

$$\begin{aligned} & \frac{256 t^4 c_3}{32 t^2 c_2} + \frac{16384 t^5 c_4}{32 t c_2} + \frac{512 t^5 c_3}{64 t^2 c_3} + \frac{128 t^3 c_2}{10649 t^4 c_4} + \frac{16 t^3 c_1}{18227 t^5 c_4} + \frac{2048 t^6 c_4}{13792 t^2 c_4} + \frac{128 t^4 c_2}{13792 t^2 c_4} + \frac{p - p -}{512 t c_4} + \frac{4 c_1 t}{8 c_4} \\ & + 3 \frac{p - p -}{p - p -} + 7 \frac{p - p -}{p - p -} + 5 \frac{p - p -}{p - p -} + 3 \frac{p - p -}{p - p -} + 4 t c_3 + \frac{1925}{178 t^5 c_4} + \frac{2475}{178 t^5 c_4} + 5 \frac{p - p -}{p - p -} + \frac{165}{p - p -} \\ & 2 = 5 t c_1 + 4 \frac{t c_3}{p - p -} + \frac{512 t c_4}{p - p -} + \frac{1}{p - p -} + \frac{2 = 21}{p - p -} + \frac{t c_3 + 55}{p - p -} + \frac{5}{p - p -} + \frac{12 t c_2}{p - p -} \\ & + 80 t c_4 + \frac{1}{5} + 4 t c_1 + 32 t c_3 + 24 t c_3 + 7 \frac{16 t^3}{3} + \frac{64 t^5}{3} \\ & = 5 \frac{1}{5} \quad (3.77) \end{aligned}$$

Equation (3.76) is collocated at equally spaced interior points on [0,1], to give 3 equations with 5 unknown constants. Two extra equations were obtained using the initial conditions. Solving these 5 linear algebraic equations together with the use of some computer software (maple 18), the values of the constants, c_i ; $i = 0; 1; \dots; 4$ are obtained as:

$$c_0 = 0.050625000; c_1 = 0.1420071560; c_2 = 0.2812100066; c_3 = 0.487599958; c_4 = 3.097698049 \cdot 10^{-9}$$

Substituting these values of into assumed approximate solution given in equation (3.65), gives

$$y_4(t) = 8.186015089 \cdot 10^{-9} + 3.23 \cdot 10^{-8} t + 1.04 \cdot 10^{-7} t^2 + 9.999998873 t^3 + 4.956316878 \cdot 10^{-8} t^4$$

Solution of example 2 by Cubic Weighted Polynomial (CWP), at $\alpha = 1.5$

Here, we assumed an approximate solution of the form:

$$y_4(t) = \sum_{j=0}^4 c_j C_j(t) \quad (3.78)$$

where c_j ; $j = 0(1)4$ are unknown constants to be determined and $C_j(t)$ are the Cubic Weighted Polynomials defined in equation (3.52). Equation (3.77) is expanded in conjunction with equation (3.52) to get

$$y_4(t) = c_0 + \frac{1}{5}c_1(10t - 2) + \frac{1}{21}c_2(84t^2 - 48t + 4) + \frac{1}{21}c_3(168t^3 - 168t^2 + 42t - 2) + \frac{1}{165}c_4(2640t^4 - 3840t^3 + 1728t^2 - 256t + 8) \quad (3.79)$$

Putting (3.78) in (3.70), and applying (1.15) and (1.18), gives

$$\begin{aligned}
 & 8:6390399 \, t^{\frac{21}{4}} c_4 + 1:7098099 \, t^{\frac{9}{4}} c_4 + 2:2040519 \, t^{\frac{9}{4}} c_3 + 0:1836709948 \, t^{\frac{5}{4}} c_3 + 2:518916500 \, t^{\frac{9}{4}} c_2 \\
 & 0:3673419897 \, t^{\frac{5}{4}} c_2 + 6:411787457 \, t^{\frac{13}{4}} c_3 + 8:39361267 \, t^{\frac{13}{4}} c_4 + 2:2040519 \, t^{\frac{9}{4}} c_1 + 0:0935052337 \, t^{\frac{5}{4}} c_4 \\
 & 3:205893728 \, t^{\frac{13}{4}} c_2 + 5:129429965 \, t^{\frac{17}{4}} c_3 + 14:921978 \, t^{\frac{17}{4}} c_4 + 1:928545447 \, t^{\frac{5}{4}} c_0 + 0:7714181783 \, t^{\frac{5}{4}} c_1 \\
 & 64 \, t_3 c_2 + 4096 \, t_5 c_4 + 256 \, t_4 c_3 + 2 \, t c_0 + 8 = 3 \, t^2 c_1 + 24 \, c_1 t + 96 \, t_2 c_2 + 104 \, t c_2 + 512 \, t_3 c_3 + 24 \, t^2 c_3 + 88 \, t c_3 \\
 & \frac{153153557211521}{6144 \, t^4 c_4} + \frac{23552 \, t^3 c_4}{275} + \frac{14848 \, t^2 c_4}{495} + \frac{16 \, t c_4}{5} + \frac{8 \, c_4}{165} + \frac{4 \, c_2}{21} = \frac{16 \, t^3}{5} + \frac{32 \, t^4}{35} + 0:641 \\
 & + \frac{17}{1787457 \, t^{\frac{17}{4}}} \quad (3.80)
 \end{aligned}$$

Equation (3.79) is collocated at equally spaced interior points on [0,1], to give 3 equations with 5 unknown constants. Two extra equations are obtained using the initial conditions. Solving these 5 linear algebraic equations together with the use of some computer software (maple 18), the values of the constants, c_i ; $i = 0, 1, \dots, 4$ are obtained as:

$$\begin{aligned}
 c_0 &= 0:02857143090; \quad c_1 = 0:1607142858; \quad c_2 = 0:2500000297; \quad c_3 = 0:354999991; \\
 c_4 &= 1:312943868 \, 10^{-9}
 \end{aligned}$$

Substituting these values of into assumed approximate solution given in equation (3.78), gives

$$\begin{aligned}
 y_4(t) &= 3:083657885 \, 10^{-9} + 1:121 \, 10^{-8} t + 4:02 \, 10^{-8} t^2 + :9999999542 t^3 \\
 &\quad + 2:100710189 \, 10^{-8} t^4
 \end{aligned}$$

Example 2 by Linear Weighted Polynomial (LWP)

Solving equation (3.70) together with the initial conditions using the LWP as basis function at $\tau = 1:5$ for $N = 5$, the constant coefficients are:

$$\begin{aligned}
 c_0 &= 0:050000000; \quad c_1 = 0:21857156; \quad c_2 = 0:271256600; \quad c_3 = 0:299999988; \\
 c_4 &= 3:097698049 \, 10^{-9}; \quad c_5 = 6:25642 \, 10^{-9}
 \end{aligned}$$

Substituting these values of into assumed approximate solution, gives:

$$\begin{aligned}
 y_5(t) &= 8:09824714 \, 10^{-9} + 6:1450 \, 10^{-8} t + 3:8071 \, 10^{-7} t^2 + :9999993753 t^3 + 3:99 \\
 &\quad 65389 \, 10^{-8} t^4 + 1:65320589 \, 10^{-8} t^5
 \end{aligned}$$

Solution of example 2 by Quadratic Weighted Polynomial (QWP)

Solving equation (3.70) using the QWP as basis function at $\tau = 1:5$ for $N = 5$, the constant

coefficients:

$$c_0 = 0.052587239; c_1 = 0.0607342858; c_2 = 0.265880025; c_3 = 0.134999991; \\ c_4 = 1.312943868 \times 10^{-9}; c_5 = 5.71430413 \times 10^{-9}$$

Substituting these values into assumed approximate solution, gives:

$$y_5(t) = 3.083657885 \times 10^{-9} + 2.621 \times 10^{-8}t + 4.02 \times 10^{-8}t^2 + .9999999542t^3 \\ + 2.100710189 \times 10^{-8}t^4 + 2.112310189 \times 10^{-8}t^5$$

Solution of example 2 by Cubic Weighted Polynomial (CWP)

Solving equation (3.70) using the CWP as basis function at $\alpha = 1.5$ for $N = 5$, the constant coefficients are:

$$c_0 = 0.028571530; c_1 = 0.1607142858; c_2 = 0.25825820; c_3 = 0.159999998; \\ c_4 = 1.312943868 \times 10^{-9}; c_5 = 0.5714285 \times 10^{-9}$$

Substituting these values into assumed approximate solution, gives:

$$y_5(t) = 2.08565750 \times 10^{-9} + 3.2351 \times 10^{-8}t + 4.602 \times 10^{-8}t^2 + .999999988t^3 \\ + 2.100710189 \times 10^{-8}t^4 + 1.0800045272 \times 10^{-8}t^5$$

Solution of example 2 by Linear Weighted Polynomial (LWP) at $\alpha = 1.5$ for $N = 6$

Solving equation (3.70) using the LWP as basis function, we have the constants:

$$c_0 = 0.02857143090; c_1 = 0.1607142858; c_2 = 0.2500650027; c_3 = 0.125999998; \\ c_4 = 1.312943868 \times 10^{-9}; c_5 = 0.6783427906 \times 10^{-8}; c_6 = 3.2001153807 \times 10^{-8}$$

Substituting these values into assumed approximate solution, gives:

$$y_6(t) = 3.3657885 \times 10^{-7} + 1.0021 \times 10^{-8}t + 3.025 \times 10^{-8}t^2 + .999998800t^3 \\ + 2.1010189 \times 10^{-8}t^4 + 2.00327932 \times 10^{-8}t^5 + 3.23675200 \times 10^{-8}t^6$$

Solution of example 2 by Quadratic Weighted Polynomial (QWP)

Solving equation (3.70) using the QWP as basis function, we have the constants:

$$c_0 = 0.02857143090; c_1 = 0.1607142858; c_2 = 0.20062490; c_3 = 0.1349999981; \\ c_4 = 1.312943868 \times 10^{-9}; c_5 = 1.501294 \times 10^{-9}; c_6 = 1.7102943 \times 10^{-9}$$

Substituting these values of into assumed approximate solution, gives:

$$y_6(t) = 3:093657885 \cdot 10^{-6} + 2:221 \cdot 10^{-8}t + 2:0264 \cdot 10^{-8}t^2 + :999999988t^3 \\ + 2:100610189 \cdot 10^{-8}t^4 + 2:03450710 \cdot 10^{-8}t^5 + 2:100088539 \cdot 10^{-8}t^6$$

Solution of example 2 by Cubic Weighted Polynomial (CWP) at $\alpha = 1:5$ for $N = 6$

Solving equation (3.70) using the CWP as basis function we have the constants:

$$c_0 = 0:0286943030; c_1 = 0:71426258; c_2 = 0:545007635; c_3 = 0:111999985; \\ c_4 = 1:312943868 \cdot 10^{-9}; c_5 = 2:00310043800 \cdot 10^{-9}; c_6 = 1:30015529438 \cdot 10^{-9}$$

Substituting these values of into assumed approximate solution, gives:

$$y_6(t) = 2:8365785 \cdot 10^{-9} + 1:155 \cdot 10^{-8}t + 8:953 \cdot 10^{-8}t^2 + :999998850t^3 \\ + 2:0710189 \cdot 10^{-8}t^4 + 1:5480921 \cdot 10^{-9}t^5 + 1:2325310061 \cdot 10^{-9}t^6$$

Solution of example 2 by Linear Weighted Polynomial (LWP) at $\alpha = 1:5$ for $N = 7$

Solving equation (3.70) using the QWP as basis function, we have the constants:

$$c_0 = :1000000125; c_1 = :3000000056; c_2 = :3214285718; c_3 = :1249999912; \\ c_4 = 1:44291 \cdot 10^{-8}; c_5 = 1:39986 \cdot 10^{-8}; c_6 = 2:73433 \cdot 10^{-8}; \\ c_7 = 1:422881 \cdot 10^{-8}$$

Substituting these values of into assumed approximate solution, gives

$$y_7(t) = 1:869943847 \cdot 10^{-7}t + 1:967979747 \cdot 10^{-8} + 0:1264105253 \cdot 10^{-8}t^2 \\ 0:1821288050 \cdot 10^{-8}t^7 + 0:7075733345 \cdot 10^{-8}t^6 + 0:11216052879 \cdot 10^{-9}t^5 + :9999954951t^3 \\ + 0:9401049622 \cdot 10^{-9}t^4$$

Solution of example 2 by Quadratic Weighted Polynomial (QWP) at $\alpha = 1:5$ for $N = 7$

Solving equation (3.70) using the QWP as basis function, we have the constants:

$$c_0 = 0:049999999419; c_1 = 0:228514719; c_2 = 0:2812499716; c_3 = :1253500068; \\ c_4 = 1:194013895 \cdot 10^{-8}; c_5 = 1:661276624 \cdot 10^{-10}; c_6 = 1:229761892 \cdot 10^{-8}; \\ c_7 = 1:33711049 \cdot 10^{-8}$$

Substituting these values of into assumed approximate solution, gives

$$y_7(t) = 9.537718015 \times 10^9 + 1.0000000065t^3 + 0.1612186407 \times 10^7 t^4 + 0.427124 \times 10^8 t^5 + 6.220628794 \times 10^8 t + 0.44544255 \times 10^8 t^6 + 2.3736 \times 10^7 t^2 + 0.1711501 \times 10^9$$

Solution of example 2 by Cubic Weighted Polynomial (CWP) at $\alpha = 1.5$ for $N = 7$

Solving equation (3.70) using the QWP as basis function, we have the constants:

$$c_0 = 0.285142528 \times 10^5; c_1 = .1657142861; c_2 = .64500000015; c_3 = .1258000001; \\ c_4 = 1.045499944 \times 10^8; c_5 = 7.473338697 \times 10^9; c_6 = 6.223001107 \times 10^9; \\ c_7 = 2.924050750 \times 10^9$$

Substituting these values of into assumed approximate solution, gives

$$y_7(t) = 3.165447624 \times 10^7 t^2 + 1.00000101t^3 + 0.2102628506 \times 10^8 t^4 + 3.7427960 \times 10^7 t^7 \\ + 0.2408652873 \times 10^9 t^5 + 5.592282557 \times 10^9 + 0.14770795 \times 10^5 t^6 + 4.42564836 \times 10^8 t$$

Solution of example 2 by Linear Weighted Polynomial (LWP) at $\alpha = 1.5$ for $N = 8$

Solving equation (3.70) using the LWP as basis function, we have the constants:

$$c_0 = 0.0223571090; c_1 = 0.15607280; c_2 = 0.25012021; c_3 = 0.1532499999; \\ c_4 = 1.3153029438 \times 10^9; c_5 = +1.70412305 \times 10^8; c_6 = 1.506309215031 \times 10^8; \\ c_7 = 2.012000045465 \times 10^8; c_8 = 2.57235309475 \times 10^8$$

Substituting these values of into assumed approximate solution, gives:

$$y_8(t) = 3.26836578 \times 10^9 + 3.2751 \times 10^8 t + 4.02 \times 10^8 t^2 + .999999999t^3 \\ + 2.0071019 \times 10^8 t^7 + 1.5230921 \times 10^8 t^5 + 1.254096201 \times 10^8 t^6 + 1.66 \\ 8021 \times 10^8 t^4 + 1.803212 \times 10^8 t^8$$

Solution of example 2 by Quadratic Weighted Polynomial (QWP) at $\alpha = 1.5$ for $N = 8$

Solving equation (3.70) using the QWP as basis function, we have the constants:

$$c_0 = 0.027143090; c_1 = 0.15672858; c_2 = 0.25000007; c_3 = 0.1249999981; \\ c_4 = 1.312943868 \times 10^9; c_5 = 1.04123230544 \times 10^7; c_6 = 1.06409213451 \times 10^8;$$

$$c_7 = 2.0004546639 \cdot 10^8; c_8 = 1.553094104 \cdot 10^8$$

Substituting these values of into assumed approximate solution, gives:

$$y_8(t) = 3:365087885 \ 10^9 + 2:401 \ 10^8 t + 5:4202 \ 10^8 t^2 + :9999999777 t^3 \\ + 2:100710189 \ 10^8 t^7 + 1:3206525309 \ 10^8 t^5 + 1:13096280 \ 10^8 t^6 \ 1:6543600 \\ 8021 \ 10^6 t^4 + 1:123800032 \ 10^8 t^8$$

Solution of example 2 by Cubic Weighted Polynomial (CWP) at $\alpha = 1:5$ for $N = 8$

Solving equation (3.70) using the CWP as basis function we have the constants:

$$c_0 = 0:357543090; c_1 = 0:163714858; c_2 = 0:250246897; c_3 = 0:179999951; \\ c_4 = 1:3129438 \ 10^9; c_5 = 1:104123 \ 10^8; c_6 = 1:225609 \ 10^8; \\ c_7 = 2:4580004 \ 10^8; c_8 = 1:0055345204 \ 10^8$$

Substituting these values of into assumed approximate solution, gives:

$$y_8(t) = 3:0807885 \ 10^9 + 2:521 \ 10^8 t + 4:02 \ 10^8 t^2 + :9999999555 t^3 \\ + 2:1101895 \ 10^8 t^7 + 1:589361 \ 10^8 t^5 + 3:1654964 \ 10^7 t^6 + 2:615436001 \ 8021 \\ 10^6 t^4 \ 1:1598000352 \ 10^8 t^8$$

SOLUTION OF EXAMPLE 3

Consider a fractional order integro-differential equation

$$D^\alpha y(t) = t(1 + e^t) + 3e^t + y^0(t) \int_0^t y(x)dx; \quad (3.81)$$

Subject to $y(0) = 1$

$3 < \alpha < 4$ and $0 < t < 1$ The exact solution is $y(t) = 1 + te^t$ at $\alpha = 4$. (Avipsita et al (2017))

Solution of example 3 by Linear Weighted Polynomial (LWP) at $\alpha = 3:25$

Here, we assumed an approximate solution of the form:

$$y_8(t) = \sum_{j=0}^8 c_j L_j(t) \quad (3.82)$$

where c_j ; $j = 0(1)8$ are unknown constants to be determined and $L_j(t)$ are the Linear Weighted Polynomials defined in equation (3.22). Equation (3.81) is expanded in conjunction with equa-

tion (3.21) to get

$$\begin{aligned}
 y_8(t) = & c_0 + \frac{1}{8} 3c_1(6t^2 - 2) + \frac{1}{5} 5c_2(20t^2 - 16t + 2) + \frac{35}{8} c_3(280t^3 - 360t^2 + 120t \\
 & + 63 c_4(1008t^4 - 1792t^3 + 1008t^2 - 192t + 8) + \frac{231}{8} c_5(7392t^5 - 16800t^4 + 13440t^3 - 4480t^2 \\
 & + 560t - 16) + \frac{3003}{1} c_6(192192t^6 - 532224t^5 + 554400t^4 - 268800t^3 + 60480t^2 - 6520t + 398) \\
 & + \frac{684585720}{1} c_7(87626972160t^7 - 256236175104t^6 + 284923312128t^5 - 149345884800t^4 + 366 \\
 & 64857600t^3 - 5728160940t^2 + 1420124840t - 116143116) + c_8(7330633674167901065379840 \\
 & t^8 - 43507905751146412173312000t^7 + 113623692198339912073605120t^6 \\
 & 17062573519926 \\
 & 9534948470784t^5 + 161104744779368919543221760t^4 - 97852290667632851030876160t^3 \\
 & + 3 \\
 & 7285336446910284794999880t^2 - 8135643125780429287106960t + \\
 & 777021914692904462637 \\
 & 658) \quad (3.83)
 \end{aligned}$$

Putting (3.82) in (3.80), and applying (1.15) and (1.18), gives

$$\begin{aligned}
 & \frac{256}{9} c_8 t^9 + \frac{18}{5} c_2 t + \frac{128}{35} c_3 t + \frac{24392}{63} c_4 t + \frac{134592}{77} c_5 t + \frac{4437506}{1001} c_6 t + \frac{128028368747}{24449490} c_7 t \\
 & + \frac{2581335721228830169308946}{11185659292858735} c_8 t + \frac{7710953044054489968742}{3389593725108707805} c_8 + \frac{9603630943}{29882710} c_7 + \frac{1613198}{3003} c_6 \\
 & + \frac{80656}{10760} c_5 + \frac{1688}{1688} c_3 + \frac{315962636068400}{315962636068400} t^8 c_8 + \frac{64}{64} t^7 c_6 \\
 & + \frac{231}{56882848} t^7 c_7 + \frac{63}{63926085932905460992} t^7 c_8 + \frac{35}{1216} t^6 c_6 + \frac{2=5}{139067488} t^6 c_7 \\
 & \frac{c_0 + 2=3}{313455} c_1 + 16 t^8 c_7 + \frac{831819491967}{313455} + \frac{7}{785634t^3} \\
 & 646854551684843 t^6 c_8 + 16 t^5 c_4 - 512 t^5 c_5 + 2784 t^5 c_6 - 624588272 t^5 c_7 + 1091022368329 t^5 c_8 + \\
 & 33895937251067051113135830512428510 \\
 & 2 t^4 c_3 + \frac{208}{9} t^4 c_4 + \frac{960}{11} t^4 c_5 + \frac{29600}{143} t^4 c_6 + \frac{1699649840}{62691} t^4 c_7 + \frac{1559397894185477954711936}{7457106195239157171} t^4 c_8 \\
 & + \frac{80}{80} t^3 c_3 + \frac{304}{304} t^3 c_4 + \frac{6400}{6400} t^3 c_5 + \frac{1112000}{1112000} t^3 c_6 + \frac{2418934656689}{2418934656689} t^3 c_7 + \frac{8623160}{8623160} t^3 c_8 \\
 & 4=3 t^3 c_2 + \frac{7}{7} + \frac{9}{9} + \frac{99}{99} + \frac{143}{143} + \frac{53788878}{53788878} + \frac{7865}{7865} \\
 & \frac{4427282536070482298328590}{22371318585717471513} t^3 c_8 + \frac{28}{28} t^2 c_2 + \frac{368}{368} t^2 c_4 + \frac{64040}{64040} t^2 c_5 + \frac{31997180}{31997180} t^2 c_6 \\
 & + \frac{671783367488}{26894439} t^2 c_7 + \frac{2117184559805647486560040}{22371318585717471513} t^2 c_8 \\
 & = t + te^t + 3 e^t \quad (3.84)
 \end{aligned}$$

Equation (3.83) is collocated at equally spaced interior points on [0,1], to give 8 equations with 9 unknown constants. One extra equation is obtained using the initial conditions. Solving these 9 linear algebraic equations together with the use of some computer software (maple 18), the values of the constants, c_i ; $i = 0; 1; \dots; 8$ are obtained as:

$$c_0 = 1.854732105; c_1 = 1.09482139; c_2 = 5.67098596; c_3 = 1.16126149; c_4 = 0.4598710^{-7};$$

$$c_5 = 1:1750250 \cdot 10^8; c_6 = 3:122556 \cdot 10^8; c_7 = 2:0058349 \cdot 10^8; c_8 = 1:05055309462 \cdot 10^{10}$$

Substituting these values of into assumed approximate solution, gives:

$$y_8(t) = :6599995645 \cdot 10^8 \cdot 1:23561391t + :99999990t^2 + 0:964591857t^3 \\ + 1:920124027t^4 + 2:100710189 \cdot 10^8 t^5 + 1:1256280 \cdot 10^8 t^6 + 1:65436008 \cdot 10^6 t^7 + 1:123800032 \cdot 10^{10} t^8$$

Solution of example 3 by Quadratic Weighted Polynomial (QWP) at $\alpha = 3:25$

Solving equation (3.80) for $N = 8$ and at $\alpha = 3:25$ the constants coefficients were found as:

$$c_0 = 1:299450284; c_1 = :1267400502; c_2 = 0:3892357846; c_3 = 0:133126117 \cdot 10^5; \\ c_4 = 5:125600540; c_5 = :1670141387; c_6 = :8906610973 \cdot 10^8; c_7 = :8234752462 \cdot 10^8; \\ c_8 = :6702490335 \cdot 10^8$$

