

GLOBAL CONTROL THEORY AND ITS APPLICATIONS

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Abstract

This chapter presents a controlled dynamic system whose state at any moment in time is described by the vector x . To control means that the systems equations are subdefinite. If this subdefiniteness with fixed t, x is represented by some non fixed element u , then controlling the system implies closing its equations by assigning the function $u(t)$ or $u(t, x)$. The assignment of $u(t)$ is called the program of control. It is the aim of control to guide the system in the assigned conditions as reflected in the constraints imposed on the state x at the end of the control period $t = t_f$, to provide process admissibility connected with fulfilment of current constraints imposed on the state $x(t)$ fulfil some qualitative dynamic requirements set for the system, such as stability. In the process of solving these problems, some new mathematical ideas were suggested, namely, the theory of optimal control, developed as a synthesis of the classical variational calculus which makes use of Euler-Lagrange variation method. The first one is connected with generalization of conditions for a local minimum of the real variable function $f(x) = 0, f(x) \geq 0$ to the problem of functional analysis. The second one is connected with imbedding the optimal process construction problem in the family of these problems for various initial conditions. The second method is more demanding and it culminates in global control theory. Methods of control improvement are discussed including computer methods or iterative techniques of control improvement, namely, the gradient method and needle method. The new idea of the theory of optimal control makes use of the approach based on sufficient conditions for global optimality of control processes and a mathematical technique for global estimates related to it. An interesting characteristic feature of the global estimates technique is the richness of the analytical medium created by this technique. Some areas of application are provided by the control of the pure inertia plant, sliding mass, sliding regimes, and electricity generation.

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1. Optimal Control Theory

It seems that optimal control theory can simply be viewed as the modern face of the calculus of variations. It should be remarked that a paradigm shift has taken place which outweighs in importance the mathematical links between the two. Also, many of the basic questions differ and different sorts of answers are required. By and large the methodology and the areas of application of the two disciplines have little in common because as pointed out in [12], the smoothness requirements of the classical variational calculus are inappropriate in optimal control theory. The three central themes common to any optimization problem are (i) Existence of a solution (ii) necessary conditions for identifying the solution(s) and sufficient conditions to confirm that a candidate solution is indeed a solution.

Consider a system of the form:

$$\dot{x} = f(x, u, t) \quad (1)$$

where

$$f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$$

$$\text{with } x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

The introduction of a given function of time $u(t)$ into the right-hand side of equation (1) yields a generally nonlinear ordinary differential equation. In many physical processes described by differential equations the time dependence of the process may be influenced in some manner. This is generally referred to as a steering of the process; it is implemented in (1) by means of the steering function or control function $u(t)$. The variable x is called the state variable in control theory. Consider, for example, a spaceship; then the component of the vector x may refer to the position and velocity and $u^T = (u_1, u_2, \dots, u_m)$ may represent the controllable thrusts of the individual engines as well as their directions of action insofar as the engines are gimbaled [5]. If a system is to be transferred from a given state x_0 to another state x_1 there often exist numerous control functions with which this may be accomplished. Not all of these will be suitable and the question arises as to how one might determine a most suitable one; that is, that one which minimizes the