Substituting these values of into assumed approximate solution, gives

$$y_8(t) = 5:75643 \cdot 10^9 + 1:140419765t + 1:000000001t^2 + 0:4382866003t^3 + 6:1057103 \cdot 10^8 t^5 \\ + 1:9082065 \cdot 10^8 t^8 + 1:14546280 \cdot 10^8 t^4 + 1:1336008021 \cdot 10^6 t^7 + 1:173879 \cdot 10^8 t^6$$

Solution of example 3 by Cubic Weighted Polynomial (CWP) at $\alpha = 3:25$

Solving equation (3.80) for $N = 8$ and at $\alpha = 3:25$ we obtained the constants coefficients as:

$$c_0 = 1:578838360; ; c_1 = :4458878618; c_2 = 0:2961080841; c_3 = 0:1331726 \cdot 10^6; \\ c_4 = 3:102125600590; c_5 = 3:157014973138; c_6 = :375066105 \cdot 10^8; c_7 = 1:223430 \cdot 10^8; \\ c_8 = 2:0703490 \cdot 10^8$$

Substituting these values of into assumed approximate solution, gives:

$$y_{10}(t) = :95468025 \cdot 10^9 + 1:04197564t + 1:00000999t^2 + :53586676009t^3 \\ + :42619523575t^4 + 1:0821202 \cdot 10^8 t^5 + 1:51456280 \cdot 10^8 t^6 + 2:6520080 \cdot 10^6 t^7 + 1:25738000325 \cdot 10^{10} t^8$$

Solution of example 3 by Linear Weighted Polynomial (LWP) at $\alpha = 3:75$

Solving equation (3.80) for $N = 8$ and at $\alpha = 3:75$ the constants coefficients were found as:

$$c_0 = 0:4458878648; c_1 = 2:1473498590; c_2 = 0:2961080791; c_3 = 0:133172615 \cdot 10^8; \\ c_4 = 1:2395854732; c_5 = 1:790948210 \cdot 10^9; c_6 = 4:65673596 \cdot 10^8; c_7 = :509116126912 \cdot 10^{10}; \\ c_8 = 0:0045987623426 \cdot 10^{10}$$

Substituting these values of into assumed approximate solution, gives:

$$y_8(t) = 1.6 \cdot 10^{-9} + 65789302t + 0.376635418t^2 + 77902331t^3 + 1.090635231t^4 + 0.1592567 \cdot 10^{-8}t^5 \\ + 1.1165096333 \cdot 10^{-8}t^6 + 0.5003786590 \cdot 10^{-10}t^7 + 0.75698245 \cdot 10^{-10}t^8$$

Solution of example 3 by Quadratic Weighted Polynomial (QWP) at $\alpha = 3:75$ Solving equation (3.80) at $\alpha = 3:75$ and $N = 8$ and following the same procedure, the constant coefficients were obtained as:

$$\begin{aligned} c_0 &= 1:094821039; c_1 = 5:67098596; c_2 = :1161261149; c_3 = 0:459876226; \\ c_4 &= :670085338; c_5 = 1:290532001 \quad 10^{-10}; c_6 = :095740105 \\ &\quad 10^{-10}; c_7 = 0:23596 \\ &\quad 7846 \quad 10^{-10}; c_8 = 0:1331726117 \quad 10^{-10} \end{aligned}$$

Substituting these values of into assumed approximate solution, gives:

$$\begin{aligned} y_8(t) &= :864395864 \quad 10^{-9} + 1:56134917t + :782359967t^2 + 0:99159857t^3 + 2:28585163t^4 + 1:001 \\ &\quad 537 \quad 10^{-8}t^5 + 1:16509106 \quad 10^{-8}t^6 + 0:05003702 \quad 10^{-10}t^7 + 1:15690170 \quad 10^{-10}t^8 \end{aligned}$$

Solution of example 3 by Cubic Weighted Polynomial (CWP)

Solving equation (3.80) at $\alpha = 3:75$ and $N = 8$ and following the same procedure, the constant coefficients were obtained as:

$$\begin{aligned} c_0 &= 1:299450284; c_1 = :8974810411; c_2 = :1267400502; c_3 = 0:3892357846; c_4 = \\ &\quad 0:1331726117; \\ c_5 &= :12670055338; c_6 = 2:1674010675 \quad 10^{-10}; c_7 = \\ &\quad 0:10090567838 \quad 10^{-10}; \\ c_8 &= 1:5741517265 \quad 10^{-10} \end{aligned}$$

Substituting these values of into assumed approximate solution, gives

$$\begin{aligned} y_8(t) &= 0:578905 \quad 10^{-10} + 1:140419765t + 1:000000001t^2 + 3:543600356t^3 + 0:4261523576t^4 \\ &\quad + 1:12758493t^5 + 1:157165010 \quad 10^{-8}t^6 + 1:0504519 \quad 10^{-10}t^7 + 1:690177 \quad 10^{-10}t^8 \end{aligned}$$

SOLUTIONS TO EXAMPLE 4

Consider the initial value problem that consists of the multi-order fractional differential equation

$$D^\alpha y(t) + D^{\frac{1}{2}} y(t) + y^3(t) + (t+1)y^{(0)}(t) = t^9 \quad (3.85)$$

Subject to $y(0) = 0$ and $y^{(0)}(0) = 0$.

The exact solution is $y(t) = t^3$. (Yin Yang (2013))

Solution of example 4 by Linear Weighted Polynomial (LWP) for $N = 4$ and at $\alpha = 2:5$

we take an assumed approximate solution:

$$\begin{aligned} y_4(t) &= c_0 + \frac{1}{3}c_1(6t-2) + \frac{1}{5}c_2(20t^2-16t+2) + \frac{1}{35}c_3(280t^3-360t^2+120t-8) \\ &\quad + \frac{1}{63}c_4(1008t^4-1792t^3+1008t^2-192t+8) \quad (3.86) \end{aligned}$$

Substituting (3.68) into (3.84) and applying (1.15) with some simplifications, we have

$$\begin{aligned}
& 237:6776210t^3c^2c \overset{13}{358:9172156t^3c^2c} \overset{24}{477:0838066t^3c^2c} + \overset{23}{1361:244557t^8c} c^2 + \overset{13}{2291:891346t^5c} c^2 \\
& + \overset{14}{8181:872045t^5c} c^2 + \overset{23}{2263:555234t^5c^2c} : \overset{124}{3601176076c} c c + \overset{123}{6482116937c} c c + \overset{13}{6:111710255tc} c^2 \\
& 275:4899698t^3c_0c_1c_3 + 541:9769995t^3c_0c_1c_4 + 529:9130596t^3c_0c_2c_3 \quad 795:1396776t^3c_0c_2c_4 + 996:70264 \\
& 71t^3c c c \quad 693:5865123t^3c c c + 884:4488443t^3c c c + 6376:674502t^6c^2c \quad 16:20529234tc^2c \\
& + 1727 \\
& \overset{034}{6:04203t^7c} c^2 + \overset{123}{907:4963712t^9c^3} + \overset{124}{90615:19321t^10c^3} \quad \overset{24}{1964:081432t^6c^2c} \quad \overset{04}{15400:22942t^6c} \\
& c^2 + 18:2309 \\
& \overset{14}{5389tc^2c} \quad \overset{3}{1977:045666t^6c} c^2 + \overset{4}{85:07778480t^4c^2c} \quad \overset{23}{4:115629801tc} c^2 \quad \overset{14}{8:334150348tc} c^2 \\
& 0:1959823 \\
& \overset{03}{715e} \quad \overset{13}{1c_3c^2_4} + 0:3527682687e \quad \overset{12}{1c^2_3c_4} + 0:3429691501e \quad \overset{04}{1c_2c^2_4} + 1111220046c_2c^2_3 + 1080352823c^2_2c_4 \quad 17: \\
& 1555696tc^2_0c_2 + 170:1555696t^5c^2_1c_3 : 1944635081c^2_2c_3 \quad 0:5716152502e \quad \overset{14}{1c_1c^2_4} : 1852033411c_1c^2_3 : 5671 \\
& 852320c_1c^2_2 + 3000980063c^2_1c_4 : 5401764114c^2_1c_3 \quad 136:1244557t^3c_0c^2_2 \quad 15:2493952t^3c^2_0c_4 + 42:5389256 \\
& 40t^3c^2_0c_3 \quad 3199:608197t^4c_3c^2_4 \quad 394:4831165t^3c_0c^2_3 \quad 1330:130396t^4c^2_2c_3 \quad 2650:465592t^4c_1c^2_4 \quad 1358:4665 \\
& 07t^4c_1c^2_3 \quad 328:9674346t^4c_1c^2_2 + 781:4552085t^4c^2_1c_4 + 2952:352917t^4c^2_3c_4 \quad 152:6898656t^2c_1c_2c_4 + 133:346 \\
& 056t^2c_1c_3c_4 + 158:1636533t^2c_1c_2c_3 + 177:1778629t^2c_0c_2c_4 \quad 163:949568t^2c_0c_3c_4 \quad 170:1555696t^2c_0c_2c_3 + \\
& 68:6222784t^7c_1c^2_3 \quad 4839:980646t^7c_0c^2_4 + 10:63472310tc^2_0c_1 \quad 96:42148944t^2c_0c_1c_2 + 145:8476311t^2c_0c_1c_3 \\
& + 422:9581302t^3c^2_2c_3 : 3086722351c_0c_3c_4 \quad 54:69286166t^2c^2_0c_3 + 85:07778480t^2c^2_0c_4 + 64:17295768t^2c^2_2c_4 \\
& 680:6222784t^7c^2_2c_3 \quad 4597:981614t^7c^2_2c_4 + 1:05538466910t^8c^3_4 \quad 1:22379522410t^9c^3_4 \quad 4589:338792t^7c^2_2c^2_3 \\
& 124:7807876510t^3c^2_1c_2 : 5251715111c^3_1 + 1:234688940tc_3c^2_4 + 85:07778480t^4c_0c^2_2 + 85:07778480t^4c^2_0c_4 + \\
& 21:26944620t^2c^2_0c_2 + 5401764114c_0c_2c_4 + 413:7383845t^3c_3c^2_4 : 9723175406c_0c_2c_3 + 5444:978227t^10c^2_3c_4 \\
& 48139:69866t^6c_3c_4^2 \quad 831:8716736t^5c_1^2c_4 + 170:1555696t^5c_1c^2_2 + \frac{1024}{5} t_2c_4 + 1361:244557t^8c_0c_4^2 \quad 1:9049 \\
& 48651tc^2_3c_4 + 29176:24841t^6c^2_2c^2_4 \quad 10587:45766t^8c_1c^2_4 \quad 23681:33388t^9c^2_3c_4 + 5:803038020tc_2c_3c_4 \quad 8:642 \\
& 822583tc_1c_3c_4 + 12:604119627tc_1c_2c_4 + 12:03821717tc_0c_3c_4 \quad 16:85350404tc_1c_2c_3 \quad 17:28564517tc_0c_2c_4 \\
& 22:36330343tc_0c_2c_3 + 24:30793851tc_0c_1c_4 + 31:19518776tc_0c_1c_2 \quad 29:16952622tc_0c_1c_3 \quad 38719:84517t^{11} \\
& c^3_4 \quad 458:0343201t^3c^2_3c_4 \quad 52721:21776t^10c_3c^2_4 \quad 105:3189666t^2c_2c_3c_4 \quad 71:75703449t^2c^2_2c_3 \quad 74:86845062 \\
& t^2c_1c^2_2 + 15423:73424t^5c_3c^2_4 \quad 75:007313t^2c_1c^2_3 \quad 55:56100232t^2c_1c^2_4 + 5444:978227t^10c_2c^2_4 \quad 332:20849
\end{aligned}$$

$$\begin{aligned}
& 30t^4c^2c + 10889:95645t^{13}c^2 + 7259:970970t^{34}c^3 + 233:6803156t^{12}c^3c^2 + 438:0059016t^{13}c^3c^2 : 90029 \\
& 191c^2c^2 + 2722:489114t^{14}c^2c^2 + 17:14453786t^{14}c^3 + 25:18302430t^{12}c^3 + 2:915237776tc^2c^2 5079:263363 \\
& t^4c^2c^2 + 85:07778480t^{14}c^3c^2 + 5358:550001t^{14}c^3c^2 + 11648:76100t^{14}c^3c^2 + 4661:568095t^{14}c^3c^2 + 11746:5 \\
& 5620t^{234}c^2c^2 + 14:17963080t^{14}c^3 + 118:4604557t^{14}c^3 + 159:3807038t^{14}c^3 + 112:5295500t^{14}c^3 + 14:17963080t^{14}c^3 \\
& c^3 + 1081:496053t^{14}c^3 + 251:8302430t^{14}c^3 + 838:1774064t^{14}c^3 + 6233:807273t^{14}c^3 + 2628:829138t^{14}c^3 + 272:2 \\
& 489114t^{14}c^3 + 23692:61375t^{14}c^3 + 5006:840038t^{14}c^3 + 113:4370464t^{14}c^3 + 5667:222236t^{14}c^3 + 60743:03753t^{14}c^3 \\
& c^3 + 971:3297894t^{14}c^3c^2 + 992:2783451t^{14}c^3c^2 + 340:3111392t^{14}c^3c^2 + 6353:400615t^{14}c^3c^2 + 875:0857865 \\
& t^{14}c^3c^2 + 4687:866969t^{14}c^3c^2 + 44131:21242t^{14}c^3c^2 + 113437464c^3 + 4:321411291tc^2c^2 + 1533:020656t^{14}c^3c^2 \\
& c^3 + 2734:733112t^{14}c^3c^2 + 4050:397069t^{14}c^3c^2 + 709:7918046t^{14}c^3c^2 + 1716:680636t^{14}c^3c^2 + 3085:48 \\
& 662t^{14}c^3c^2 + 15882:79159t^{14}c^3c^2 + 170:1555696t^{14}c^3c^2 + 718:4346272t^{14}c^3c^2 + 5056:051211t^{14}c^3c^2 + 3085:48 \\
& 1646:457702t^{14}c^3c^2 + 4603:023261t^{14}c^3c^2 + 9541:676131t^{14}c^3c^2 + 12892:21035t^{14}c^3c^2 + 30189:3792 \\
& t^{14}c^3c^2 + 340:31192t^{14}c^3c^2 + 340:3111392t^{14}c^3c^2 + 1754:4965484t^{14}c^3c^2 + 1361:244557t^{14}c^3c^2 + 136 \\
& 1:244557t^{14}c^3c^2 + 9247:820164t^{14}c^3c^2 + 34113:22074t^{14}c^3c^2 + 680:62784t^{14}c^3c^2 + 4170:161896t^{14}c^3c^2 \\
& c^4 + 680:62284t^{14}c^3c^2 + 3962:734154t^{14}c^3c^2 + 178:2582158t^{14}c^3c^2 + 5444:978227t^{14}c^3c^2 + 2722:4891 \\
& 14t^{14}c^3c^2 + 21036:63017t^{14}c^3c^2 + 3500:343146t^{14}c^3c^2 + 1361:244557t^{14}c^3c^2 + 10:77651941tc^2c^2 + 94530872 \\
& 00c^2c^2 + 0:8574228753e^2c^2 + 1c^2c^2 + 27050116c^2c^2 + 8507778480c^2c^2 + 2:363271800c^2c^2 + 6752205143c^2c^2 \\
& 1:215396926c^2c^2 + 2:126944620c^2c^2 + 0:3629303176e^2c^2 + 2c^3 + 1:10647335310t^{14}c^3c^2 + 2057814901c^2c^2 \\
& 3:54490770c^2c^2 + 2613098287tc^3 + 9524743254tc^3 + 2:722489114tc^3 + 4:726543600tc^3 + 7:643312488t^{14}c^3c^2 \\
& c^3 + 0:2116609612e^2c^2 + 1c^3 + 2304:752689t^{14}c^3c^2 + 854:250106t^{14}c^3c^2 + 45849:21349t^{14}c^3c^2 + 45648:91950t^{14}c^3c^2 \\
& c + 1969:915337t^{14}c^3c^2 + 2988:221611t^{14}c^3c^2 + 1954:080451t^{14}c^3c^2 + 87:50857865t^{14}c^3c^2 + 71:4653393t^{14}c^3c^2 \\
& 9:002940191tc^2c^2 + 11:34370464tc^2c^2 + 13:23432208tc^2c^2 + 83:72734377t^{14}c^3c^2 + 98745:48270t^{14}c^3c^2 + 62 \\
& :78393262t^{14}c^3c^2 + 41:91083014t^{14}c^3c^2 + 40:95639600t^{14}c^3c^2 \\
& 2:835926160c^2c^2 + 1:620529234c^2c^2 + 31 \\
& :21999178t^{14}c^3c^2 + 21:26620t^{14}c^3c^2 + 1:77245385c^3 + 340:3111392t^{14}c^3c^2 + 7024:694133t^{14}c^3c^2 + 77:7854032 \\
& 5t^{14}c^3c^2 + 13:61244557tc^2c^2 + 14:17963080tc^2c^2 + 466:4380442t^{14}c^3c^2 + 63:338424t^{14}c^3c^2 + 70:991407t^{14}c^3c^2 \\
& 471:1572997t^{14}c^3c^2 + 1:320277810t^{14}c^3c^2 + 1234688940c^2c^3c^4 + 4:22636176tc^2c^2 + 1:9206257241tc^2c^2 + 3434
\end{aligned}$$

Equation (3.86) is collocated at equally spaced points in the interval under consideration to give 5 system of linear equations. The equations are solved to obtain the values of constant

coefficients as:

$$c_0 = :1; c_1 = :3000002501; c_2 = :3214285715; c_3 = :125; c_4 = 1:98436022 \quad 10^{11}$$

Remark The other values containing imaginary parts were ignored.

Substituting the coefficients into the assumed approximate solution, we got the required approximate solution:

$$y_4(t) = 1:251982301 \quad 10^{11} + 3:65470 \quad 10^9 t^2 + 1:000000000 t^3 + 3:17497699510 \quad 10^{10} t^4$$

Solution of example 4 by Quadratic Weighted Polynomial (QWP)

Solving equation (3.84) using Quadratic Weighted Polynomial (QWP) at $\alpha = 2:5$ and $N = 4$, we obtain the constant coefficients as:

$$c_0 = 0:499999999; c_1 = :2142857143; c_2 = :2812499999; c_3 = :125500000; c_4 = 1:2601096 \quad 10^{11}$$

Remark The other values containing imaginary parts were ignored.

Substituting the coefficients into the assumed approximate solution, gives:

$$y_4(t) = 1:109600770 \quad 10^{10} + 5:9054 \quad 10^{10} t + 1:43296 \quad 10^9 t^2 + 1:000000000 t^3 + 2:01616175 \quad 10^{10} t^4$$

Solution of example 4 by Cubic Weighted Polynomial (CWP)

Solving equation (3.84) using Cubic Weighted Polynomial (CWP) at $\alpha = 2:5$ and $N = 4$, we obtain the constant coefficients as:

$$c_0 = 0:2857285764; c_1 = :1847142857; c_2 = :2500000000; c_3 = :126000500; c_4 = 9:672444740 \quad 10^{12}$$

Remark The other values containing imaginary parts were ignored.

Substituting the coefficients into the assumed approximate solution, we got the required approximate solution:

$$y_4(t) = 1:046896702 \quad 10^{11} + 2:33265 \quad 10^{10} t + 4:8251 \quad 10^{10} t^2 + 1:000000000 t^3 + 1:547591158 \quad 10^{10} t^4$$

SOLUTION OF EXAMPLE 5:

Consider the initial value problem that consists of the multi-order fractional differential equation.

$$D^{\frac{1}{2}} y(t) + t y^0(t) + y(t) D^{\frac{1}{2}} y(t) = g(t) \quad (3.88)$$

where

$$g(t) = \left(\frac{4}{\sqrt{2}}\right) t^{\frac{11}{2}} + \left(\frac{4}{\sqrt{2}}\right) t^{\frac{3}{2}} \quad t \in [0; 1]$$

with exact solution $y(t) = t^3 + 1$, Subject to $y(0) = 0$.

[see Abdullah and Prabbat (2015)]

Solution of example 5 by Linear Weighted Polynomial (LWP) for $N = 4$ and at

$\tau = 1:5$

we take an assumed approximate solution

$$y_4(t) = c_0 + \frac{1}{3} c_1 (6t^2 - 16t + 2) + \frac{1}{5} c_2 (20t^2 - 16t + 2) + \frac{1}{35} c_3 (280t^3 - 360t^2 + 120t - 8) + \frac{1}{63} c_4 (1008t^4 - 1792t^3 + 1008t^2 - 192t + 8) \quad (3.89)$$

Putting (3.88) into (3.87) and applying (1.15), with some simplifications, gives

$$\begin{aligned} & \frac{677888}{1024} c_5 - \frac{104064}{1024} c_7 - \frac{7296}{384} c_3 - \frac{1856}{384} c_3 - \frac{2385920}{31744} c_{11} - \frac{65536}{29824} c_{13} - \frac{1024}{2048} c_5 \\ & + \frac{2835}{1024} t_2 c_4^2 + 5 t_2 c_3^2 + 245 t_2 c_3^2 + 75 t_2 c_2^2 + 7 t_2 c_4^2 + 1 t_2 c_4^2 + 5 t_2 c_4^2 + 5 t_2 c_4 \\ & + \frac{1024}{64} t_2 c_4^2 + 5 t_2 c_3^2 + 8 c_1^2 t_2 - 3 c_1^2 t_2 - 32768 t_2 c_4^2 + 1323 t_2 c_4^2 + 5 t_2 c_3^2 + 9 t_2 c_4 - 35 \\ & + \frac{1323}{1664} t_2 c_2^2 + \frac{1323}{40448} t_2 c_4^2 + 245 t_2 c_3^2 + 35 t_2 c_4^2 + 64 t_2 c_4 + 3 t_2 c_2^2 + 5 t_2 c_2^2 + 7 c_3 t_2 + 64 t_2 \\ & + \frac{1323}{671488} t_2 c_4^2 + \frac{35}{6144} t_2 c_4^2 + \frac{1}{408320} t_2 c_2^2 + \frac{7}{5604352} c_3 t_2 + 64 t_2 \\ & + \frac{7}{3} c_3 + 16 c_2 t_2 + 5 c_3 t_2 c_4 - 735 t_2 c_3 c_4 + 1575 t_2 c_3 c_4 + 7 t_2 c_3 c_4 + 7 t_2 c_3 c_4 + 5 t_2 c_3 c_4 + 5 t_2 \\ & + \frac{1024}{5696} c_2 t_2 - \frac{6016}{42496} c_2 t_2 + \frac{736}{32} c_2 t_2 + \frac{74752}{2816} t_2 c_2 c_4 + \frac{54272}{88} t_2 c_2 c_4 + \frac{4522}{61696} c_2 t_2 \\ & + \frac{496}{3328} t_2 c_2 c_4 + 1575 t_2 c_2 c_4 + 5 t_2 c_2 c_3 + 5 t_2 c_2 c_4 + 7 c_1 t_2 c_4 + 5 t_2 c_2 c_3 + 15 c_1 t_2 c_2 + 175 \\ & + \frac{496}{6} t_2 c_2 c_3 + \frac{1}{6} c_1 t_2 c_4 + 1 c_1 t_2 c_4 + 3 c_1 t_2 c_3 + 3 c_1 t_2 c_4 + 1 c_1 t_2 c_3 + 7 c_1 t_2 c_3 + 315 c_1 t_2 \end{aligned}$$

$$c_4 + \frac{112}{3} c_1 t_2 c_2 + \frac{4096}{45} c_0 t_2 c_4 - \frac{128}{21} c_0 t_2 c_4 - \frac{1472}{45} c_1 t_2 c_2 + \frac{416}{5} c_1 t_2 c_3 + 4 c_0 c_1 \sqrt{t_2} + \frac{32}{3} c_0 t_2 c_2 + \frac{48}{7} c_0 t_2 c_3 + 5 c_0 t_2^3 + 5 c_0 t_2 c_2 + 7 c_0 t_2 c_3 + 5 c_0 t_2^4 + 3 c_0 t_2 c_4 = 5 t_2 + 8 t_2 \quad (3.90)$$

Equation (3.89) is collocated at equally spaced points in the interval under consideration to give 5 system of linear equations. The equations are solved to obtain the values of constant coefficients as:

$$c_0 = 0.9999999907 \cdot 10^{-9}; c_1 = 2.9999999986; c_2 = 3.24285716; c_3 = 1.2500000002;$$

$$c_4 = 1.077672604 \cdot 10^{-10}$$

Remark The other values containing imaginary parts were ignored.

Substituting the values of the constants into the assumed approximate solution, gives

$$y_4(t) = 6.31526852 \cdot 10^{-12} - 3.0 \cdot 10^{-9}t + 1.9999999989t^3 + 1.724276166 \cdot 10^{-9}t^4$$

Now, solving the same problem for $N = 5$ and $\Delta t = 1:5$, following the same procedure, the constant coefficients are

$$c_0 = :1000000006; c_1 = :3000000008; c_2 = :342957122; c_3 = :1999999999;$$

$$c_4 = 1.345503357 \cdot 10^{-10}; c_5 = -2.707766991 \cdot 10^{-10}$$

Substituting the values of the constants into the assumed approximate solution, gives

$$y_5(t) = 5.695793 \cdot 10^{-11} + 1.098 \cdot 10^{-9}t - 2.8650 \cdot 10^{-9}t^2 + 1.000000011t^3 + 1.754007748 \cdot 10^{-8}t^4$$

$$+ 8.664854374 \cdot 10^{-9}t^5$$

Estimation of Error for Problem using LWP

The error estimate was done using the formula $|y_{i+1}(t) - y_i(t)|$ where $i = 4$ Solution of example 5 by Quadratic Weighted Polynomial (QWP)

Solving equation (3.87) using Quadratic Weighted Polynomial at $\Delta t = 1:5$ for $N = 4$ the constant coefficients were obtained as:

$$c_0 = 0.5000000377 \cdot 10^{-8}; c_1 = :2285755215; c_2 = :2865499991; c_3 = :199999999;$$

$$c_4 = -3.059869926 \cdot 10^{-10}$$

Remark The other values containing imaginary parts were ignored.

Substituting the values of the constants into the assumed approximate solution, gives

$$y_4(t) = 2.331329467 \cdot 10^{-11} + 1.54 \cdot 10^{-8}t + 1.00000000t^3 - 4.895791881 \cdot 10^{-9}t^4$$

Now, solving the same problem for $N = 5$ and $\Delta t = 1:5$, following the same procedure, gives

$$c_0 = 0.5000000308 \cdot 10^{-9}; c_1 = :2825571980; c_2 = :2657499980; c_3 = :2000009981;$$

$$c_4 = 6.34047722 \cdot 10^{-10}; c_5 = 1.398240046 \cdot 10^{-10}$$

Substituting the values of the constants into the assumed approximate solution, gives

$$y_5(t) = 1.604273 \cdot 10^{-11} + 1.30643 \cdot 10^{-8}t + 1.00885 \cdot 10^{-9}t^2 + 1.00000008t^3 + 1.461372 \cdot 10^{-8}t^4$$

$$+ 4.474368148 \cdot 10^{-9}t^5$$

Estimation of Error for Problem using QWP

The error estimate was done using the formula $|y_{i+1}(t) - y_i(t)|$ where $i = 4$ Solution of example 5 by Cubic Weighted Polynomial (CWP)

Solving equation (3.87) using Cubic Weighted Polynomial (CWP) at $\Delta t = 1:5$ for $N = 4$ The constant coefficients were obtained as:

$$c_0 = 0.57142707 \times 10^{-9}; c_1 = 0.167542820; c_2 = 0.259500000; c_3 = 0.574200343; \\ c_4 = 2.3867352 \times 10^{-10}$$

Remark The other values containing imaginary parts were ignored.

Substituting the values of the constants into the assumed approximate solution, gives

$$y_4(t) = 5.2675284 \times 10^{-12} + 4.6795 \times 10^{-9}t + 1.8650 \times 10^{-9}t^2 + 9.99999999t^3 + 4.861872563 \times 10^{-9}t^4$$

Now, solving the same problem for $N = 5$ and $\Delta t = 1:5$, following the same procedure, The constant coefficients were obtained as:

$$c_0 = 0.285714 \times 10^{-9}; c_1 = 0.1614276936; c_2 = 0.2699999993; c_3 = 0.1359999991; \\ c_4 = 5.0807180 \times 10^{-10}; c_5 = 2.01540843 \times 10^{-10}$$

Substituting the values of the constants into the assumed approximate solution, gives

$$y_5(t) = 3.55129273 \times 10^{-10} + 2.656 \times 10^{-8}t + 3.4050 \times 10^{-9}t^2 + 1.000000013t^3 + 2.053165233 \times 10^{-8}t^4 \\ + 6.454930699 \times 10^{-9}t^5$$

Estimation of Error for Problem using CWP

The error estimate was done using the formula $|y_{i+1}(t) - y_i(t)|$ where $i = 4$

Chapter 4

Perturbed Collocation Method for Solving Multi-order Fractional Differential and Integro-differential Equations

4.1 Preamble

In this chapter, we present the methodology of the Perturbed Collocation Method for the general class of multi-order fractional differential and integro-differential equations and then numerical solutions of examples 1-5 presented in chapter three.

4.2 The Perturbed Collocation Method

Substituting an assumed approximate solution of the form equation (3.55) into a slightly perturbed multi-order fractional integro-differential equation:

$$D(y(t)) + \sum_{i=0}^n c_i y^{(i)}(t) + \int_0^t (y(t)) = f(t) + H_M(t) \quad (4.1)$$

gives

$$\sum_{j=0}^N D c_j Q_j(t) + \sum_{i=0}^N c_i Q_j^{(i)}(t) + \int_0^t \sum_{j=0}^N c_j Q_j(t) = f(t) + H_M(t) \quad (4.2)$$

subject to conditions:

$$y_{N^k}^{(k)}(0) = k; \quad k = 1; 2; \dots; N \quad (4.3)$$

and

$$H_M(t) = \sum_{p=1}^n T_p(t); \quad t \in [a; b] \quad (4.4)$$

is called the perturbation term, $p(p = 1(1)n)$ are the free tau parameters to be determined, T_p are the Chebyshev polynomials defined in equation (2.19) and c_j are the unknown constants also to be determined. $Q_j(t)$ is as defined earlier. Substituting equation (4.3) into equation (4.2), we get:

$$\sum_{j=0}^N c_j Q_j(t) + \sum_{j=0}^N c_j Q_j^{(j)}(t) + \sum_{j=0}^N c_j Q_j^{(j)}(t) = f(t) + \sum_{k=1}^n T_{N+n+k}(t) \quad (4.5)$$

Thus, equation (4.5) is rewritten as:

$$\sum_{j=0}^N c_j \frac{(j+1)}{(j+1)} t^j + \sum_{r=0}^N c_j \frac{(j+1)}{(j+1)} t^r + \sum_{r=0}^N c_j \frac{(j+1)}{(j+1)} t^r = f(t) + \sum_{k=1}^n T_{N+n+k}(t) \quad (4.6)$$

Substituting equations (6) and (7) into (4.6) gives

$$\sum_{j=0}^N c_j \frac{(j+1)}{(j+1)} t^j + \sum_{j=0}^N c_j \frac{(j+1)}{(j+1)} t^j + \sum_{j=0}^N c_j \frac{(j+1)}{(j+1)} t^j = f(t) + \sum_{k=1}^n T_{N+n+k}(t) \quad (4.7)$$

Equation (4.7) is further simplified and then collocated at equally spaced interior points, $t = t_e$ on $[a, b]$; $t_e = a + \frac{(b-a)e}{(N+1)}$; $e = 1, 2, \dots, N$ to obtain $(N + n + 1)$ linear system of algebraic equations, including the n initial conditions for the unique determination of the $(N + n + 1)$ parameters.

The linear system of algebraic equations $(N + n + 1)$ are put in matrix form as: $Ax = B$ where,

Substituting equation (4.12) into (4.9) then apply (1.15) and (1.18), gives:

$$\begin{aligned}
 & \frac{16t_3c_1}{15315934653153465245} - \frac{4}{3}t_2c_0 + \frac{512t_5c_3}{15315934653153465245} + \frac{8t_2c_1}{15315934653153465245} + \frac{8192t_6c_4}{15315934653153465245} + \frac{16384t_5}{15315934653153465245} + \frac{234496t_6}{15315934653153465245} + \frac{768t_4c_3}{15315934653153465245} \\
 & + \frac{53248t^5c_4}{2835} + \frac{128t^3c_2}{75} + \frac{128t^4c_2}{105} + \frac{152t^2c_2}{15} + \frac{32tc_2}{5} + \frac{832t^3c_3}{35} + \frac{2848t^2c_3}{105} + \frac{48tc_3}{7} \\
 & + \frac{512t_4c_4}{10563189213515363} + \frac{5632t_3c_4}{10563189213515363} + \frac{8032t_2c_4}{10563189213515363} + \frac{128tc_4}{10563189213515363} + \frac{16384t_4}{10563189213515363} + \frac{12032t_3}{10563189213515363} + \frac{1276t_2}{10563189213515363} + \frac{64t_1}{10563189213515363} + \frac{8c_4}{10563189213515363} \\
 & \frac{8c_3}{35} + 2 = 5c_2 + c_0 \quad 2 = 3c_1 = \frac{16t^3}{5} \quad \frac{64t^5}{315} \quad (4.12)
 \end{aligned}$$

Equation (4.13) is collocated at equally spaced interior points on [0,1], to give 4 equations with 5 unknown constants. One extra equation is obtained using the initial condition. Solving these 5 linear algebraic equations together with the use of some computer software (maple 18), the values of the constants, c_i ; $i = 0; 1; \dots; 4$ are obtained as:

$$\begin{aligned}
 c_0 &= :1000000083; c_1 = :3000000291; c_2 = :3214286093; c_3 = :1250000225; \\
 c_4 &= 8:75438 \quad 10^9; \quad 1 = 7:346572 \quad \frac{10}{10}
 \end{aligned}$$

Substituting the values of the constants into the assumed approximate solution given in equation (4.11) gives

$$y_4(t) = 4:920635 \cdot 10^7 + 1:3 \cdot 10^8 t + 6:150 \cdot 10^8 t^2 + 9999999999 t^3 + 1:404800000 \cdot 10^{10} t^4$$

Solution of example 1 by Quadratic Weighted Polynomial (QWP)

Here, we assumed an approximate solution of the form:

$$y_4(t) = \sum_{j=0}^4 c_j Q_j(t) \quad (4.13)$$

where c_j ; $j = 0(1)4$ are unknown constants to be determined and $Q_j(t)$ are the Quadratic Weighted Polynomials defined in equation (3.36). Equation (4.14) is expanded in conjunction with equation (3.36) to get

$$\begin{aligned}
 y_4(t) &= c_0 + \frac{1}{2}c_1(4t-1) + \frac{1}{15}c_2(60t^2-40t+4) + \frac{1}{7}c_3(56t^3-63t^2+18t-1) + \frac{1}{105}c_4(16 \\
 & \quad 80t^4-2688t^3+1344t^2-224t+8) \quad (4.14)
 \end{aligned}$$

$$\begin{array}{c}
\frac{512t_5c_3 + 128t_3c_2}{3} + \frac{16t_3c_1}{1} + \frac{2048t_6c_4}{4} + \frac{256t_4c_3}{3} + \frac{16384t_5c_4}{4} + \frac{128t_4c_2}{2} + p p_{tc0+4c1t+} \\
\frac{3151051534651053465105}{32t^2c_2} \quad \frac{32tc_2}{32} \quad \frac{368t^3c_3}{64} \quad \frac{64t^2c_3}{106496t^4c_4} \quad \frac{182272t^3c_4}{13792t^2c_4} \quad \frac{512tc_4}{8c_4} \\
\frac{1}{9} \quad \frac{2}{4} \quad \frac{4}{7} \quad \frac{1}{5} \quad \frac{165}{1728t^{5=2}c_4} \quad \frac{8}{165} \quad \frac{1}{32} \quad \frac{1}{7} \\
\frac{3}{7} \quad \frac{1}{5} \quad \frac{3}{5} \quad \frac{1024}{11} \quad \frac{2=21tc_3+}{55} \quad \frac{1728}{165} \quad \frac{8}{165} \quad \frac{1}{32} \quad \frac{1}{7} \\
\frac{2=5}{t^{5=2}c_2 + 80} \quad \frac{p - 9=2}{t^{9=2}c_4} \quad \frac{512p - t^{3=2}c_4}{165} \quad \frac{p - 3=2}{t^{3=2}c_1 + 32} \quad \frac{p - 7=2}{t^{7=2}c_3} \quad \frac{p - 5=2}{t^{5=2}c_3} \quad \frac{p - 1}{t^{1=1}c_3} \\
\frac{2}{1} \quad \frac{1}{5} \quad \frac{1}{5} \quad \frac{1}{5} \quad \frac{1}{5} \quad \frac{1}{5} \quad \frac{1}{5} \quad \frac{1}{5} \quad \frac{1}{5} \\
= 5 \quad 5 \quad + \quad t_1 \quad (4.15)
\end{array}$$
$$c_0 = :1000008888; c_1 = :1428171495; c_2 = :12812499998; c_3 = :1524999995;$$

$$c_4 = 2:793295182 \cdot 10^9; {}_1 = 6:0999 \cdot 10^{10}$$
$$y_4(t) = 6.0129662 \cdot 10^9 + 2.1452 \cdot 10^8 t + 7.4 \cdot 10^8 t^2 + 999999895 t^3 + 6.069272291 \cdot 10^9 t^4$$

Here, we assumed an approximate solution of the form:

$$y_4(t) = \sum_{j=0}^4 c_j C_j(t) \quad (4.16)$$

$$y_4(t) = c_0 + \frac{1}{2}c_1(4t - 1) + \frac{1}{15}c_2(60t^2 - 40t + 4) + \frac{1}{7}c_3(56t^3 - 63t^2 + 18t - 1) + \frac{1}{105}c_4(16$$

$$80t^4 - 2688t^3 + 1344t^2 - 224t + 8) \quad (4.17)$$

$$\begin{array}{cccccccc}
\underline{13792\ t^2 c_4} & \underline{512\ t c_4} & \underline{512\ t^5 c_3} & \underline{128\ t^3 c_2} & \underline{16\ t^3 c_1} & \underline{2048\ t^6 c_4} & \underline{256\ t^4 c_3} & \underline{16384\ t^5 c_4} & \underline{128\ t^4 c_2} \\
4 & 1 & 3 & 1 & & 4 & 1 & 4 & \\
9 & 6 & 1 & 0 & 1 & 6 & 0 & 6 & \\
5 & 5 & 5 & +5 & 5 & 5 & +5 & +5 & 105 \\
p_p- & & \underline{32\ t^2 c_2} & \underline{32\ t c_2} & \underline{368\ t^3 c_3} & \underline{64\ t^2 c_3} & \underline{106496\ t^4 c_4} & \underline{182272\ t^3 c_4} & \underline{8\ c_4} \\
+ & t c_0 + 4\ c_1 t + 3 & & 7 & +5 & 3 & +4\ t c_3 + 5 & 5 & +165\ 2=5 \\
p_p- & p_3=2 & \underline{1024\ p_t^{7=2} c_4} & & p_p- & \underline{1728\ p_t^{5=2} c_4} & \underline{8\ p_p-_{t c_4}} & & p_5=2 \\
& & 1 & & & 5 & 6 & & \\
& t c_1 + 4 & t\ c_3 & 1 & 2=21 & t c_3 + 5 & +5 & +12\ t\ c_2 + & \\
p_9=2 & \underline{512\ p_t^{3=2} c_4} & p_3=2 & & p_7=2 & p_5=2 & \underline{32\ p_t^{3=2} c_2} & \underline{4\ p_p\ t c_2} & \\
& 1 & & & & & & & \\
& 6 & & & & & & & 2 \\
80\ t\ c_4 & 5 & +4\ t\ c_1 + 32\ t\ c_3 & 24 & t\ c_3 & 7 & & +1 & \\
16\ t^3 & & & & & & \underline{64\ t^5} & p_p- & \\
& & & & & =5 & \underline{3} & & \\
& & & & & & 1 & & \\
& & & & & & 5 & + & t\ 1\ (4.18)
\end{array}$$
$$c_0 = 0.999999970; c_1 = .1542857149; c_2 = .201239998; c_3 = .102499996; c_4 = .793295182 \cdot 10^8;$$
$$y_4(t) = 5.01296562 \cdot 10^9 \cdot 1.4425 \cdot 10^8 t + 7.14 \cdot 10^8 t^2 + 9999998795 t^3 + 5.069272291 \cdot 10^9 t^4$$

Solving equation (4.9) using the LWP at $\alpha = 0.5$ for $N = 5$, following the same procedure as above, we obtain the constant values as:

$$C_0 = 0:1000000333; c_1 = 0:11412855131; c_2 = 0:282500017; c_3 = 0:1249999986; c_4 = :852717963 \cdot 10^6; c_5 = 1:79563263 \cdot 10^6; \mathbf{1} = 0:15945627515 \cdot 10^8$$

Substituting these values of into assumed approximate solution, gives:

$$y_5(t) = 2.5065284 \cdot 10^{-9} + 1.176 \cdot 10^{-8}t + 2.8 \cdot 10^{-6}t^2 + 9.99999975t^3 \\ + 1.096434874 \cdot 10^{-8}t^4 + 3.200796 \cdot 10^{-8}t^5$$

Solution of example 1 by Quadratic Weighted Polynomial (QWP)

Solving equation (4.9) using the QWP at $\tau = 0.5$ for $N = 5$, following the same procedure

as above, we obtain the constant values as:

$$c_0 = 0:00009999; c_1 = 1:42571311; c_2 = 0:815500019; c_3 = 0:16599996; c_4 = 5:8527963 \cdot 10^8;$$

$$c_5 = 0:23117963 \cdot 10^8; {}_1 = 0:17945527513 \cdot 10^{10}$$

Substituting these values of into assumed approximate solution:

$$y_5(t) = 2:158084 \cdot 10^8 + 3:18650 \cdot 10^8 t + 2:5638 \cdot 10^8 t^2 + :99999998 t^3$$

$$+ 1:6434874 \cdot 10^8 t^4 + 1:450961 \cdot 10^8 t^5$$

Solution of example 1 by Cubic Weighted Polynomial (CWP)

Solving equation (4.9) using the CWP at $\alpha = 0:5$ for $N = 5$, following the same procedure as above, we obtain the constant values as:

$$c_0 = 0:0000002857; c_1 = 0:1607142845; c_2 = 0:250000267; c_3 = 0:1249999983; c_4 = 1:163219$$

$$61 \cdot 10^7; c_5 = 1:032551961 \cdot 10^7; {}_1 = 0:0894562733 \cdot 10^8$$

Substituting these values of into assumed approximate solution, gives:

$$y_5(t) = 2:6398657 \cdot 10^7 + 1:9948 \cdot 10^8 t + 3:27 \cdot 10^8 t^2 + 1:9999999592 t^3$$

$$+ 1:8651384 \cdot 10^8 t^4 + 7:562219 \cdot 10^8 t^5$$

Solution of example 1 by Linear Weighted Polynomial (LWP)

Solving equation (4.9) using the LWP at $\alpha = 0:5$ for $N = 6$, following the same procedure as above, we obtain the constant values as:

$$c_0 = 0:000000085; c_1 = 0:160742845; c_2 = 0:250000576; c_3 = 0:128999983; c_4 = 1:1631961 \cdot 10^7;$$

$$c_5 = 9:00983643 \cdot 10^6; c_6 = 0:0000017 \cdot 10^7; {}_1 = 0:85945625679 \cdot 10^8$$

Substituting these values of into assumed approximate solution, gives:

$$y_6(t) = 1:652757 \cdot 10^8 + 3:18575 \cdot 10^8 t + 3:27 \cdot 10^8 t^2 + :9999999592 t^3$$

$$+ 1:1415184 \cdot 10^8 t^4 + 2:51809 \cdot 10^8 t^5 + 1:30804 \cdot 10^8 t^6$$

Solution of example 1 by Quadratic Weighted Polynomial (QWP)

Solving equation (4.9) using the QWP as basis function at $\alpha = 0:5$ for $N = 6$, following the same procedure as above, we obtain the constant values as:

$$c_0 = 0:000000003; c_1 = 0:2542857131; c_2 = 0:2812500017; c_3 = 0:1249999986;$$

$$c_4 = 6:852717963 \cdot 10^7; c_5 = 7:010983543 \cdot 10^6; c_6 = 0:066000017 \cdot 10^7;$$

$${}_1 = 0:5627562450 \cdot 10^8$$

Substituting these values of into assumed approximate solution, gives:

$$y_6(t) = 2:0268101 \ 10^9 + 0:169 \ 10^8 t + 2:95 \ 10^8 t^2 + :999999715 t^3 \\ + 1:096854 \ 10^8 t^4 + 1:10018 \ 10^8 t^5 + 1:89516584 \ 10^9 t^6$$

Solution of example 1 by Cubic Weighted Polynomial (CWP)

Solving equation (4.9) using the CWP as basis function at $\alpha = 0:5$ for $N = 6$, following the same procedure as above, we obtain the constant values as:

$$c_0 = 0:0000000333; c_1 = 0:2132857131; c_2 = 0:2812500017; c_3 = 0:1249999986; \\ c_4 = 6:852717963 \ 10^8; c_5 = 7:04517963 \ 10^7; c_6 = 0:1257179 \ 10^7; \\ c_7 = 0:825945625076 \ 10^8$$

Substituting these values of into assumed approximate solution, gives:

$$y_6(t) = 2:521798 \ 10^9 + 2:0059 \ 10^8 t + 2:85 \ 10^8 t^2 + :999999887 t^3 + 1:904348 \\ 10^8 t^4 \\ + 1:96208 \ 10^8 t^5 + 1:138954 \ 10^{10} t^6$$

Solution of example 1 by Linear Weighted Polynomial (LWP)

Solving equation (4.9) using the LWP at $\alpha = 0:5$ for $N = 7$, following the same procedure as above, we obtain the constant values as:

$$c_0 = 0:100000058; c_1 = 0:2999999970; c_2 = 0:3214285716; c_3 = 0:1249999899; c_4 = 1:241103 \\ 152 \ 10^8; c_5 = 1:382852837 \ 10^8; c_6 = 2:124688474 \ 10^8; c_7 = 9:182353437 \ 10^7; \\ c_8 = 0:4562173253 \ 10^{10}$$

Substituting these values of into assumed approximate solution, gives:

$$y_7(t) = 1:7759548 \ 10^8 + 0:9663983 \ 10^9 t^6 + 0:7996736 \ 10^9 t^4 + 0:1568341 \\ 10^9 t^7 + 0:8029793173 \ 10^5 t^5 + :9999963680 t^3 + 0:1050728 \ 10^9 t^2 + 6:52638 \ 10^{10} t$$

Solution of example 1 by Quadratic Weighted Polynomial (QWP)

Solving equation (4.9) using the QWP at $\alpha = 0:5$ for $N = 7$, following the same procedure as above, we obtain the constant values as:

$$c_0 = 0:0000000035; c_1 = 0:2142858388; c_2 = 0:2812502336; c_3 = 0:1250001658; \\ c_4 = 8:744925602 \ 10^8; c_5 = 2:725905309 \ 10^8; c_6 = 2:264275176 \ 10^8; \\ c_7 = 1:211991229 \ 10^8; c_8 = 0:577902527690 \ 10^{10}$$

Substituting these values of into assumed approximate solution, gives

$$y_7(t) = :9999951289t^3 \quad 0:1029984644e^4t^5 + 0:9248082086e^5t^4 + 2:747139760 \quad 10^8 \\ + 0:6200141731 \quad 10^5t^6 + 0:1522838014 \quad 10^5t^2 \quad 2:687548688 \quad 10^7t \quad 0:1551348773e^5t^7$$

Solution of example 1 by Cubic Weighted Polynomial (CWP)

Solving equation (4.9) using the CWP as basis function at = 0:5 for N = 7, following the same procedure as above, we obtain the constant values as:

$$c_0 = 0:00000000035; c_1 = :1607142651; c_2 = :2499999341; c_3 = :1249999762; c_4 = 6:85609 \quad 10^8; \\ c_5 = 3:938068453 \quad 10^8; c_6 = 2:88 \quad 10^8; \\ 2483665 \quad 10^8; c_7 = 1:180688538 \quad 10^8; \\ 1 = 0:569800005325 \quad 10^{10}$$

Substituting these values of into assumed approximate solution, gives:

$$y_7(t) = 0:9985674 \quad 10^{10}t^4 + 5:152813 \quad 10^9t^7 + 0:1572545 \quad 10^9t^2 + 0:68420 \quad 10^9t^6 \\ + 1:005031t^3 + 0:10589822 \quad 10^9t^5 + 2:6107645 \quad 10^9t + 2:5891325 \quad 10^{10}$$

Solution of example 1 by Linear Weighted Polynomial (LWP)

Solving equation (4.9) using the LWP as basis function at = 0:5 for N = 8, following the same procedure as above, we obtain the constant values as:

$$c_0 = 0:000000000086; c_1 = 0:107142845; c_2 = 0:25800000; c_3 = 0:1249999983; c_4 = 1:16321 \quad 961 \quad 10^9; \\ c_5 = 1:02103152 \quad 10^8; c_6 = 1:25610152 \quad 10^8; c_7 = 2:25509720 \quad 10^8; c_8 = 0:0075 \quad 1352 \quad 10^8; \\ 1 = 0:1866656666 \quad 10^{10}$$

Substituting these values of into assumed approximate solution, gives:

$$y_8(t) = 6:8719567 \quad 10^{10}t^4 + 4:165132 \quad 10^9t^7 + 0:30725 \quad 10^{10}t^2 + 0:280729 \quad 10^{10}t^6 + 2:0 \quad 5031t^3 \\ + 1:10588 \quad 10^{10}t^5 + 2:076664 \quad 10^9t + 2:085325 \quad 10^{10} + 2:12817 \quad 10^{10}t^8$$

Solution of example 1 by Quadratic Weighted Polynomial (QWP)

Solving equation (4.9) using the QWP at = 0:5 for N = 8, following the same procedure as above, we obtain the constant values as:

$$c_0 = :1000000058; c_1 = :29999970; c_2 = :3214285716; c_3 = :1249999899; c_4 = 1:241103152 \quad 10^8;$$

$$c_5 = 1:382852837 \cdot 10^8; c_6 = 2:124688474 \cdot 10^8; c_7 = 9:182353437 \cdot 10^9; c_8 = 1:103103152 \cdot 10^{10}; c_1 = 0:1666666627 \cdot 10^{10}$$

Substituting these values of into assumed approximate solution, gives:

$$y_8(t) = 0.000001704759008 + 0.000479669837 t^6 + 0.000007129966938 t^4 - 0.00000175341240 t^7 \\ 0.000008029793173 t^5 + 0.9999963680 t^3 + 0.00000175507298 t^2 - 0.000000175263788 + 2.000000 \\ 5603278 t^8 t$$

Solution of example 1 by Cubic Weighted Polynomial (CWP)

Solving equation (4.9) using the CWP as basis function at $\alpha = 0.5$ for $N = 8$, following the same procedure as above, we obtain the constant values as:

$$c_0 = 0.0000000033; c_1 = 0.2142857131; c_2 = 0.2812500017; c_3 = 0.6666666666; \\ c_4 = 0.4531857131; c_5 = 0.27502812500017; c_6 = 0.1249999986; c_7 = 6.82352717963 \\ 10^{10}; \\ c_8 = 5.000082539963 \quad 10^{10}; c_1 = 4.0000562921 \quad 10^{10}$$

Substituting these values of into assumed approximate solution, gives:

$$y_8(t) = 2.502284 \quad 10^{10} + 1.7436 \quad 10^8 t + 2.8 \quad 10^8 t^2 + .999999715 t^3 + 1.0934 \\ 874 \quad 10^8 t^4 - 0.000001175341240 t^7 + 0.01280000802 t^5 + 0.99999213680 t^3 + 0.000001 \\ 075507298 t^2 - 0.000456900012 \quad 2.00325005603278 t^8 t$$

SOLUTION OF EXAMPLE 2

Consider a slightly perturbed initial value problem that consists of the multi-order fractional integro-differential equation

$$D^\alpha y(t) + (t-1)y^{(0)}(t) - e^t y^{(0)}(t) = \frac{(3)}{(3.5)} t^{2.5} - \frac{(3)}{(4.5)} t^{3.5} - \frac{(3)}{(9.5)} t^{7.5} + J^{0.5} y(t) + J^{0.75} y(t) + {}_1T_3(t) \\ + {}_2T_4(t) \quad (4.19)$$

Here

$$T_4(t) = 128t^4 - 256t^3 + 160t^2 - 32t + 1 \quad (4.20)$$

$$T_3(t) = 32t^3 - 42t^2 + 18t - 1 \quad (4.21)$$

Subject to $y(0) = 0$, $y(1) = 1$ The exact solution is $y(t) = t^3$ at $\alpha = 2$. (Omar (2014))

Solution of example 2 by Linear Weighted Polynomial (LWP) at $\alpha = 1.5$ for $N = 4$

Taking an assumed approximate solution:

$$y_4(t) = \sum_{j=0}^4 c_j L_j(t) \quad (4.22)$$

Equation (4.23) is expanded to get

$$y_4(t) = c_0 + \frac{1}{3}c_1(6t^2 - 8) + \frac{1}{5}c_2(20t^2 - 16t + 2) + \frac{35}{35}c_3(280t^3 - 360t^2 + 120t + 63) + \frac{1}{63}c_4(1008t^4 - 1792t^3 + 1008t^2 - 192t + 8) \quad (4.23)$$

Equation (4.24) is substituted into (4.20) then applying (1.15) and (1.18), gives.

$$\begin{aligned} & \frac{8c_4}{63} + \frac{8c_3}{35} + 2 = 5c_2 + c_0 + 16 = 3c_1 + \frac{224t^2c_2}{15} + \frac{36tc_2}{5} + \frac{256t^3c_3}{7} + \frac{256tc_3}{35} + \frac{26624t^4c_4}{315} \\ & 4864t_3c_4 + 2944t_2c_4 + 400tc_4 + 8 = 3t_2c_1 + \frac{4096t_5c_4}{256} + \frac{t_4c_3}{64} + \frac{t_3c_2}{2} + tc_0 + 3.526483t_9 = 4c_2 \\ & \frac{4563633153515}{21} \\ & 8:639039941 t_4 c_4 + 1:92854544 t_5 = 4c_0 + 0:771418178 t_5 = 4c_2 + 3:3585553 t_9 = 4c_4 + 2:204051939 t_9 = 4c_1 \\ & 12:82357491 t_4 c_4 + 8:243726730 t_4 c_3 + 5:1294299 t_4 c_3 + 0:24489465 t_5 = 4c_4 + 3:778374750 t_9 = 4c_3 \\ & + 0:4408103875 t_5 = 4c_3 + 18:23797322 t_4 c_4 + 1:2856969 t_5 = 4c_1 + 3:20589372 t_4 c_2 = \frac{16t^3}{5} + \frac{32t^4}{35} + 0:6 \\ & 411787457 t_4 + 124:0717695 t_4 = 2 + 248:143539 t_4 = 2 + 159:5208465 t_5 = 2 + 35:449077 t_4 = 2 + 1:77 \\ & 245385 p t_2 + 496:2870780 t_7 = 2 + 638:0833860 t_5 = 2 + 212:6944620 t_3 = 2 + 14:17963080 p t_1 \\ & \quad (4.24) \end{aligned}$$

Equation (4.25) is collocated and solved to obtain the constants coefficients:

$$c_0 = 5:45422941; c_1 = :7798657066; c_2 = 3:014629633; c_3 = 1:501715366;$$

$$c_4 = 0:556436005e^{-1}; {}_1 = 7:008531256 \cdot 10^{-6}; {}_2 = 1:20085190312 \cdot 10^{-8}$$

Substituting these values of into assumed approximate solution, gives:

$$y_4(t) = 8:35478300 \cdot 10^{-9}t + 5:925526 \cdot 10^{-8}t^2 + :999998990t^3 + 5:035 \cdot 10^{-10}t^4$$

Solution of example 2 by Quadratic Weighted Polynomial (QWP) at $\alpha = 1:5$

Taking an assumed approximate solution:

$$y_4(t) = \sum_{j=0}^4 c_j Q_j(t) \quad (4.25)$$

Equation (4.26) is expanded to get

$$y_4(t) = c_0 + \frac{1}{2}c_1(4t - 1) + \frac{1}{15}c_2(60t^2 - 40t + 4) + \frac{1}{7}c_3(56t^3 - 63t^2 + 18t - 1) + \frac{1}{105}c_4(1680t^4 - 2688t^3 + 1344t^2 - 224t + 8) \quad (4.26)$$

Equation (4.27) is substituted into (4.20) then applying (1.15) and (1.18), gives.

$$\begin{aligned}
 & \frac{4096 t_5 c_4}{31515351059155} + \frac{64 t_3 c_2}{2 t c_0} + \frac{256 t_4 c_3}{8=3 t^2 c_1} + \frac{464 t c_4}{5 c_1 t} + \frac{128 t_2 c_2}{2} + \frac{88 t c_2}{2} + \frac{176 t_3 c_3}{2} \\
 & \frac{192 t^2 c_3}{7} + \frac{38 t c_3}{7} + \frac{2048 t^4 c_4}{25} + \frac{7168 t^3 c_4}{75} + \frac{1664 t^2 c_4}{45} + \frac{17}{5:129429965 t^4 c_3} + \frac{2:204051939 t^{9=4} c_1 + 2:9}{13} \\
 & 38735917 t^{9=4} c_2 + 2:350988733 t^{9=4} c_4 + 2:833781064 t^{9=4} c_3 + 0:2755064922 t^{5=4} c_3 + 10:25885993 t^{13} c_4 \\
 & + 16:41417589 t^{17} c_4 + 8:639039941 t^{21} c_4 + 7:2132889 t^{13} c_3 + 3:205893728 t^{13} c_2 + 0:5142787856 t^{5=4} c_2 \\
 & + 0:9642727230 t^{5=4} c_1 + 1:928545447 t^{5=4} c_0 + 0:1469367959 t^{5=4} c_4 = \frac{16 t^3}{5} - \frac{32 t^4}{35} + 0:6411787457 t^{17} c_4 \\
 & + 226:8740928 t^{9=2} c_1 + 453:7481856 t^{7=2} c_1 + 283:5926160 t^{5=2} c_1 + 56:71852320 t^{3=2} c_1 + 1:77245385 p t_1 \\
 & + 56:71852320 t^{7=2} c_2 + 85:07778480 t^{5=2} c_2 + 31:90416930 t^{3=2} c_2 + 1:77245385 p t_2 \quad (4.27)
 \end{aligned}$$

Equation (4.28) is collocated and solved to obtain the constants coefficients:

$$\begin{aligned}
 c_0 &= 9:769997101; c_1 = 0:7855148 & 10^8; c_2 &= :2376052899; c_3 \\
 &= & 0:779580 & 10^8; \\
 c_4 &= 1:732301477 \cdot 10^8; c_1 = 9:617278113 \cdot 10^6; c_2 = 2:5640980056 \cdot 10^8
 \end{aligned}$$

Substituting these values of into assumed approximate solution, gives:

$$\begin{aligned}
 y_4(t) &= 2:0467802 \cdot 10^8 + 2:875437 \cdot 10^8 t + 4:493 \cdot 10^8 t^2 + 1:000000011 t^3 + \\
 & \quad 1:540670764 \cdot 10^{10} t^4
 \end{aligned}$$

Solution of example 2 by Cubic Weighted Polynomial (CWP) at $\alpha = 1:5$ for $N = 4$

Taking an assumed approximate solution:

$$y_4(t) = \sum_{j=0}^4 c_j C_j(t) \quad (4.28)$$

Equation (4.29) is expanded to get

$$\begin{aligned}
 y_4(t) &= c_0 + \frac{1}{2} 5 c_1 (10 t^2) + \frac{21}{2} c_2 (84 t^2 + 48 t + 4) + \frac{21}{2} c_3 (168 t^3 + 168 t^2 + 42 t \\
 & \quad + \frac{1}{165} c_4 (2640 t^4 + 3840 t^3 + 1728 t^2 + 256 t + 8) \quad (4.29)
 \end{aligned}$$

Equation (4.30) is substituted into (4.20) then applying (1.15) and (1.18), gives.

$$\begin{aligned}
& 2:5189165 t^{\frac{9=4}{13}} c_2 + 14:921978 t^{\frac{17}{4}} c_4 - 5:129429965 t^{\frac{17}{4}} c_3 - 2:204051939 t^{\frac{9=4}{13}} c_3 - 0:09350523374 t^{\frac{5=4}{13}} c_4 \\
& 3:20589378 t^{\frac{5=4}{13}} c_2 - 0:3673419897 t^{\frac{5=4}{13}} c_2 + 0:183679948 t^{\frac{5=4}{13}} c_3 - 8:3936267 t^{\frac{5=4}{13}} c_4 + 0:7714181783 t^{\frac{5=4}{13}} c_1 \\
& 1:928545447 t^{\frac{5=4}{13}} c_0 + 6:411787457 t^{\frac{5=4}{13}} c_3 - 2:204051939 t^{\frac{9=4}{13}} c_1 + 1:709809988 t^{\frac{9=4}{13}} c_4 - 8:63903994 t^{\frac{5=4}{13}} c_4 \\
& 256 t^{\frac{5=4}{13}} c_3 - 4096 t^{\frac{5=4}{13}} c_4 - 2 t c_0 - 8 = 3 t^2 c_1 - 64 t^3 c_2 + 8 c_4 - 2 = 21 c_3 + 4 c_2 + c_0 - 2 = 5 c_1 + 96 t^2 c_2 - 104 t c_2 + \\
& \frac{3531515}{15} t^3 c_3 - \frac{16521721}{21} t c_3 + \frac{6144 t^4 c_4}{77} - \frac{23552 t^3 c_4}{275} + \frac{14848 t^2 c_4}{495} - \frac{16 t c_4}{5} + \frac{24 c_1 t}{5} - \frac{16 t^3}{5} - \frac{32 t^4}{350} : 64 \\
& 11787457 t^{\frac{17}{4}} + 226:8740928 t^{\frac{9=2}{1}} - 453:7481856 t^{\frac{7=2}{1}} + 283:5926160 t^{\frac{5=2}{1}} - 56:71852320 t^{\frac{3=2}{1}} + 1:7 \\
& 7245385 p t_1 + 56:71852320 t^{\frac{7=2}{2}} - 85:07778480 t^{\frac{5=2}{2}} + 31:90416930 t^{\frac{3=2}{2}} - 1:77245385 p t_2 \quad (4.30)
\end{aligned}$$

Equation (4.31) is collocated and solved to obtain the constants coefficients:

$$c_0 = 1:578838358; c_1 = 1:092294070; c_2 = :4458878648; c_3 = :1473498590;$$

$$c_4 = 0:2961080791 \cdot 10^{-8}; {}_1 = 2:6527039015 \cdot 10^{-6}; {}_2 = 0:453285316 \cdot 10^{-8}$$

Substituting these values of into assumed approximate solution, gives:

$$\begin{aligned}
y_5(t) = & 1:0000 \cdot 10^{-8} + 1:072565698t + :999999999t^2 + 0:976635418t^3 + :7023443 \cdot 10^{-8}t^4 \\
& + :170970 \cdot 10^{-9}t^5
\end{aligned}$$

Solution of example 2 by Linear Weighted Polynomial (LWP) at $\alpha = 1:5$ for $N = 5$ Solving equation (4.20) using Linear Weighted Polynomial (LWP) for $N = 5$ the constants coefficient obtained are:

$$\begin{aligned}
c_0 = & 2:88357835; c_1 = 1:400922970; c_2 = :864345884; c_3 = :17344980; c_4 = 1:61080791 \cdot 10^{-8}; \\
c_5 = & 0:1331726156 \cdot 10^{-8}; {}_1 = 1:96432539015 \cdot 10^{-8}; {}_2 = 1:4568985350 \cdot 10^{-8}
\end{aligned}$$

Substituting these values of into assumed approximate solution, gives:

$$\begin{aligned}
y_5(t) = & 0:00001080185261 - 0:000000008 t + 0:000000031 t^2 + 0:9999999556 t^3 + 0:00000002 \\
& 144334293 t^4 + 0:000000000025633342782 t^5
\end{aligned}$$

Solution of example 2 by Quadratic Weighted Polynomial (QWP) at $\alpha = 1:5$ for $N = 5$

Solving equation (4.20) using Quadratic Weighted Polynomial (QWP) for N = 5 the constants coefficient obtained are:

$$c_0 = 0.10100000; c_1 = 0.20500000; c_2 = 0.36142851720; c_3 = 0.16249999992; c_4 = 0.1522050564; c_5 = 2.340208933 \times 10^{-9}; \quad {}_1 = 0.00652927831 \times 10^{-8}; \quad {}_2 = 4.46153285370 \times 10^{-9}$$

Substituting these values of into assumed approximate solution, gives:

$$y_5(t) = 0.0000001090186161 - 0.0000000088t + 0.000000035t^2 + 0.9999999555t^3 + 0.00000001094334293t^4 + 0.00000001995334278t^5$$

Solution of example 2 by Cubic Weighted Polynomial (CWP) at = 1:5 for N = 5 Solving equation (4.20) using Cubic Weighted Polynomial (CWP) for N = 5 the constants coefficient obtained are:

$$c_0 = 0.019900000; c_1 = 0.0545100000; c_2 = 0.851743120; c_3 = 0.109169999; c_4 = 0.152620600; c_5 = 1.20895603 \times 10^{-9}; \quad {}_1 = 0.0004529086 \times 10^{-8}; \quad {}_2 = 4.6750532850 \times 10^{-9}$$

Substituting these values of into assumed approximate solution, gives:

$$y_5(t) = 0.000000001080005262 - 0.00000000899t + 0.0000000040t^2 + 0.9999999996t^3 + 0.0000000200934293t^4 + 0.00000002570104278t^5$$

Solution of example 2 by Linear Weighted Polynomial (LWP) at = 1:5 for N = 6 Solving equation (4.20) using Linear Weighted Polynomial (LWP) for N = 6 the constants coefficient obtained are:

$$c_0 = 0.005660000; c_1 = 0.99999999988; c_2 = 0.100000000; c_3 = 1.90060096; c_4 = 6.15039690 \times 10^{-10}; c_5 = 3.59690459 \times 10^{-10}; c_6 = 1.0053965902 \times 10^{-10}; \quad {}_1 = 1.80132870 \times 10^{-9}; \quad {}_2 = 0.51324539 \times 10^{-9}$$

Substituting these values of into assumed approximate solution, gives:

$$y_6(t) = 1.4296349 \times 10^{-10}t^6 + 6.740902 \times 10^{-11}t + 6.16800089 \times 10^{-10} + 6.30752017610 \times 10^{-8}t^5 + 0.999999999t^3 + 3.659530907 \times 10^{-8}t^4 + 1.901694023 \times 10^{-8}t^2$$

Solution of example 2 by Quadratic Weighted Polynomial (QWP) at = 1:5 for N = 6

Solving equation (4.20) using Quadratic Weighted Polynomial (QWP) for N = 6 the constants coefficient obtained are:

$$c_0 = :100500000; c_1 = :999999845; c_2 = :999999988; c_3 = 1:890055596; c_4 = 6:1503960 \cdot 10^{-10}; c_5 = 3:3969052 \cdot 10^{-10}; c_6 = 1:16539902 \cdot 10^{-10}; \quad {}_1 = 1:01351270 \cdot 10^{-9}; \quad {}_2 = 1:9073156 \cdot 10^{-9}$$

Substituting these values of into assumed approximate solution, gives:

$$y_6(t) = 6:74097702 \cdot 10^{-11}t + 6:1680089 \cdot 10^{-12} \cdot 6:30751176 \cdot 10^{-8}t^5 + :999999988t^3 + 3:659530907 \cdot 10^{-9}t^2 + 1:901694023 \cdot 10^{-8}t^4 + 3:429124339 \cdot 10^{-8}t^6$$

Solution of example 2 by Cubic Weighted Polynomial (CWP) and N = 6

Solving equation (4.20) using Cubic Weighted Polynomial (CWP) for N = 6 the constants coefficient obtained are:

$$c_0 = :1500000001; c_1 = :249999998; c_2 = :3999999979; c_3 = 1:890005590; c_4 = 8:14739690 \cdot 10^{-10}; c_5 = 3:3969052 \cdot 10^{-10}; c_6 = 1:1053968902 \cdot 10^{-10}; \quad {}_1 = 1:2580135 \cdot 10^{-9}; \quad {}_2 = 0:486925850 \cdot 10^{-9}$$

Substituting these values of into assumed approximate solution, gives:

$$y_6(t) = :999999988t^3 + 3:6585309 \cdot 10^{-8}t^4 + 1:8694023 \cdot 10^{-9}t^2 + 3:42914339 \cdot 10^{-8}t^6 + 9:74097982 \cdot 10^{-11}t + 8:16800089 \cdot 10^{-10} \cdot 6:307176 \cdot 10^{-10}t^5$$

Solution of example 2 by Linear Weighted Polynomial (LWP) at = 1:5 for N = 7 Solving equation (4.20) using Linear Weighted Polynomial (LWP) for N = 7 the constants coefficient obtained are:

$$c_0 = 0:0499999988; c_1 = 0:2192857150; c_2 = 0:2812499716; c_3 = :1250000069; c_4 = 1:1850895 \cdot 10^{-8}; c_5 = 1:6802624 \cdot 10^{-10}; c_6 = 1:229761892 \cdot 10^{-10}; c_7 = 1:3371104 \cdot 10^{-10}; \quad {}_1 = 0:2579105320 \cdot 10^{-10}; \quad {}_2 = 0:537281090 \cdot 10^{-9}$$

Substituting these values of into assumed approximate solution, gives:

$$y_7(t) = 2:7585307 \cdot 10^{-8}t^4 + 1:85093623 \cdot 10^{-9}t^2 + 5:42954339 \cdot 10^{-8}t^6 + 1:7330638 \cdot 10^{-8}t^7$$

$$+ :9999999887t^3 + 9:74097982$$

$$6:35205076 \quad 10^8 t^5$$

$$10^{11}t + 8:16965089 \ 10^{10} +$$

Solution of example 2 by Quadratic Weighted Polynomial (QWP) at $\alpha = 1:5$ for $N = 7$

Solving equation (4.20) using Quadratic Weighted Polynomial (QWP) for $N = 7$ the constants coefficient obtained are:

$$\begin{aligned} c_0 &= 0:0499999998; c_1 = 0:21328589; c_2 = 0:28499706; c_3 = :135000685; \\ c_4 &= 1:94013895 \cdot 10^{-9}; c_5 = 1:6115624 \cdot 10^{-9}; c_6 = 1:2986199 \cdot 10^{-10}; \\ c_7 &= 1:371049 \cdot 10^{-10}; {}_1 = 0:2479105352 \cdot 10^{-10}; {}_2 = 1:513752860 \cdot 10^{-9} \end{aligned}$$

Substituting these values of into assumed approximate solution, gives:

$$\begin{aligned} y_7(t) &= 1:8296338 \cdot 10^{-8}t^7 + :9999999887t^3 + 3:658530907 \cdot 10^{-8}t^4 + 1:8673623 \cdot 10^{-9}t^2 + \\ &5:429524339 \cdot 10^{-8}t^6 + 5:7097982 \cdot 10^{-10}t + 8:165089 \cdot 10^{-10} + 6:3975476 \cdot 10^{-9}t^5 \end{aligned}$$

Solution of example 2 by Cubic Weighted Polynomial (CWP) at $\alpha = 1:5$ for $N = 7$ Solving equation (4.20) using Cubic Weighted Polynomial (CWP) for $N = 7$ the constants coefficient obtained are:

$$\begin{aligned} c_0 &= 0:0499999999; c_1 = 0:2282857149; c_2 = 0:27812499716; c_3 = :1350000068; \\ c_4 &= 1:193895 \cdot 10^{-8}; c_5 = 1:6651220 \cdot 10^{-10}; c_6 = 1:2289170 \cdot 10^{-10}; \\ c_7 &= 1:33711049 \cdot 10^{-10}; {}_1 = 0:15753120 \cdot 10^{-10}; {}_2 = 2:8955336 \cdot 10^{-9} \end{aligned}$$

Substituting these values of into assumed approximate solution, gives:

$$\begin{aligned} y_7(t) &= 1:9999999887t^3 + 9:74097982 \cdot 10^{-10}t + 8:16965089 \cdot 10^{-10} \\ &+ 6:3845276 \cdot 10^{-8}t^5 + 2:75 \cdot 10^{-10}t^4 + 1:8506933 \cdot 10^{-9}t^2 + 5:429524339 \cdot 10^{-8}t^6 + 1:723633 \cdot 10^{-10}t^7 \end{aligned}$$

Solution of example 2 by Linear Weighted Polynomial (LWP) at $\alpha = 1:5$ for $N = 8$ Solving equation (4.20) using Linear Weighted Polynomial (LWP) for $N = 8$ the constants coefficient obtained are:

$$\begin{aligned} c_0 &= :100000062; c_1 = :150000437; c_2 = :3999999967; c_3 = 0:9563219805; c_4 = 5:00800432; \\ c_5 &= 1:25803275631; c_6 = 0:0075325 \cdot 10^{-9}; c_7 = 3:128092262 \cdot 10^{-9}; c_8 = 1:58657800148 \cdot 10^{-10}, \\ {}_1 &= 0:918203745 \cdot 10^{-10}; {}_2 = 6:9835537289 \cdot 10^{-10} \end{aligned}$$

Substituting these values of into assumed approximate solution, gives:

$$\begin{aligned} y_8(t) &= 0:1152344 \cdot 10^{-7}t + 0:1334573 \cdot 10^{-8}t^5 + 0:1740476236 \cdot 10^{-8}t^7 + 0:802375 \cdot 10^{-8}t^2 + \\ &:99999565t^3 + 2:405763341t^4 + 0:1075586347 \cdot 10^{-8}t + 0:73805243 \cdot 10^{-8}t^6 + 0:3190573 \cdot 10^{-10}t^8 \end{aligned}$$

Solution of example 2 by Quadratic Weighted Polynomial (QWP) at $\alpha = 1:5$ for $N = 8$

Solving equation (4.20) using Quadratic Weighted Polynomial (QWP) for $N = 8$ the constants coefficient obtained are:

$$c_0 = :100000999; c_1 = :150000250; c_2 = :3999999987; c_3 = 0:1563219805; c_4 = 5:0106980045; \\ c_5 = 1:250352756; c_6 = 0:0095532589; c_7 = 2:1528092262 \quad 10^9; c_8 = 1:25864571 \quad 10^{10}; \\ c_1 = 0:9018745 \quad 10^{10}; c_2 = 0:726985816 \quad 10^{10}$$

Substituting these values of into assumed approximate solution, gives:

$$y_8(t) = 0:10758634 \quad 10^8 + 0:14152934 \quad 10^8 t + 0:74047 \quad 10^8 t^2 + 1:8023799 \quad 10^8 t^3 + 0:331904 \quad 10^7 t^4 + :9956946565 t^5 :2405763341 t^6 + 0:73805210 \quad 10^{10} t^7 + 0:4651904 \quad 10^{10} t^8$$

Solution of example 2 by Cubic Weighted Polynomial (CWP) at $\alpha = 1:5$ for $N = 8$ Solving equation (4.20) using Cubic Weighted Polynomial (CWP) for $N = 8$ the constants coefficient obtained are:

$$c_0 = :1999999999; c_1 = :100050250; c_2 = 2:9999999987; c_3 = 1:1005639805; c_4 = 3:10695865; \\ c_5 = 1:265580352865; c_6 = 0:0095745325; c_7 = 2:14529262 \quad 10^9; c_8 = 1:15864741 \quad 10^{10}; \\ c_1 = 0:365020537 \quad 10^{10}; c_2 = 1:17452895 \quad 10^{10}$$

Substituting these values of into assumed approximate solution, gives:

$$y_8(t) = 0:217404762 \quad 10^8 t^7 + 0:8023795 \quad 10^8 t^2 + :999999565 t^3 :2402576341 t^4 + 0:10763 \quad 10^8 \\ 0:7380524 \quad 10^8 t^6 + 0:15319045 \quad 10^8 t^8 + 0:52019113 \quad 10^{10} t + 0:136319045 \quad 10^9 t^5$$

SOLUTION OF EXAMPLE 3

Consider a slightly perturbed fractional order integro-differential equation

$$D^\alpha y(t) = t(1 + e^t) + 3e^t + y^0(t) \int_0^t y(x)dx + H_M(t); \quad (4.31)$$

$3 < \alpha < 4$ and $0 < t < 1$ The exact solution is $y(t) = 1 + te^t$ at $\alpha = 4$. Subject to $y(0) = 1$. (Avipsita et al (2017))

Solution of example 3 by Linear Weighted Polynomial (LWP) at $\alpha = 3:25$

Here, we assumed an approximate solution of the form:

$$y_8(t) = \sum_{j=0}^8 c_j L_j(t) \quad (4.32)$$

where c_j ; $j = 0(1)8$ are unknown constants to be determined and $L_j(t)$ are the Linear Weighted Polynomials defined in equation (3.21).

Equation (4.33) is expanded in conjunction with equation (3.21) to get

$$\begin{aligned}
 y_8(t) = & c_0 + \frac{1}{8} 3c_1(6t^2 - 2) + \frac{1}{8} 5c_2(20t^2 - 16t + 2) + \frac{35}{8} c_3(280t^3 - 360t^2 + 120t \\
 & + 63) c_4(1008t^4 - 1792t^3 + 1008t^2 - 192t + 8) + \frac{1}{231} c_5(7392t^5 - 16800t^4 + 13440t^3 - 4480t^2 \\
 & + 560t - 16) + \frac{1}{3003} c_6(192192t^6 - 532224t^5 + 554400t^4 - 268800t^3 + 60480t^2 - 6520t + 398) \\
 & + \frac{1}{67} c_7(87626972160t^7 - 256236175104t^6 + 284923312128t^5 - 149345884800t^4 + 366684585720 \\
 & 64857600t^3 - 5728160940t^2 + 1420124840t - 116143116) + c_8(7330633674167901065379840t^8 \\
 & 43507905751146412173312000t^7 + 113623692198339912073605120t^6 - 17062573519926 \\
 & 9534948470784t^5 + 161104744779368919543221760t^4 \\
 & 97852290667632851030876160t^3 + 3 \\
 & 7285336446910284794999880t^2 \\
 & 8135643125780429287106960t \\
 & + 777021914692904462637 \\
 & 658) \quad (4.33)
 \end{aligned}$$

Putting (4.34) in (4.32), and applying (1.15) and (1.18), gives

$$\begin{aligned}
 & \frac{1}{072} t^7 + c_0 t + 16 t^8 c_7 + \frac{64 t^7 c_6}{7} + 16 = 3 t^6 c_5 + \frac{16 t^5 c_4}{5} + 2 t^4 c_3 + 4 = 3 t^3 c_2 + t^2 c_1 + \frac{65536 c_8 t^9}{9} + \frac{72 t^2 c_3}{7} \\
 & \frac{19 t c_3}{751515105533399} + \frac{112 t^4 c_4}{76196 t c_5} + \frac{448 t^3 c_4}{640 t^6 c_6} + \frac{208 t^2 c_4}{17856 t^5 c_6} + \frac{40552 t c_4}{16000 t^4 c_6} + \frac{192 t^5 c_5}{7762 t^3 c_6} + \frac{1456 t^4 c_5}{761472 t^2 c_6} + \frac{160 t^3 c_5}{148160 t c_6} + \frac{191600 t^2 c_5}{184 t^7 c_7} \\
 & + \frac{78708736 t^7 c_8}{153} + \frac{59981824 t^6 c_8}{51} + \frac{3612815360 t^5 c_8}{153} + \frac{12374878208 t^4 c_8}{153} + \frac{25118932 t^3 c_8}{21879} + \frac{1980 t^2 c_8}{2431} \\
 & + \frac{209799296528 t c_8}{7293} + \frac{44 x c_2}{15} + \frac{2264216528 c_8}{561} + \frac{369608 c_7}{715} + \frac{1267264 c_6}{3003} \\
 & + \frac{28804 c_5}{99} + \frac{16136 c_4}{105} + \frac{337 c_3}{7} + \frac{4 c_2}{15} c_0 + 1 = 2 c_1 = t + t e^t + 3 e^t \quad (4.34)
 \end{aligned}$$

Equation (4.35) is collocated and solved to obtain the coefficients:

$$\begin{aligned}
 c_0 = & 1:854732105; c_1 = 1:094821039; c_2 = 5:67098596; c_3 = 1:161261149; c_4 = 3:226924018; \\
 c_5 = & 6:70085338; c_6 = 1:17502504010 \cdot 10^{-8}; c_7 = 2:00583489 \cdot 10^{-8}; c_8 = 1:505153094 \cdot 10^{-10}; \\
 & 1 = 1:67540109 \cdot 10^{-10}
 \end{aligned}$$

Substituting these values of into assumed approximate solution, gives:

$$y_8(t) = 2:1718619 \ 10^{10} + 1:5618491t + 2:999999967t^2 + 0:6916591857t^3 + 1:012924027t^4 \\ + :789999999t^5 + 1:20628309 \ 10^{10}t^6 + 1:612230960 \ 10^{10}t^7 + 0:43656802 \ 10^{10}t^8$$

Solution of example 3 by Quadratic Weighted Polynomial (QWP) at = 3:25 Solving equation (4.32) for = 3:25 and N = 8 and following the same procedure, the constant coe cients were obtained as:

$$c_0 = 2:945942845; c_1 = 3:74818904101; c_2 = :13640010849; c_3 = 0:3173526 \ 10^8; c_4 = 5:15 \\ 607050; c_5 = 5:1695701413; c_6 = :89066973 \ 10^8; c_7 = :84752146 \ 10^8; c_8 = :2490345 \ 10^8; \\ 1 = 0:573197 \ 10^{10}$$

Substituting these values of into assumed approximate solution, gives

$$y_8(t) = 6:171970 \ 10^8 + 0:999999999t + 1:438656345t^2 + 1:44782102t^3 + 0:47261752356t^4 + 1:98 \\ 2065263 \ 10^8t^5 + 1:147409026280 \ 10^{10}t^6 + 1:3364580 \ 10^{10}t^7 + 1:800034 \ 10^{10}t^8$$

Solution of example 3 by Cubic Weighted Polynomial (CWP) at = 3:25

Solving equation (4.32) for = 3:25 and N = 8 and following the same procedure, the constant coe cients were obtained as:

$$c_0 = 1:58358360; c_1 = 1:229054076; c_2 = :14349585123; c_3 = 0:61080841567; c_4 = \\ 0:731 \\ 7276 \ 10^8; c_5 = 3:199014973338; c_6 = :7704565 \ 10^8; c_7 = 1:95235230 \ 10^8; c_8 = 2:07 \\ 034918 \ 10^8; 1 = 1:567275401 \ 10^{10}$$

Substituting these values of into assumed approximate solution, gives:

$$y_8(t) = :5666968939 \ 10^{10} + 1:108419764t + 1:00000005t^2 + 1:385666009t^3 + 5:1571390 \ 10^8t^4 \\ + 1:809823065 \ 10^{10}t^5 + 1:274543260 \ 10^{10}t^6 + 2:523008 \ 10^{10}t^7 + 1:278003025 \ 10^{10}t^8$$

Solution of example 3 by Linear Weighted Polynomial (LWP)

Solving equation (4.32) for = 3:75 and N = 8 and following the same procedure, the constant coe cients were obtained as:

$$c_0 = 5:6909259634; c_1 = 3:61261449; c_2 = 2:4598766 \ 10^8; c_3 = 0:326792418; c_4 = \\ :67053387; \\ c_5 = 1:102540 \ 10^8; c_6 = 3:1225933 \ 10^8; c_7 = 2:158389 \ 10^{10}; c_8 = 1:505153094 \ 10^{10}; \\ 1 = 1:6757519 \ 10^{10}$$

Substituting these values of into assumed approximate solution, gives:

$$y_8(t) = :266598670 \cdot 10^8 + 1:239349453t + 6:999999967t^2 + 0:94538056959t^3 + 1:9212027t^4 + 1:89988t^5 + 2:8755604163t^6 + :9999999542t^3 + 2:100710189 \cdot 10^8 t^7 + 1:3206525309 \cdot 10^9 t^5 + 1:12 \cdot 3800032 \cdot 10^{10} t^8$$

Solution of example 3 by Quadratic Weighted Polynomial (QWP)

Solving equation (4.32) for N = 8 and at $\alpha = 3:75$ the constants coefficients were found as:

$$c_0 = 1:299450284; c_1 = :1267400502; c_2 = 0:3892357846 \cdot 10^8; c_3 = 0:1331726117 \cdot 10^9; c_4 = 5:25600540; c_5 = :1670141387; c_6 = :89068173 \cdot 10^8; c_7 = :8234752462 \cdot 10^{10}; c_8 = 0:67024335 \cdot 10^{10}; c_1 = 0:7536973 \cdot 10^{10}$$

Substituting these values of into assumed approximate solution, gives

$$y_8(t) = 0:9999654 \cdot 10^8 + 1:104197865t + 1:000000099t^2 + 1:432860403t^3 + 1:4623017284t^4 + 6:1903977 \cdot 10^{10} t^7 + 1:98206525 \cdot 10^{10} t^{10} + 1:1236021 \cdot 10^{10} t^9 + 1:27538035 \cdot 10^{10} t^8$$

Solution of example 3 by Cubic Weighted Polynomial (CWP)

Solving equation (4.32) for N = 8 and at $\alpha = 3:75$ we obtained the constants coefficients as:

$$c_0 = 1:2445818761; c_1 = :1973496578; c_2 = 0:29654121 \cdot 10^{10}; c_3 = 0:1289172 \cdot 10^{10}; c_4 = 3:18125640; c_5 = 3:1570173138; c_6 = :3567197 \cdot 10^{10}; c_7 = 1:234395240 \cdot 10^8; c_8 = 2:0703490331 \cdot 10^{10}; c_1 = 1:275675401 \cdot 10^{10}$$

Substituting these values of into assumed approximate solution, gives:

$$y_8(t) = :96431956 \cdot 10^{10} + :380846079t + 1:435019290t^2 + :42619743575t^3 + 5:1571396 \cdot 10^{10} t^4 + 1:082126 \cdot 10^{10} t^5 + 1:545489576 \cdot 10^{10} t^6 + 2:527336 \cdot 10^{10} t^7 + 1:25800035 \cdot 10^{10} t^8$$

SOLUTION OF EXAMPLE 4

Consider the slightly perturbed initial value problem that consists of the multi-order fractional differential equation

$$D^\alpha y(t) + D^{\frac{1}{2}} y(t) + y^3(t) + (t+1)y^{(0)}(t) = t^9 + H_M(t) \quad (4.35)$$

Subject to $y(0) = 0$ and $y^{00}(0) = 0$.

The exact solution is $y(t) = t^3$. (Yin Yang (2013))

Solution of example 4 by Linear Weighted Polynomial (LWP) = 2:5 for N = 4 Solving equation (4.36) for = 2:5 and N = 4, we assumed an approximate solution of the form:

$$y_4(t) = \sum_{j=0}^4 c_j L_j(t) \quad (4.36)$$

where c_j ; $j = 0(1)4$ are unknown constants to be determined and $L_j(t)$ are the Linear Weighted Polynomials defined in equation (3.21).

Equation (4.37) is expanded in conjunction with equation (3.21) to get

$$y_8(t) = c_0 + \frac{1}{8} 3c_1(6t^2) + \frac{1}{8} 5c_2(20t^2 - 16t + 2) + \frac{1}{35} c_3(280t^3 - 360t^2 + 120t + 63) + \frac{1}{63} c_4(1008t^4 - 1792t^3 + 1008t^2 - 192t + 8) \quad (4.37)$$

Putting (4.38) in (4.36), and applying (1.15), gives

$$\begin{aligned} & 5444:978227 t^9 c_2 c_3 c_4 + 2722:489114 t^8 c_1 c_3 c_4 - 21036:63017 t^8 c_2 c_3 c_4 + 1361:244557 t^7 c_0 c_3 c_4 + 1361:244557 t^7 c_1 c_2 c_4 \\ & \quad 9247:820164 t^7 c_1 c_3 c_4 + 34113:22074 t^7 c_2 c_3 c_4 + 680:6222784 t^6 c_0 c_2 c_4 \\ & \quad 4170:161896 t^6 c_0 c_3 c_4 + 680:6222784 t^6 c_1 c_2 c_3 - 3962:734154 t^6 c_1 c_2 c_4 + 12892:21035 t^6 c_1 c_3 c_4 - 30189:37928 t^6 c_2 c_3 c_4 \\ & \quad + 340:3111392 t^5 c_0 c_1 c_4 + 340:3111392 t^5 c_0 c_2 c_3 - 1754:492984 t^5 c_0 c_2 c_4 + 5056:051211 t^5 c_0 c_3 c_4 \\ & \quad 1646:457702 t^5 c_1 c_2 c_3 + 4603:023261 t^5 c_1 c_2 c_4 - 9541:676131 t^5 c_1 c_3 c_4 + 15882:79159 t^5 c_2 c_3 c_4 + 170:1555696 t^4 c_0 c_1 c_3 \\ & \quad 718:4346272 t^4 c_0 c_1 c_4 - 709:7918046 t^4 c_0 c_2 c_3 + 1716:680636 t^4 c_0 c_2 c_4 - 3085:487662 t^4 c_0 c_3 c_4 \\ & \quad + 1533:020656 t^4 c_1 c_2 c_3 - 2734:733112 t^4 c_1 c_2 c_4 + 4050:397069 t^4 c_1 c_3 c_4 - 5079:263363 t^4 c_2 c_3 c_4 \\ & \quad + 85:07778480 t^3 c_0 c_1 c_2 - 275:4899698 t^3 c_0 c_1 c_3 + 541:9769995 t^3 c_0 c_1 c_4 + 529:9130596 t^3 c_0 c_2 c_3 - 795:1396776 t^3 c_0 c_2 c_4 \\ & \quad + 996:7026471 t^3 c_0 c_3 c_4 - 693:5865123 t^3 c_1 c_2 c_3 + 884:4488443 t^3 c_1 c_2 c_4 - 992:2783451 t^3 c_1 c_3 c_4 \\ & \quad + 971:3297894 t^3 c_2 c_3 c_4 - 96:42148944 t^2 c_0 c_1 c_2 + 145:8476311 t^2 c_0 c_1 c_3 - 178:2582158 t^2 c_0 c_1 c_4 - 170:1555696 t^2 c_0 c_2 c_3 \\ & \quad + 177:1778629 t^2 c_0 c_2 c_4 - 163:9049568 t^2 c_0 c_3 c_4 + 158:1636533 t^2 c_1 c_2 c_3 - 152:6898656 t^2 c_1 c_2 c_4 \\ & \quad + 133:3464056 t^2 c_1 c_3 c_4 - 105:3189666 t^2 c_2 c_3 c_4 + 31:19518776 t c_0 c_1 c_2 - 29:16952622 t c_0 c_1 c_3 \\ & \quad + 24:30793851 t c_0 c_1 c_4 + 22:36330343 t c_0 c_2 c_3 - 17:28564517 t c_0 c_2 c_4 + 12:03821717 t c_0 c_3 c_4 - 16:85350404 t c_1 c_2 c_3 \\ & \quad + 12:60411627 t c_1 c_2 c_4 - 8:642822583 t c_1 c_3 c_4 + 5:803038020 t c_2 c_3 c_4 + \frac{1024 t^5 c_4}{31:219991785} \\ & \quad t^2 c_3 c_4^2 + 10:63472310 t c_0^2 c_1 - 17:01555696 t c_0^2 c_2 + 18:23095389 t c_0^2 c_3 - 16:20529234 t c_0^2 c_4 - 14:17963080 t c_0 c_1^2 \\ & \quad - 13:61244557 t c_0 c_2^2 - 8:334150348 t c_0 c_3^2 - 4:115629801 t c_0 c_4^2 - 13:23432208 t c_1^2 c_2 + 11:34370464 t c_1^2 c_3 \\ & \quad - 9:002940191 t c_1^2 c_4 + 10:77651941 t c_1 c_2^2 + 6:111710255 t c_1 c_3^2 + 2:915237776 t c_1 c_4^2 + \end{aligned}$$

6:028368751 $tc_2^2c_3$ 4:321411291 $tc_2^2c_4$ 4:222636176 $tc_2c_3^2$ 1:920627241 $tc_2c_4^2$ 1:904948651 tc_3^2
 $c_4+1:234688940$ $tc_3c_4^2$ 2:835926160 $c_0c_1c_2+1:620529234$ $c_0c_1c_3$ 0:9002940191 $c_0c_1c_4$ 0:97231754
06 $c_0c_2c_3+0:5401764114$ $c_0c_2c_4$ 0:3086722351 $c_0c_3c_4+0:6482116937$ $c_1c_2c_3$ 0:3601176076 $c_1c_2c_4+0:2$
057814901 $c_1c_3c_4$ 0:1234688940 $c_2c_3c_4+340:3111392$ $t^6c_1^2c_4$ 1977:045666 $t^6c_1c_3^2$ 15400:22942 t^6c_1
 c_4^2 1964:081432 $t^6c_2^2c_3+6376:674502$ $t^6c_2^2c_4+6353:400615$ $t^6c_2c_3^2+29176:24841$ $t^6c_2c_4^2+28999:139$
14 $t^6c_3^2c_4$ 48139:69866 $t^6c_3c_4^2$ 875:0857865 $t^5c_0c_3^2$ 5358:550001 $t^5c_0c_4^2+170:1555696$ $t^5c_1^2c_3$ 831
716736 $t^5c_1^2c_4+170:1555696$ $t^5c_1c_2^2+2291:891346$ $t^5c_1c_3^2+8181:872045$ $t^5c_1c_4^2+2263:555234$ $t^5c_2^2c_3$
4687:866969 $t^5c_2^2c_4$ 4661:568095 $t^5c_2c_3^2$ 11746:55620 $t^5c_2c_4^2$ 11648:76100 $t^5c_3^2c_4+153:73424$ t^5c_3
 $c_4^2+85:07778480$ $t^4c_0^2c_4+85:07778480$ $t^4c_0c_2^2+854:2504106$ $t^4c_0c_3^2+2304:752689$ $t^4c_0c_4^2+85:077784$
 $t^4c_1^2c_2$ 332:2084930 $t^4c_1^2c_3+781:4552085$ $t^4c_1^2c_4$ 328:9674346 $t^4c_1c_2^2$ 1358:466507 $t^4c_1c_3^2$ 2650:4
5592 $t^4c_1c_4^2$ 1330:130396 $t^4c_2^2c_3+1969:915337$ $t^4c_2^2c_4+1954:080451$ $t^4c_2c_3^2+2988:211$ $t^4c_2c_4^2+295$
2:352917 $t^4c_3^2c_4$ 3199:608197 $t^4c_3c_4^2+42:53889240$ $t^3c_0^2c_3$ 151:2493952 $t^3c_0^2c_4$ 136:124457 $t^3c_0c_2^2$
394:4831165 $t^3c_0c_3^2$ 556:9818998 $t^3c_0c_4^2$ 124:7807510 $t^3c_1^2c_2+237:6776210$ $t^3c_1^2c_3$ 358:91756 $t^3c_1^2$
 $c_4+233:6803156$ $t^3c_1c_2^2+438:0059016$ $t^3c_1c_3^2+513:3104947$ $t^3c_1c_4^2+422:9581302$ $t^3c_2^2c_3$ 477:08366 t^3
 $c_2^2c_4$ 471:1572997 $t^3c_2c_3^2$ 466:4380442 $t^3c_2c_4^2$ 458:0343201 $t^3c_3^2c_4+413:7383845$ $t^3c_3c_4^2+21:26946$
20 $t^2c_0^2c_2$ 54:69286166 $t^2c_0^2c_3+85:07778480$ $t^2c_0^2c_4+21:26944620$ $t^2c_0c_1^2+71:46533923$ $t^2c_0c_2^2+87:5$
7865 $t^2c_0c_3^2+70:99461407$ $t^2c_0c_4^2+63:33568424$ $t^2c_1^2c_2$ 77:78540325 $t^2c_1^2c_3+83:72734377$ $t^2c_1^2c_4$ 74:
86845062 $t^2c_1c_2^2$ 75:00735313 $t^2c_1c_3^2$ 55:56100232 $t^2c_1c_4^2$ 71:75703449 $t^2c_2^2c_3+64:172957$ $t^2c_2^2c_4+$
62:78393262 $t^2c_2c_3^2+41:91083014$ $t^2c_2c_4^2+40:95639600$ $t^2c_3^2c_4+10889:95645$ $t^{11}c_3c_4^2+5444:978227$ t^{10}
 $c_2c_4^2+5444:978227$ $t^{10}c_3^2c_4$ 52721:21776 $t^{10}c_3c_4^2+2722:489114$ $t^9c_1c_4^2$ 23715:90517 $t^9c_2c_4^2$ 23681:3
3388 $t^9c_3^2c_4+110647:3353$ $t^9c_3c_4^2+1361:244557$ $t^8c_0c_4^2$ 10587:45766 $t^8c_1c_4^2+1361:244557$ $t^8c_2^2c_4+13$
61:244557 $t^8c_2c_3^2+44131:21242$ $t^8c_2c_4^2+44004:31384$ $t^8c_3^2c_4$ 132027:7578 $t^8c_3c_4^2$ 4839:980646 $t^7c_0c_4^2$
+680:6222784 $t^7c_1c_3^2+17276:04203$ $t^7c_1c_4^2+680:6222784$ $t^7c_2^2c_3$ 4597:981614 $t^7c_2^2c_4$ 4589:338792 t^7
 $c_2c_3^2$ 45849:21349 $t^7c_2c_4^2$ 45648:91950 $t^7c_3^2c_4+98745:48270$ $t^7c_3c_4^2+340:3111392$ $t^6c_0c_3^2+7024:6941$
33 $t^6c_0c_4^2+1:77245385$ c_0^3 0:5251715111 $c_1^3+0:1134370464$ c_2^3 0:02116609612 $c_3^3+0:003629303176$
 $c_4^3+7:643312488$ $t^2c_4^3+4:726543600$ tc_1^3 2:722489114 $tc_2^3+0:9524743254$ tc_3^3 0:2613098287 tc_4^3 3:
54490770 $c_0^2c_1+2:126944620$ $c_0^2c_2$ 1:215396926 $c_0^2c_3+0:6752205143$ $c_0^2c_4+2:363271800$ $c_0c_1^2+0:8507$

$$\begin{aligned}
& 778480 c_0 c_2^2 + 0.2778050116 c_0 c_3^2 + 0.08574228753 c_0 c_4^2 + 0.9453087200 c_1^2 c_2 + 0.5401764114 c_1^2 c_3 + \\
& 0.3000980063 c_1^2 c_4 + 0.5671852320 c_1 c_2^2 + 0.1852033411 c_1 c_3^2 + 0.05716152502 c_1 c_4^2 + 0.1944635081 c_2^2 \\
& c_3 + 0.1080352823 c_2^2 c_4 + 0.1111220046 c_2 c_3^2 + 0.03429691501 c_2 c_4^2 + 0.03527682687 c_3^2 c_4 + 0.01959823 \\
& 715 c_3 c_4^2 + 7259.970970 t^{12} c_4^3 + 38719.84517 t^{11} c_4^3 + 90615.19321 t^{10} c_4^3 + 907.4963712 t^9 c_3^3 + 122379.5 \\
& 224 t^9 c_4^3 + 3500.343146 t^8 c_3^3 + 105538.4669 t^8 c_4^3 + 5667.222236 t^7 c_3^3 + 60743.03753 t^7 c_4^3 + 113.4370464 \\
& t^6 c_2^3 + 5006.840038 t^6 c_3^3 + 23692.61375 t^6 c_4^3 + 272.2489114 t^5 c_2^3 + 2628.829138 t^5 c_3^3 + 6233.807273 t^5 c_4^3 \\
& + 251.8302430 t^4 c_2^3 + 838.1774064 t^4 c_3^3 + 1081.496053 t^4 c_4^3 + 14.17963080 t^3 c_1^3 + 112.5295500 t^3 c_2^3 + 15 \\
& 9.3807038 t^3 c_3^3 + 118.4604557 t^3 c_4^3 + 14.17963080 t^2 c_1^3 + 25.18302430 t^2 c_2^3 + 17.14453786 t^2 c_3^3 + 680.62 \\
& 22784 c_4 = 1.77245385 t^9 + 124.0717695 t^4 + 248.1435390 t^3 + 35.44907700 t^3 + 159.5208465 t^2 + 1.77245385 \\
& 3.17361550 t^2 + 35.44907700 t + 21.26944620 t + 1.77245385 + 1.77245385 \\
& 2 \quad (4.38)
\end{aligned}$$

Equation (4.39) is collocated and solved to obtain the constants coefficients:

$$c_0 = 1.0009999909; c_1 = 3.594019754; c_2 = .5364618003; c_3 = 0.933687613 \times 10^{-8};$$

$$c_4 = 0.7438978622 \times 10^{-10}; \quad \tau_1 = 0.5705723554 \times 10^{-9}; \quad \tau_2 = 0.8117174623 \times 10^{-10}$$

Remark: The other values containing imaginary parts were ignored

Substituting the coefficients into the assumed approximate solution, gives:

$$y_4(t) = 1.564590 \times 10^{-10} t + 1.5458 \times 10^{-10} t^2 + 1.000000000 t^3 + 1.568324 \times 10^{-10} t^4$$

Solution of example 4 by Quadratic Weighted Polynomial (QWP) $\alpha = 2:5$ for $N = 4$

Solving equation (4.36) for $N = 4$ and $\alpha = 2:5$, the constant coefficients obtained were:

$$c_0 = 1.9999999998; c_1 = 0.2842875807; c_2 = .2842863714; c_3 = 0.7316477042;$$

$$c_4 = 0.3362821829 \times 10^{-10}; \quad \tau_1 = 0.2181019482 \times 10^{-10}; \quad \tau_2 = 0.2491822719 \times 10^{-10}$$

Substituting the coefficients into the assumed approximate solution, we got the required approximate solution:

$$y_4(t) = 1.000099999 \times 10^{-10} t + 1.6652 \times 10^{-10} t^2 + .9999999999 t^3 + 2.0985 \times 10^{-11} t^4$$

Solution of example 4 by Cubic Weighted Polynomial (CWP) and $N = 4$

Solving equation (4.36) for $N = 4$ and $\alpha = 2:5$, the constant coefficients obtained were:

$$c_0 = 2.382554934; c_1 = 1.956406679; c_2 = .1208736130; c_3 = .3305545732;$$

$$c_4 = .1122009070; \quad \tau_1 = .1311822251; \quad \tau_2 = 0.5425795490 \times 10^{-10}$$

Substituting the coefficients into the assumed approximate solution, gives:

$$y_4(t) = 1.3554698 \cdot 10^{-9}t + 1.99998 \cdot 10^{-9}t^2 + 1.000000000t^3 + 5.656589 \cdot 10^{-9}t^4$$

SOLUTION OF EXAMPLE 5

Consider the slightly perturbed initial value problem that consists of the multi-order fractional differential equation.

$$D^{\frac{1}{2}} y(t) + y(t) D^{\frac{1}{2}} y(t) + t y^0(t) = g(t) + H_M(t) \quad (4.39)$$

where

$$g(t) = \left(\frac{4}{7}\right)t^{\frac{11}{2}} + \left(\frac{4}{5}\right)t^{\frac{3}{2}} \quad 2 \in [0; 1]$$

with exact solution $y(t) = t^3 + 1$, Subject to $y(0) = 1$.

[see Abdollah Sadeghinia (2015)]

Solution of example 5 by Linear Weighted Polynomial (LWP)

Solving equation (4.40) at $\alpha = 1.5$ and $N = 4$, we assumed an approximate solution of the form:

$$y_4(t) = \sum_{j=0}^4 c_j L_j(t) \quad (4.40)$$

where c_j ; $j = 0(1)4$ are unknown constants to be determined and $L_j(t)$ are the Linear Weighted Polynomials defined in equation (3.21). Equation (4.41) is expanded in conjunction with equation (3.21) to get

$$y_4(t) = c_0 + \frac{1}{3}c_1(6t - 2) + \frac{1}{5}c_2(20t^2 - 16t + 2) + \frac{1}{35}c_3(280t^3 - 360t^2 + 120t - 8) + \frac{1}{63}c_4(1008t^4 - 1792t^3 + 1008t^2 - 192t + 8) \quad (4.41)$$

Putting (4.42) in (4.40), and applying (1.15), gives

$$\begin{array}{r}
 \begin{array}{cccccc}
 1509376 t^{7=2} c_4^2 & 104064 t^{7=2} c_3^2 & 128 t^{7=2} c_2^2 & 65536 t^{13=2} c_4^2 & 3897344 t^{9=2} c_4^2 & 1024 p \bar{t} c_4^2 \\
 \hline
 1 & & & & 1 & 1 \\
 3 & 2 & & & 3 & 3 \\
 2 & 4 & & 2 & 2 & 2 \\
 3 & +5 & +3 & 1 & 3 & 3 \\
 2p- & 1024 t^{5=2} c_4 & p- & 32768 t^{15=2} c_4^2 & 29824 t^{5=2} c_3^2 & p- & 896 t^{5=2} c_2^2 \\
 \hline
 & & 3 & & 1 & & 1 \\
 8=3 c_1 t+5 & +64 & t c_4+5 & +8 c_1 t & 5 & +16 & t c_2 & 5 \\
 2048 t^{3=2} c_4 & 7296 t^{3=2} c_3^2 & 1024 t^{11=2} c_3^2 & 384 p \bar{t} c_3 & 2385920 t^{11=2} c_4^2 & 677888 t^{5=2} c_4^2 \\
 \hline
 & 2 & & 2 & 5 & 2 & & 2 \\
 & 4 & & 4 & 6 & 8 & & 8 \\
 & 9 & +5 & +5 & +64 t^{3=2} c_3 & 5 & +7 & 5 \\
 1856 t^{3=2} c_2^2 & 31744 t^{3=2} c_4^2 & 288 p \bar{t} c_3 & 64 p \bar{t} c_2^2 & 16896 t^{9=2} c_3^2 & 6016 t^{3=2} c_2 c_3 & 54272 t^{3=2} c_2 c_4 \\
 \hline
 & 1 & & & & & & 1 \\
 & 3 & & & & & & 3 \\
 & 2 & & 2 & & & & 2 \\
 + & 75 & +3 & 7 & 5 & 3 & 5 & 5 \\
 736 c_2 p \bar{t} c_3 & 1024 c_2 p \bar{t} c_4 & 6144 t^{13=2} c_3 c_4 & 5506048 t^{11=2} c_3 c_4 & 408320 t^{9=2} c_3 c_4 & 5604352 t^{7=2} c_3 c_4 \\
 \hline
 & 3 & & 2 & & & & 3 \\
 & 1 & & 2 & & 1 & & 6 \\
 + & 175 & 5 & +7 & 5 & +7 & & 7 \\
 671488 t^{5=2} c_3 c_4 & 40448 t^{3=2} c_3 c_4 & 1664 c_3 p \bar{t} c_4 & p & 32 c_0 t^{3=2} c_2 & 32 c_0 p \bar{t} c_2 & 128 c_0 t^{5=2} c_3 \\
 \hline
 & 7 & 7 & & & & & 7 \\
 & 3 & 3 & & & & & 3 \\
 + & 1575 & 5 & +5 & +4 c_0 c_1 & t+3 & 5 & +5 \\
 192 c_0 t^{3=2} c_3 & 48 c_0 p \bar{t} c_3 & 2048 c_0 t^{7=2} c_4 & 4096 c_0 t^{5=2} c_4 & 128 c_0 t^{3=2} c_4 & 128 c_0 p \bar{t} c_4 & 112 c_1 t^{5=2} c_2 \\
 \hline
 7 & +7 & +5 & 5 & +3 & 1 & +3 & 3 \\
 1472 c_1 t^{3=2} c_2 & 416 c_1 t^{7=2} c_3 & 1696 c_1 t^{5=2} c_3 & 320 c_1 t^{3=2} c_3 & 6336 c_1 t^{9=2} c_4 & 10572 c_1 t^{7=2} c_4 & 28 c_1 t^{5=2} c_4 \\
 \hline
 4 & & 1 & & 3 & & & 1 \\
 5 & +5 & 5 & +7 & +5 & 5 & +135 & 5 \\
 3328 c_1 t^{3=2} c_4 & 88 c_1 p \bar{t} c_2 & 192 c_1 p \bar{t} c_3 & 32 c_1 p \bar{t} c_4 & 2816 t^{9=2} c_2 c_3 & 61696 t^{7=2} c_2 c_3 & 5696 t^{5=2} c_2 c_3 \\
 \hline
 & 1 & 3 & & 1 & 7 & 2 & 1 \\
 63 & +5 & 5 & +7 & +5 & 5 & +5 & 5 \\
 42496 t^{11=2} c_2 c_4 & 4522496 t^{9=2} c_2 c_4 & 1319936 t^{7=2} c_2 c_4 & 74752 t^{5=2} c_2 c_4 & 16 t^{11=2} & & & 16 \\
 \hline
 1 & 7 & 5 & 2 & & & & 1 \\
 0 & 2 & 7 & 2 & & & & 2 \\
 5 & 5 & +5 & 5 & =5 & +8 t^{3=2} & +226:874092 & 5
 \end{array}
 \end{array}$$

$$8 t^4_1 \quad 453:7481856 t^3_1 + 283:5926160 t^2_1 \quad 56:71852320 t_1 + 1:77245385_1 \quad (4.42)$$

Equation (4.43) is collocated and solved to obtain the constants coefficients:

$$c_0 = 1:000299999; c_1 = 3:25585999; c_2 = :1329746886; c_3 = :3196259048;$$

$$c_4 = 0:3234420591 \cdot 10^{10}; c_5 = 0:46302689 \cdot 10^{10}$$

Remark: The other values containing imaginary parts were ignored

Substituting the coefficients into the assumed approximate solution, gives:

$$y_4(t) = 3:818859 \cdot 10^{10} + 2:045 \cdot 10^9 t + 8:95595 \cdot 10^{10} t^2 + 1:00000005 t^3 + 7:04 \cdot 10^{10} t^4$$

Solving same problem for $\alpha = 1:5$ and $N = 5$ following the same procedure, the constant coefficients were obtained as:

$$c_0 = 2:59999999; c_1 = 0:8618045727; c_2 = :1029746865; c_3 = :3598259048;$$

$$c_4 = 0:323442 \cdot 10^{10}; c_5 = 0:863254 \cdot 10^{10}; c_6 = 5:0748352 \cdot 10^{10}$$

Substituting the coefficients into the assumed approximate solution, gives:

$$y_5(t) = 3:980 \cdot 10^{10} + 2:10556 \cdot 10^{10} t + 9:35654 \cdot 10^{10} t^2 + 1:000000045 t^3 + 6:12049 \cdot 10^{10} t^4 + 4:10425 \cdot 10^{10} t^5$$

Estimation of Error for Problem using LWP

The error estimate was done using the formula $|y_{i+1} - y_i|$ where $i = 4$ and the result shown on the table

Solution of example 5 by Quadratic Weighted Polynomial (QWP) at $\lambda = 1.5$ and $N = 4$

Solving equation (4.40) for $\lambda = 1.5$ and $N = 4$ following the same procedure, the constant coefficients were obtained as:

$$c_0 = 3.9512920453; c_1 = 1.284914547148; c_2 = 0.39275386856; c_3 = 0.1714600752; \\ c_4 = 0.12995822 \times 10^{-10}; c_5 = 0.1624882348 \times 10^{-10};$$

Substituting the coefficients into the assumed approximate solution, gives

$$y_4(t) = 8.654736 \times 10^{-10} + 4.67105 \times 10^{-9}t - 1 \times 10^{-10}t^2 + 1.56430 \times 10^{-10}t^3 + 8.9010 \times 10^{-10}t^4$$

Solving same problem for $\lambda = 1.5$ and $N = 5$ and following the same procedure, the constant coefficients were obtained as:

$$c_0 = 0.56283482558; c_1 = 1.5658232799; c_2 = 0.24304374886; c_3 = 0.5464148102; \\ c_4 = 0.704197579 \times 10^{-9}; c_5 = 0.6783452 \times 10^{-10}; c_6 = 0.5773659 \times 10^{-10}$$

Substituting the coefficients into the assumed approximate solution, gives

$$y_5(t) = 6.738535 \times 10^{-10}t^4 + 2.1615 \times 10^{-10}t^3 + 2.9652 \times 10^{-10}t^2 + 8.612763 \times 10^{-10} + 4.76 \times 10^{-10}t \\ + 6.809356 \times 10^{-10}t^5$$

Estimation of Error for Problem using QWP

The error estimate was done using the formula $|y_{i+1} - y_i|$ where $i = 4$ and the result shown on table

Solution of example 5 by Cubic Weighted Polynomial (CWP) at $\lambda = 1.5$ and $N = 4$ solving equation (4.40) for $\lambda = 1.5$ and $N = 5$ and following the same procedure, the constant coefficients were obtained as:

$$c_0 = 9.0727656526; c_1 = 1.3286571656; c_2 = 8.2749840533; c_3 = 1.6586293 \times 10^{-9}; \\ c_4 = 4.2934384 \times 10^{-10}; c_5 = 0.94034282 \times 10^{-10}$$

Substituting the coefficients into the assumed approximate solution, we got the required approximate solution:

$$y_4(t) = 5.01256999 \times 10^{-10} + 1.64050 \times 10^{-8}t + 8 \times 10^{-10}t^2 + 0.9999999999t^3 + 1.3845 \times 10^{-10}t^4$$

solving same problem for $\lambda = 1.5$ and $N = 5$ and following the same procedure, the constant coefficients were obtained as:

$$c_0 = 7.30099194; c_1 = 0.8618045727 \cdot 10^{-9}; c_2 = 0.1029746886; c_3 = 0.3196259048;$$

$$c_4 = 0.3234420591 \cdot 10^{-9}; c_5 = 0.000353645 \cdot 10^{-9}; \lambda = 0.5195027963 \cdot 10^{-10};$$

Substituting the coefficients into the assumed approximate solution, gives

$$y_5(t) = 1.39635 \cdot 10^{-10}t + 9.999999999t^3 + 1.21865 \cdot 10^{-8}t^4 + 2.65439 \cdot 10^{-10} + 5.6953 \cdot 10^{-10}t^2$$

$$+ 1.0085 \cdot 10^{-10}t^5$$

Estimation of Error for Problem using CWP

The error estimate was done using the formula $|y_{N+1} - y_N|$ where $N = 4$ and the result shown on table

4.4 Table of Results

Table 1: Error Results for Example 1 using LWP

x	$y_5(t) - y_4(t)$	$y_6(t) - y_5(t)$	$y_7(t) - y_6(t)$	$y_8(t) - y_7(t)$
0.0	3.5245e-4	3.8562e-6	4.6711e-7	5.0221e-8
0.1	3.3690e-4	3.7123e-6	3.8481e-7	4.8153e-8
0.2	2.9923e-4	2.9032e-6	3.8234e-7	4.6325e-8
0.3	1.8800e-4	2.6022e-6	2.7656e-7	4.3261e-8
0.4	3.6734e-3	2.4876e-6	5.9001e-6	3.5056e-8
0.5	3.6512e-3	2.3001e-6	3.9890e-6	3.2351e-8
0.6	3.5053e-3	2.1762e-6	3.5543e-6	4.7963e-7
0.7	3.3971e-3	2.0845e-6	2.7512e-6	3.6762e-7
0.8	3.0035e-3	4.9952e-5	2.6881e-6	3.4066e-7
0.9	2.7680e-3	3.8331e-5	2.1310e-5	3.2712e-7
1.0	2.6790e-3	2.5461e-5	3.3257e-4	5.7231e-6

Table 2: Error Results for Example 1 using QWP

x	$y_5(t) - y_4(t)$	$y_6(t) - y_5(t)$	$y_7(t) - y_6(t)$	$y_8(t) - y_7(t)$
0.0	3.4545e-4	4.6562e-6	5.3711e-7	4.6221e-8
0.1	3.0690e-4	3.6123e-6	4.5111e-7	4.4123e-8
0.2	2.9231e-4	2.6032e-6	3.9734e-7	3.7325e-8
0.3	2.6800e-4	2.3922e-6	3.6356e-7	3.5061e-8
0.4	2.4134e-4	1.9876e-5	3.4601e-7	3.3156e-8
0.5	2.2512e-4	1.7901e-5	2.8907e-7	2.6051e-8
0.6	1.7353e-4	1.3762e-5	2.2438e-7	1.8563e-8
0.7	5.5271e-3	1.1145e-5	2.1513e-7	1.5762e-8
0.8	3.6035e-3	4.9195e-4	3.8281e-6	3.7066e-7
0.9	2.4080e-3	3.9931e-4	2.3109e-6	3.5412e-7
1.0	2.2907e-3	1.7460e-4	2.1570e-6	1.3931e-7

Table 3: Error Results for Example 1 using CWP

x	$y_5(t) - y_4(t)$	$y_6(t) - y_5(t)$	$y_7(t) - y_6(t)$	$y_8(t) - y_7(t)$
0.0	3.9545e-5	5.0582e-6	4.1211e-7	6.9221e-8
0.1	4.4690e-4	4.5123e-6	3.6711e-7	4.4623e-8
0.2	3.7923e-4	3.5632e-6	3.0754e-7	3.4025e-8
0.3	2.5800e-4	3.1922e-6	2.3656e-7	2.7661e-8
0.4	2.2414e-4	4.6876e-5	3.0164e-7	2.1356e-8
0.5	2.1012e-4	3.7901e-5	1.6807e-7	1.8251e-8
0.6	5.9353e-3	3.3962e-5	4.7238e-6	1.4213e-8
0.7	5.1121e-3	2.1645e-5	3.7113e-6	3.8672e-7
0.8	2.8031e-3	1.9455e-5	4.8181e-5	3.4066e-7
0.9	2.2180e-3	3.6031e-4	3.2319e-5	3.1152e-7
1.0	2.1967e-3	1.7607e-4	2.3701e-5	1.6191e-6

From tables 1-3, it can be seen that the errors obtained get smaller as the value of N (the degree of the approximant) increases. It can also be noticed that the Quadratic Weight Polynomial (QWP) performed slightly better than the other two.

Table 4: Error Results for Example 2 using LWP

x	$y_5(t) - y_4(t)$	$y_6(t) - y_5(t)$	$y_7(t) - y_6(t)$	$y_8(t) - y_7(t)$
0.0	5.710e-5	4.820e-6	3.984e-8	5.790e-8
0.1	5.222e-5	4.479e-6	3.232e-8	5.308e-8
0.2	4.704e-5	3.993e-6	2.939e-8	4.647e-8
0.3	3.536e-5	2.451e-6	2.505e-8	3.788e-8
0.4	2.790e-5	5.475e-5	2.282e-8	3.200e-8
0.5	1.220e-5	3.204e-5	2.672e-7	2.950e-8
0.6	4.670e-4	2.288e-5	2.125e-7	2.183e-8
0.7	3.870e-4	1.895e-5	3.646e-6	1.721e-7
0.8	2.164e-4	5.706e-4	3.287e-5	1.413e-7
0.9	2.087e-4	4.916e-4	3.150e-5	1.076e-7
1.0	3.957e-3	1.124e-4	6.391e-4	4.100e-6

Table 5: Error Results for Example 2 using QWP

x	$y_5(t) - y_4(t)$	$y_6(t) - y_5(t)$	$y_7(t) - y_6(t)$	$y_8(t) - y_7(t)$
0.0	5.638e-6	3.170e-8	5.592e-9	4.788e-9
0.1	5.503e-6	2.886e-8	2.942e-9	4.404e-9
0.2	4.595e-6	4.459e-7	4.564e-8	3.950e-9
0.3	3.931e-6	3.106e-7	3.780e-8	2.212e-9
0.4	3.498e-6	2.805e-7	3.165e-8	6.031e-8
0.5	3.136e-6	4.528e-6	3.090e-8	5.745e-8
0.6	6.903e-5	4.079e-6	2.658e-8	3.498e-8
0.7	6.680e-5	3.849e-6	1.150e-8	2.185e-8
0.8	5.883e-5	3.609e-6	6.092e-7	1.008e-8
0.9	3.415e-5	2.415e-6	4.252e-7	5.329e-7
1.0	2.583e-5	3.583e-5	3.483e-7	3.609e-7

Table 6: Error Results for Example 2 using CWP

x	$y_5(t) - y_4(t)$	$y_6(t) - y_5(t)$	$y_7(t) - y_6(t)$	$y_8(t) - y_7(t)$
0.0	4.360e-6	5.373e-7	4.504e-8	5.519e-8
0.1	3.536e-6	5.910e-6	3.865e-8	4.900e-8
0.2	3.096e-6	3.508e-6	3.018e-8	4.000e-8
0.3	2.986e-6	3.256e-6	2.935e-8	2.456e-8
0.4	1.753e-6	2.582e-6	1.252e-8	1.275e-8
0.5	1.512e-6	2.000e-6	3.747e-7	3.683e-7
0.6	4.100e-5	1.755e-6	3.222e-7	2.999e-7
0.7	3.557e-5	6.830e-5	3.999e-6	2.739e-7
0.8	2.831e-5	5.409e-5	3.763e-6	1.776e-7
0.9	4.692e-4	5.175e-5	2.350e-6	3.867e-6
1.0	3.053e-4	1.998e-5	4.097e-5	2.745e-6

From tables 4-6, it is also observed that the errors obtained decreases as the value of N (the degree of the approximant) increases. Perhaps for the double integrals on the right hand side of the problem considered, the errors obtained for example 2 are generally lower compared to that of example 1. It can be noticed also that the three polynomials performed well here.

Table 7: Results for Example 3, by SCM at $\alpha = 3:25$

x	FDTM	CPSM	Avipsita et al.,	LWP Appr	QWP Appr	CWP Appr
0.0	1.000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.1208247	1.1271325	1.1218062	1.12187900	1.1218137	1.1217978
0.2	1.2635120	1.2741287	1.2652165	1.2653000	1.2652341	1.2653009
0.3	1.4309200	1.4441923	1.4330692	1.4330393	1.4330682	1.4330395
0.4	1.6267534	1.6421688	1.6290869	1.6290709	1.6290002	1.6290880
0.5	1.8555465	1.8698097	1.8578339	1.8578000	1.8578010	1.8578009
0.6	2.1227071	2.1367908	2.1247544	2.1246528	2.1247999	2.1246320
0.7	2.4345869	2.4453183	2.4362410	2.4366687	2.4361559	2.4360177
0.8	2.7985942	2.8067522	2.7997254	2.7999921	2.7996898	2.79969009
0.9	3.2232029	3.2274548	3.2237874	3.2238000	3.2236999	3.2237700
1.0	3.7182818	3.7182818	3.7182818	3.7182818	3.7182819	3.7182818

Table 8: Results for Example 3, by PCM at $\alpha = 3:25$

x	FDTM	CPSM	Avipsita et al.,	LWP Appr	QWP Appr	CWP Appr
0.0	1.000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.1208247	1.1271325	1.1218062	1.1218070	1.1217437	1.1217978
0.2	1.2635120	1.2741287	1.2652165	1.2653000	1.2654341	1.2653009
0.3	1.4309200	1.4441923	1.4330692	1.4339393	1.4330682	1.4339395
0.4	1.6267534	1.6421688	1.6290869	1.6290799	1.6320862	1.6290780
0.5	1.8555465	1.8698097	1.8578339	1.8578340	1.8578335	1.8578336
0.6	2.1227071	2.1367908	2.1247544	2.1247541	2.1247544	2.1247546
0.7	2.4345869	2.4453183	2.4362410	2.4362412	2.4362409	2.4362490
0.8	2.7985942	2.8067522	2.7997254	2.7997091	2.7997198	2.7996011
0.9	3.2232029	3.2274548	3.2237874	3.2237860	3.2237999	3.2237700
1.0	3.7182818	3.7182818	3.7182818	3.7182818	3.7182819	3.7182818

Table 9: Results for Example 3, by SCM at $\epsilon = 3:75$

x	FDTM	CPSM	Avipsita et al.,	LWP Appr	QWP Appr	CWP Appr
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.1208247	1.1271325	1.1218062	1.1218070	1.1217437	1.1217978
0.2	1.2635120	1.2741287	1.2652165	1.2652070	1.2652164	1.2652909
0.3	1.4309200	1.4441923	1.4330692	1.4330688	1.4330682	1.4330395
0.4	1.6267534	1.6421688	1.6290869	1.6290799	1.6290862	1.6290780
0.5	1.8555465	1.8698097	1.8578339	1.8578200	1.8578339	1.8578099
0.6	2.1227071	2.1367908	2.1247544	2.1247909	2.1247910	2.1247909
0.7	2.4345869	2.4453183	2.4362410	2.4362399	2.4362559	2.4362099
0.8	2.7985942	2.8067522	2.7997254	2.7997201	2.7997298	2.7996211
0.9	3.2232029	3.2274548	3.2237874	3.2238000	3.2237999	3.2237700
1.0	3.7182818	3.7182818	3.7182818	3.7182818	3.7182819	3.7182818

Table 10: Results for Example 3, by PCM at $\alpha = 3:75$

x	FDTM	CPSM	Avipsita et al.,	LWP Appr	QWP Appr	CWP Appr
0.0	1.000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.1208247	1.1271325	1.1218062	1.1218060	1.1218061	1.1218078
0.2	1.2635120	1.2741287	1.2652165	1.2652165	1.2652161	1.2652169
0.3	1.4309200	1.4441923	1.4330692	1.4339393	1.4330682	1.4339395
0.4	1.6267534	1.6421688	1.6290869	1.6290799	1.6328702	1.6298770
0.5	1.8555465	1.8698097	1.8578339	1.8578330	1.8578339	1.8578340
0.6	2.1227071	2.1367908	2.1247544	2.1247548	2.1247544	2.1247560
0.7	2.4345869	2.4453183	2.4362410	2.4362399	2.4362411	2.4362418
0.8	2.7985942	2.8067522	2.7997254	2.7997251	2.7997280	2.7997211
0.9	3.2232029	3.2274548	3.2237874	3.2237809	3.2237874	3.2237799
1.0	3.7182818	3.7182818	3.7182818	3.7182818	3.7182819	3.7182818

From tables 7-10, it can be observed that the results obtained with our methods, SCM and PCM fairly agreed with that of the previous methods used. The results at $\alpha = 3:75$ are better than obtained at $\alpha = 3:25$ perhaps because 3.75 is closer to 4 in the interval of consideration.

Table 11: Apprx. Results for Example 4 for $N = 4$, $\alpha = 2:5$

x	Exact	LWP app	QWP app	CWP app
0.0	0.000000000000	0.00000000001252	-0.00000000011096	-0.00000000001047
0.1	0.001000000000	0.00100000000300	0.00099999994900	0.00100000000600
0.2	0.008000000000	0.00799999997400	0.00799999998900	0.00800000001400
0.3	0.027000000000	0.02699999992000	0.02700000001000	0.02700000001000
0.4	0.064000000000	0.06399999986000	0.06400000002701	0.06400000001000
0.5	0.125000000000	0.12499999980000	0.1249999971200	0.12499999701200
0.6	0.216000000000	0.21599999970000	0.21599999990000	0.21600000000000
0.7	0.343000000000	0.34299999960000	0.34299999990000	0.34299999991230
0.8	0.512000000000	0.51199999950000	0.51199999970000	0.51199999980000
0.9	0.729000000000	0.72899999940000	0.72899999960000	0.72899999970000
1.0	1.000000000000	0.99999999939000	0.99999999941450	0.99999999963500

Table 12: Error Results for Example 4 for $n = 4$, $\alpha = 2:5$

x	LWP err	QWP err	CWP err	Yang err
0.0	1.252e-11	1.110e-11	1.047e-11	0.000e+00
0.1	2.999e-11	5.100e-11	5.599e-11	6.400e-05
0.2	2.600e-11	1.099e-11	1.399e-11	8.000e-05
0.3	7.491e-11	7.407e-11	1.228e-11	8.400e-05
0.4	1.394e-10	3.879e-11	1.569e-11	1.120e-04
0.5	2.176e-10	2.879e-10	2.988e-10	2.000e-04
0.6	3.063e-10	7.709e-11	5.453e-10	3.840e-04
0.7	4.001e-10	1.000e-10	4.770e-10	7.000e-04
0.8	5.000e-10	2.735e-10	1.699e-10	1.184e-03
0.9	5.892e-10	4.232e-10	2.560e-10	1.872e-03
1.0	6.100e-10	5.854e-10	3.650e-10	3.800e-03

From tables 11-12, it is observed that the results obtained with our methods is better than that of the previous method of Legendre Pseudo-Spectral Method (LPSM) used by Yang at $\alpha = 2:5$.

Table 13: Estimated Error for Example 5 using LWP

x	SCM, y ₅ (t)	SCM, y ₄ (t)	PCM, y ₅ (t)	PCM, y ₄ (t)	SCM Error	PCM Error
0.0	0.000000005167	0.000000000632	0.000000000036	0.000000000527	4.535e-10	3.073e-10
0.1	0.101000001270	0.100999996928	0.101000001637	0.100999924010	2.397e-09	4.342e-10
0.2	0.008000000291	0.007999999388	0.001000001637	0.000999992401	2.397e-09	3.034e-10
0.3	0.127000000460	0.126999999070	0.127000004870	0.126999997700	1.410e-09	7.169e-09
0.4	0.064000000610	0.063999998760	0.064000006470	0.063999996900	2.556e-09	1.845e-09
0.5	0.725000000800	0.724999998500	0.725000008000	0.724999996100	1.188e-08	2.284e-09
0.6	0.456000000900	0.455999998200	0.456000009300	0.455999995300	1.406e-08	2.665e-09
0.7	0.583000001000	0.582999997900	0.583000010700	0.582999994700	1.598e-08	2.966e-09
0.8	0.595000000800	0.595999997700	0.594000011500	0.594999994000	1.753e-08	3.178e-09
0.9	0.929000000900	0.928999997600	0.929000012000	0.928999993500	1.856e-08	3.323e-09
1.0	1.000000001000	0.999999997600	1.000000011000	0.999999993200	1.896e-08	3.455e-09

Table 14: Estimated Error for Example 5 using QWP

x	SCM, y ₅ (t)	SCM, y ₄ (t)	PCM, y ₅ (t)	PCM, y ₄ (t)	SCM Error	PCM Error
0.0	0.000003500000	0.000000356000	0.000000052100	0.000000037500	3.144e-6	3.229e-8
0.1	0.000032109067	0.000000000632	0.000000525000	0.000000001529	3.210e-6	5.234e-7
0.2	0.831600000900	0.83169998200	0.000000060230	0.000000254123	9.999e-6	2.4810e-7
0.3	0.295850000008	0.295999997700	0.294000011500	0.294999954000	4.411e-6	3.012e-7
0.4	0.528900000080	0.524999998500	0.525000008000	0.524999996100	1.902e-6	3.613e-7
0.5	0.371900000100	0.382999997900	0.372500001070	0.372999994700	2.402e-6	2.285e-7
0.6	0.434700001000	0.432999997900	0.434000010700	0.432999994700	3.600e-5	3.147e-7
0.7	0.189200000080	0.185999997700	0.189200001150	0.184999994000	5.001e-5	4.582e-7
0.8	0.776400000800	0.775999997700	0.775900011500	0.774999994000	3.205e-5	2.637e-6
0.9	0.566800001000	0.562999997900	0.570000010700	0.562999994700	2.483e-5	1.416e-6
1.0	1.000000001000	0.999999997600	1.000000011000	0.999999993200	3.682e-5	2.326e-6

Table 15: Estimated Error for Example 5 using CWP

x	SCM, $y_5(t)$	SCM, $y_4(t)$	PCM, $y_5(t)$	PCM, $y_4(t)$	SCM Error	PCM Error
0.0	1.00000000100	0.99999999760	1.00000001100	0.99999999320	2.000e-6	1.000e-7
0.1	0.62300000100	0.82999999790	0.68800001070	0.58299999470	3.136e-6	5.720e-7
0.2	0.90400000100	0.48299999790	0.58300001070	0.58299999470	3.236e-6	6.694e-7
0.3	0.78530000010	0.29999999790	0.53400001070	0.59299999470	4.010e-6	4.092e-7
0.4	0.72500000080	0.2499999985	0.72500000800	0.79499999610	1.092e-6	3.303e-7
0.5	0.46200000100	0.26299999790	0.58300001070	0.78299999470	2.402e-5	4.365e-7
0.6	0.64500000100	0.58299999790	0.58300001070	0.68299999470	4.640e-5	5.907e-7
0.7	0.68900000100	0.58299999790	0.58300001070	0.58299999470	4.041e-5	5.912e-6
0.8	0.57500000100	0.68299999790	0.78300001070	0.59299999470	2.566e-5	4.027e-6
0.9	0.99300000100	0.88299999790	0.18300001070	0.56299999470	3.088e-5	2.146e-6
1.0	1.0000000010	0.9999999979	0.00001070000	0.80999999470	2.782e-5	1.216e-6

From tables 13-15, it is observed that the estimated error results obtained gets better as the value of N (the degree of the approximant) increases. Besides, estimated error results obtained using the PCM are fairly better than that of the SCM.

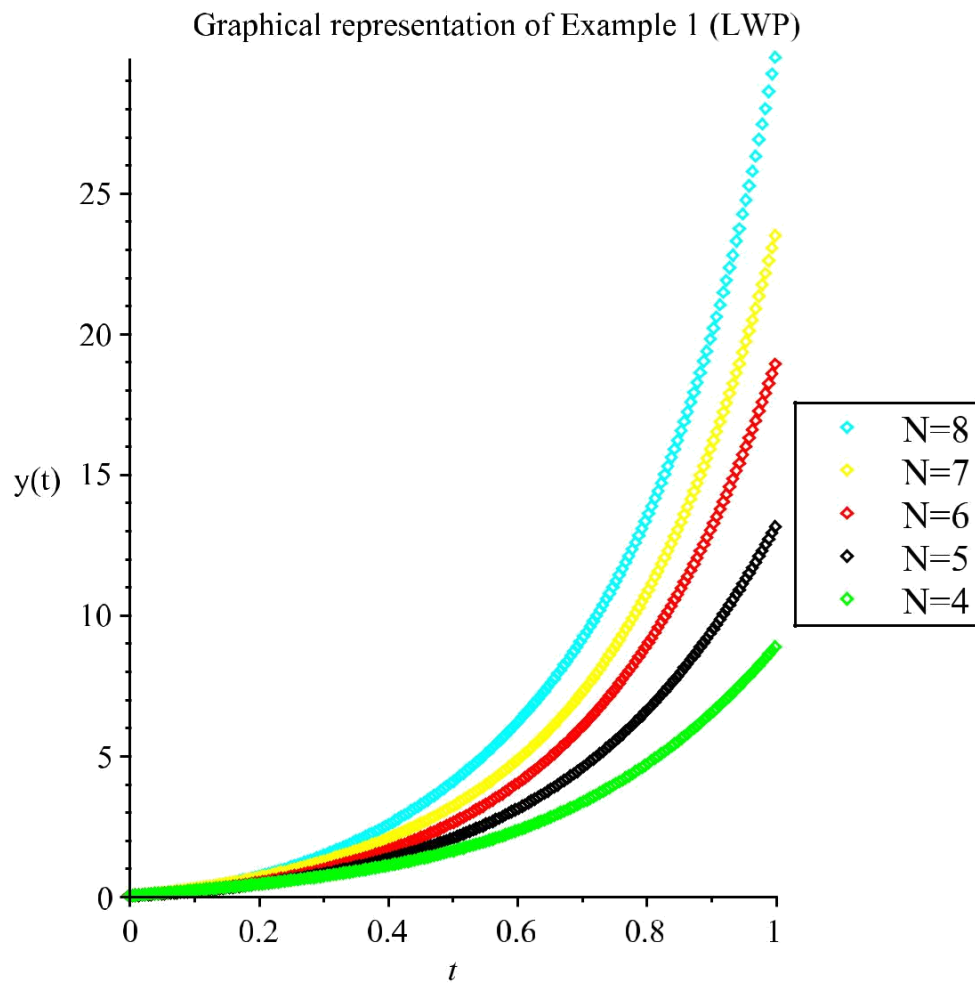


Figure 4.1:

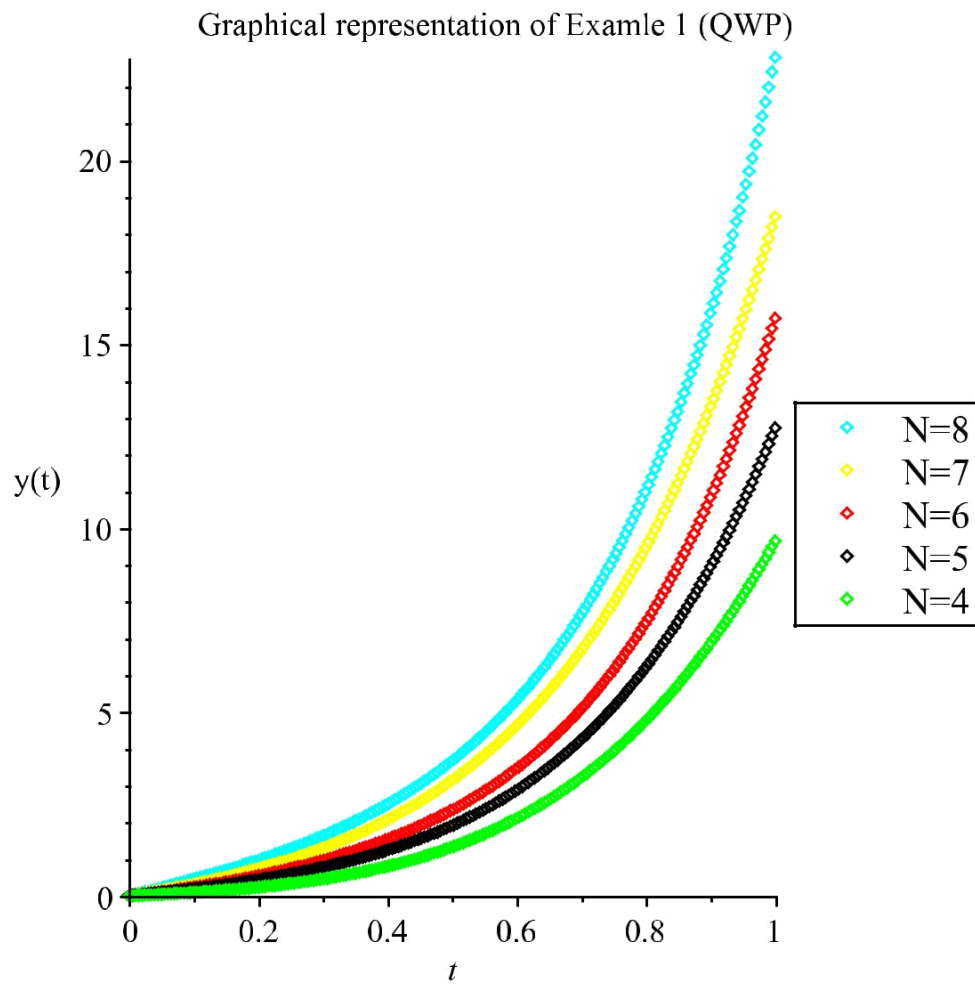


Figure 4.2:

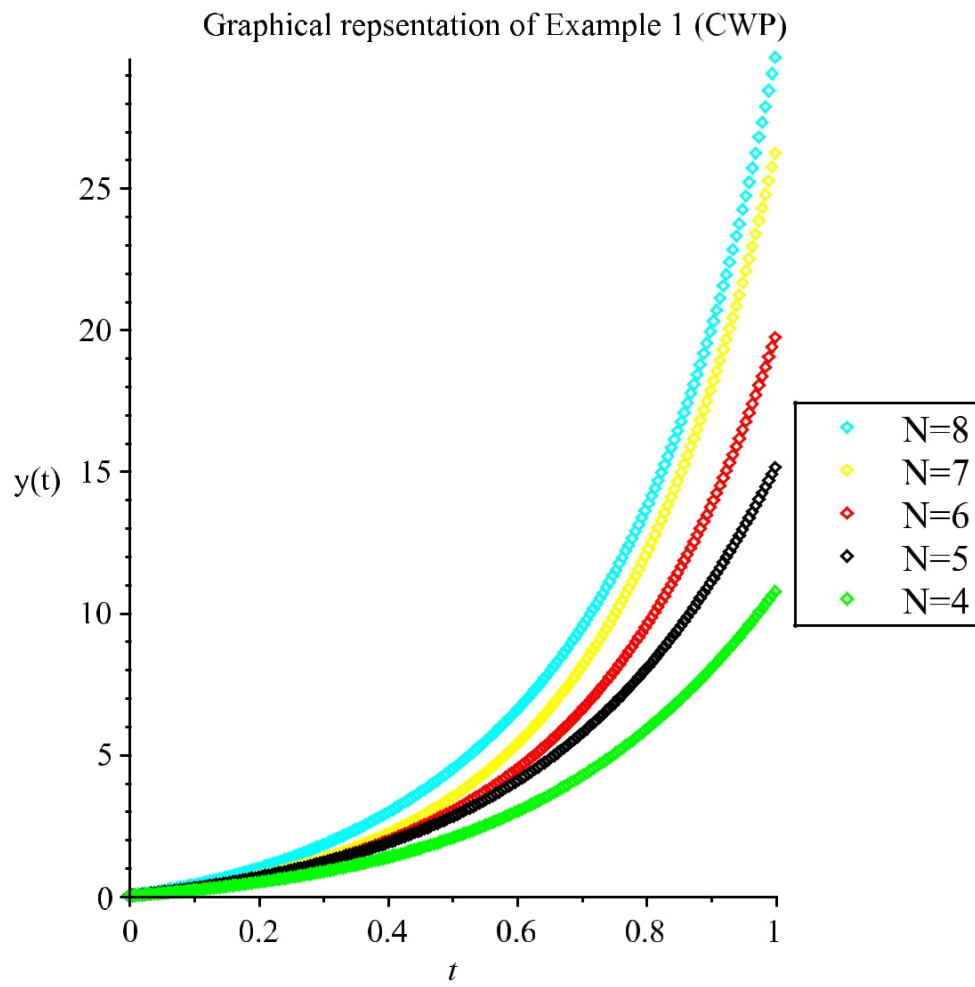


Figure 4.3:

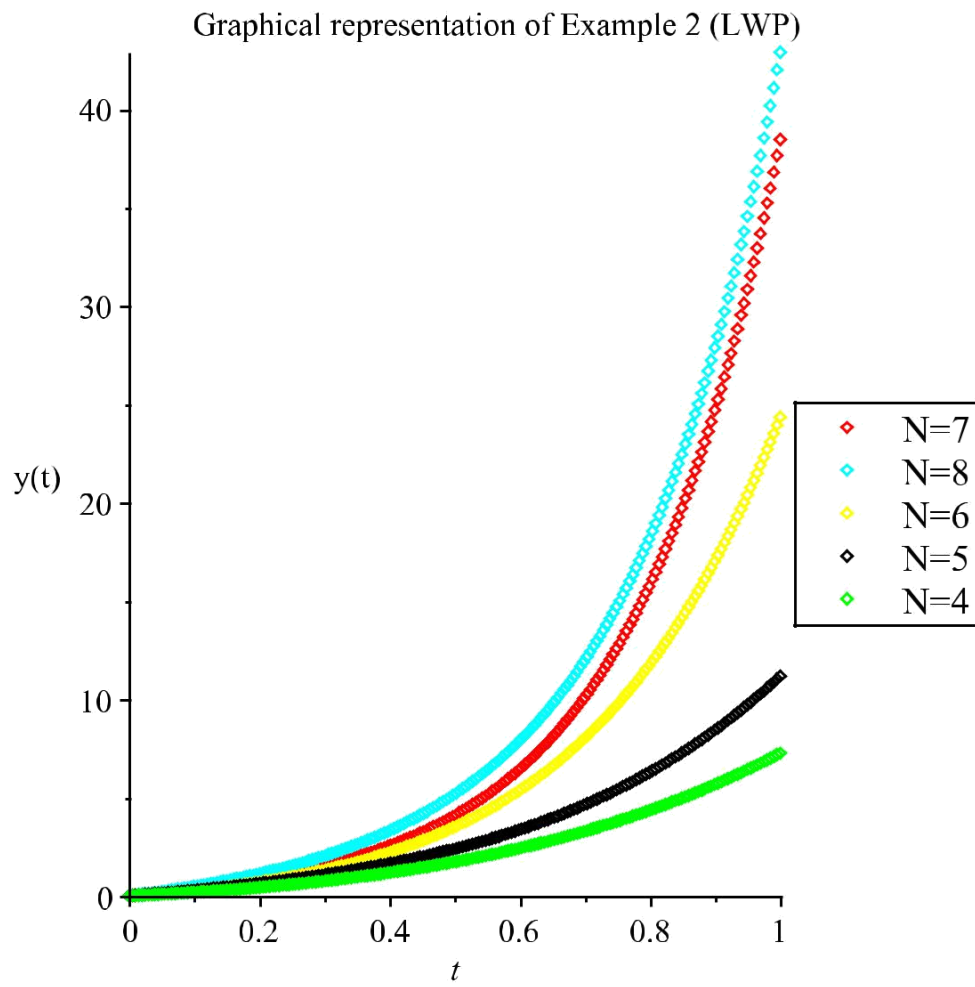


Figure 4.4:

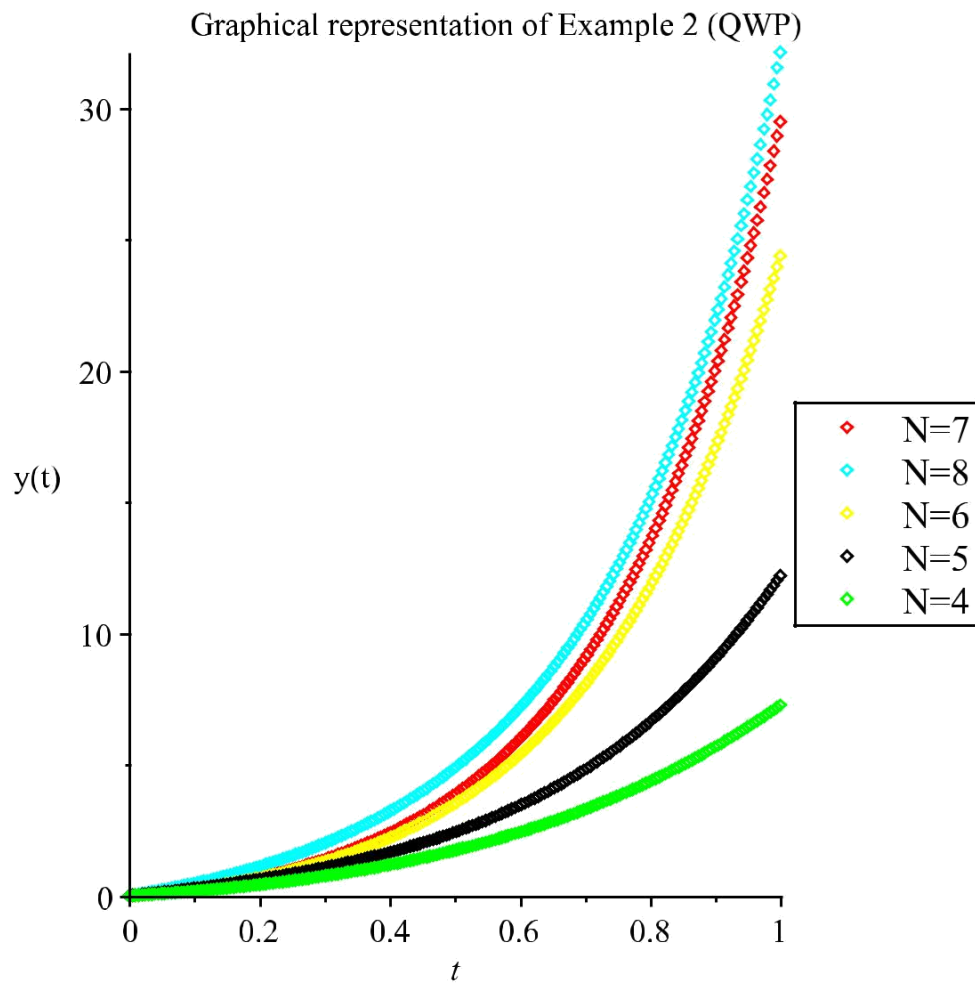


Figure 4.5:

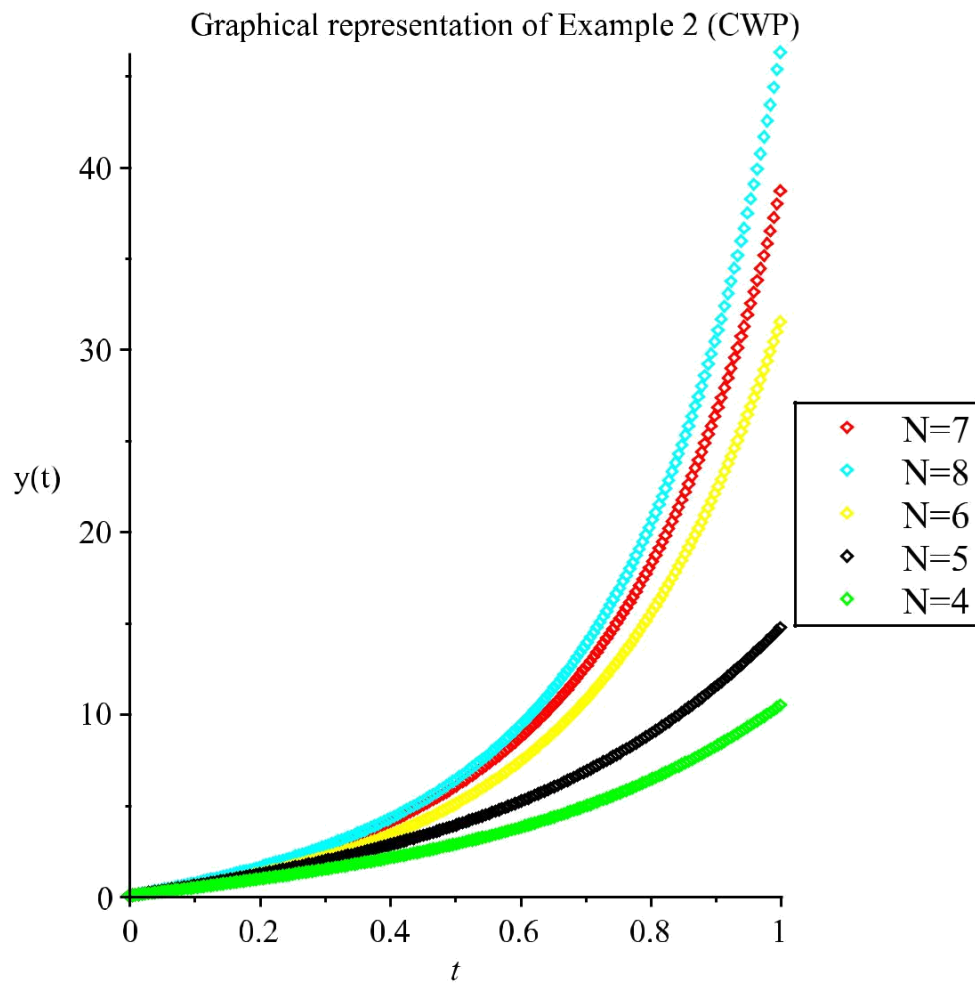


Figure 4.6:

Chapter 5

Conclusion, Contribution to Knowledge and Recommendations

5.1 Conclusion

Some constructed orthogonal polynomials were applied as basis functions to solve multi-order fractional differential and integro-differential equations. The orthogonal polynomials were generated using special weighted functions that satisfied the property of non negative criteria of weighted functions in the interval under consideration. Some examples of multi order fractional differential equations as well as multi order fractional integro-differential equations were solved using the two proposed methods; the Standard Collocation Method (SCM) and Perturbation Collocation Method (PCM) and the results show that the constructed orthogonal polynomials are good tools for solving the two types of problems. Besides, it is seen that the proposed methods are suitable. The results obtained highly agreed with other researchers' work in the literature. So in comparison to some existing results in the literature, the performances of the constructed orthogonal polynomials were found to be okay as shown in the tables of errors. It was also observed that the performances of our orthogonal polynomials is generally better for the Quadratic Weighted Polynomials (QWP).

The suitability of the constructed polynomials is considered satisfactory as the results obtained agreed closely with the results gotten where existing polynomials were used. Furthermore, it is interesting to note that the constructed polynomials and the methods used are efficient and okay for solving both linear and non-linear multi-order fractional differential and integro-differential equations.

Besides, it was also observed that the performance of the methods improved as the values of

N (the degree of the approximant) is increasing. Error estimation was carried out in cases where the exact solutions were unknown using the formula:

$$\text{Error} = \|Y_{(i+1)}(x) - Y_i(x)\|$$

and throughout the thesis, the estimated error were considered acceptable zero value if $\|y_{i+1} - y_i\| = 10^{-k}$.

5.2 Discussion of Results

Five problems were solved in this work using Standard Collocation and Perturbed Collocation Methods. Using the SCM, approximate solutions were obtained which were very closed to the exact solutions. However, using the PCM, the obtained approximate solutions improved. Thus it is easy to say that PCM performed better than the SCM in this work. On the whole, the proposed methods proved to be suitable and accurate in solving the multi-order fractional differential and integro-differential equations for which they were applied. The fact that Chebyshev polynomials were used to perturb the problems in this study and good results were obtained shows that my constructed polynomials are compatible with the Chebyshev polynomials. It is observed that generally the estimated error results obtained gets better as the value of N (the degree of the approximant) increases. Besides, estimated error results obtained using the PCM are fairly better than that of the SCM. That is the effect of adding the perturbation term.

5.3 Contribution to Knowledge

The modest contributions to knowledge of this work are in line with the stated objectives. They are as follows:

- (i) some new orthogonal polynomials valid in the interval [0,1] are constructed which are not in existence in the literature;
- (ii) solutions of linear and non-linear multi-order fractional differential equations were obtained using Standard Collocation Method;
- (iii) solutions of nonlinear multi-order fractional integro-differential equations were obtained using Perturbed Collocation Method.

5.4 Recommendations for Further Study

For further study, this research work can be extended to multi-order fractional partial differential equations and multi-order fractional partial integro-differential equations. Besides, the polynomial can further be generated using higher values of m ; $m > 3$

REFERENCES

- Abbasbandy, S (2006). Application of He's Homotopy Perturbation method for Laplace Transform. *Chaos Solution and Fractals*. 30 (5), 1206-1212.
- Abdulaziz, O., Hashim, I. and Momani, S. (2008). Application of Homotopy Perturbation to Fractional IVPs. *Journal of Comput. and Applied Math*. 216, 572-584.
- Abdullah Sadeghinia and Prabhat Kumar (2015) One Solution of Multi-Term Fractional Differential Equations by Adomian Decomposition Method. *International Journal of Scientific and Innovative Mathematical Research (IJSIMR)*. 3, (6) 14-21
- Arikoglu.A and Ozkol.I (2009). Solution of fractional Integro differential equations by using fractional Differential Transform Method, *Chaos,Solitos and Frantals*. 40, 521-529
- Atabakzadeh.M.H, Arami.M.H and Erjaee.G.H (2012). Chebyshev Operational Matrix Method for solving multi order fractional ordinary differential equations.*Applied Math-ematical Modeling*.. 37, 8903-8911
- Avipsita Chatterjee, Uma Basu and Mandal B. N. (2017) Numerical Solution of Volterra Type Fractional Order Integro-Differential equations in Bernstein Polynomial Basis. *Applied Mathematical Sciences*. HIKARI Ltd. 11 (6), 249 - 264.
- Batiha, B, Noorani, M.S.M and Hashim. I. (2008). Numerical Solutions of the Non-linear Integro-differential Equations. *International Journal of Open Problems Comput. Maths*. 1(1). 34 - 42.
- Brumer, H. (2004). Collocation Methods for Volterra Integral and Related Functional Differential Equations. Cambridge University Press. New York, U.S.A.
- Caputo, M (1967) Linear models of dissipation whose Q is almost frequency independent 11. *Geophys J. Royal astr. Soc*. 13, 529-539
- Changqing,Y and Jianhua, H (2013). Numerical solution of integro differential equations of fractional order by Laplace Decomposition Method. *WSEAS Transactions on Mathe-matics*. 12 (12), 1173-1183.
- Chen, X and Wang, L (2010). Variational iteration method for solving anatural fractional differential equation with proportional delay. *Comput. Math. Appl.*, 59, 2696-2702

- Dass, H. K (2000). Advanced Engineering Mathematics. S. Chand and Company limited. AN ISO 9001:2000 Company. Ram Nagar, New Delhi-110055, 1158-1164
- Davaeifar.S, Rashidnia.J and Amirfakhrian. M (2016) Bernstein polynomial approach for the solution of higher-order mixed linear Fredholm integro-differential equations with variable coefficients. TWMS Journal Appl. Math. 7 (1) 46-62
- Davood. R, Moshen. A, Hussein. J and Dumitru. B (2013): Solving multi term fractional integro differential equations, operational matrices of BPs with convergence analysis. Ramanian report in Physics, 65 (2), 334-349
- Fesojaye, M. O (2016). Perturbed Collocation least-squares methods for solving fractional order differential and fractional order integro-differential equations. Unpublished Ph.D. Thesis, University of Ilorin, Ilorin.
- Gegele, O. A (2016) Spline collocation method for numerical solution of higher order linear and nonlinear integro-differential equations. Unpublished Ph.D. Thesis, University of Ilorin, Ilorin.
- Gradimir, V.M (1991): Orthogonal polynomial systems and some applications. Mathematical subject classification. Email- grade@efnis.efak.ni.ac.yu, 2-50
- Guo-Chang, W (2011) Fractional Variational Iterative Method for Fractional Nonlinear Differential Equations. Rom. Journal. Physics. 58 (7), 873-880
- Hashim. L, Abdulaaziz, O, and Momeni.S (2009): homotopy Analysis Method for fractional IVPs. Commun. Nonlinear, Sci. Numer. Simulation. 14, 674-684
- He, J.H (1999):Some applications of nonlinear fractional differential equations and their approximations. Bull. Sci. Technol. 15(2): 86-90.
- He, J.H (2003). A simple perturbation approach to Blasius equation. Appl. Math. Comput., 140, 217-222.
- He, J. H (2006). Variational Iteration Method- somerecent results and interpretations. J.Comput. Appl. Math.doi., 10, 10-16
- Hesameddini, E and Asadollahifard, E (2015) Numerical Solution of multi-order fractional differential equations via the sinc collocation method. Iranian Journal of Numerical Analysis and Optimization. 5, 37-48

- Ibtisan, K.H (2011): The Homotopy Analysis Method for solving multi fractional order integro differential equations. Journal of Al-Nahrain University. 14, (3), 139-143.
- Jianping .L, Xia Li, Liyong Xu and Yufa Shen (2012): Solution of fractional differential equations by Bernstein Polynomials Method. Journal of Convergence Information Technology (JCIT). 7 (23), 820-826.
- Jafari, H and Daftardar-Gejji, V (2006). Solving linear and nonlinear fractional diffusion and wave equations by Adomian decomposition method. Appl. Math. and Comput. 180, 488-497.
- Jafari, H and Sei , S (2004). Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation. Common nonlinear SciNamer. Simul. 14 (5) 2006-2012
- Jafari, H and Momani, S (2007). Solving fractional diffusion and wave equations by modified Homotopy Perturbation method. Journal of Comput. and Applied Math. 370, 388-396.
- Khader, M.M (2011): Numerical solutions of non linear multi-order fractional differential equations by Implementation of the Operational Matrix of fractional derivative. Studies in Nonlinear Sciences. 2 (1) 5-12.
- Kilbas A. A, Srivastava H. M and Trujillo J. J (2006) Theory and Applications of fractional differential equations. ELSEVIER B.V Radarweg 29. PO Box 211, 1000 A.E Amsterdam, The Netherlands. 24-60
- Khuri, S.A (2001). Laplace decomposition algorithm applied to a class of nonlinear differential equations. Journal of Maths. Appl. 1, 141-155.
- Lacroix, S. F (1819) Traite du calcul differentiel et du calcul integral. Imprimeur-Libraire pour Les sciences, Paris. 409-410
- Li,Y (2010) Solving a Nonlinear Fractional Differential Equation Using Chebyshev Wavelets. Communications in Nonlinear Science and Numerical Simulation 15 (9), 2284-2292.
- Mahnaz Seifollahi and Ali Salimi Shamloo (2013): Numerical Solution of nonlinear multi-order fractional differential equations by operational matrix of Chebyshev polynomials. World Applied Programming Journal. 3 (3), 85-92.

- Mahdy, A. M and Shwayye, R. T (2016). Numerical solution of fractional integro-differential equations by least squares method and shifted Laguerre polynomials pseudo-spectral method. *Intc. Journal of Scientific and Engineering Research*. 7(4), 1589-1596.
- Mohammed, D.sh (2014): Numerical solutions of fractional integro differential equations by least square method and shifted Chebyshev polynomials. *Journal of Mathematical problems in Engineering*. Hindawi publishing cooperation, ID431965 4, 1-5
- Momani, S and Odibat, Z (2007). Numerical approach to differential equations of fractional order. *J. Computer Appt. Match*, 207, 96-110.
- Mittal, R.C and Nigam R (2014): Solution of Multi fractional order integro differential equations by Adomian Decomposition Method. *International Journal of Applied Math-ematics and Mechanics*. 4, 87-94.
- Miller, K.S and Ross.B (1993). *An introduction to Fractional Calculus and Fractional Differential Equations*. Wiley, New York.
- Neda. N, Mehrara. A and Sohrab. E (2012). Solution of higher order integro differential equations by Variational Iteration Method. *Austrahan Journal of Basic Applied Sciences*. 6 (2), 175-181
- Odetunde, O.S (2013): Numerical Approximation of fractional differential and integro-differential equations by an Iterative Decomposition Method. Unpublished Ph.D. Thesis. University of Ilorin, Ilorin.
- Odibat, Z and Momani, S (2008) Modified Homotopy Perturbation Method: Application to Quadratic Riccati Differential Equation of Fractional Order. *Chaos, Solutions and Fractals*. 36 (1) 167-174.
- Omar I.K (2014): Variational Iteration Method for solving multi fractional integro differential equations. *Iraqi Journal of Science*. 55, (3A), 1086-1094.
- Podlubny.I (1999). *Fractional differential equation*. Academic Press. San Diego.
- Rawashdeh.E. A (2006). Numerical Solution of fractional Integro Differential equations by Collocation Method. *Appl. Mathematical Computations*. 176, 1-6

- Sastry S. S (2010): Introductory Methods of Numerical Analysis. fourth edition. PHI Learning Private Limited. New Delhi-110001.
- Shaqul, I and Azizur, R (2013). Solution of linear and nonlinear Volterra Integral equations using Hermite and Chebyshev polynomials. International J. of Computers and Technology. 11 (8) 2910-2919
- Shehab, S. N, Ali, H. A and Yaseen, H. M (2010) Least squares method for solving integral equations with multiple time lags. Engineering and Technology Journal. 28 (10), 1893-1899.
- Sweilam, N. H and Khader, M. M (2009): A Chebyshev pseudo-spectral method for solving fractional-order integro-differential equations. ANZIAM J., 51, 464-475.
- Taiwo, O. A (1986) A computational method for ordinary differential equation and error estimation. M.Sc dissertation, Unpublished, University of Ilorin, Ilorin.
- Taiwo, O. A and Odetunde, O. S (2013). Approximation of Multi-Order Fractional Differential Equations by Iterative decomposition method. American J. of Engineering Science and Technology Research. 1 (2), 10-18.
- Taiwo, O. A, Odetunde, O. S and Adekunle, Y. A (2009). Numerical Approximation of One Dimensional Biharmonic equations by an iterative Decomposition method. Int. J. Math. Sciences. 20 (1), 37-44.
- Taiwo, O.A and Uwaheren, O.A (2015). Efficient Numerical Solution of Multi-Order Fractional Integro Differential Equations. International Journal of Applied Science and Mathematical Theory 1 (3) 1-11
- Uwaheren, O .A and Taiwo, O. A (2016). A Collocation Technique Based on an Orthogonal Polynomial for solving Multi-Fractional order Integro Differential Equations. ABACUS 43 (2) 215-223
- Wazwaz, M. A (2010). The combined Laplace transform-Adomian decomposition method for handling nonlinear Volterra integro-differential equations. Appl. Math. Comput. 216, 1304-1309.
- Wazwaz, M.A (2011). Linear and Nonlinear Integral Equations: Methods and Applications. Higher Education Press, Beijing and Springer-Heidelberg.

- Weibeer M (2005). Efficient Numerical Methods for Fractional Differential Equations and their Analytical Background. A Ph.D Thesis. Von der Carl-Friedrich-Gaub-Fakultat für Mathematik und Informatik der Technischen Universität Braunschweig, 11-129
- Yin Yang (2013) Solving a Nonlinear Multi-Order Fractional Differential Equation using Legendre Pseudo-Spectral Method. Applied Math., 4, 113-118
- Yusufoglu, E (2006). Numerical solution of Du'ng equation by Laplace decomposition algorithm. Appl. Math. Comput. 177, 572-580.
- Zadunaisky, P. (1976) On the estimate of errors in Numerical Integration of Ordinary Differential Equations. Numerical Mathematics, 27, 21-39.
- Zuppa, C (2003). Error estimates for moving least squares approximations. Bulletin of the Brazillian Mathematical Society 34 (2) 231-249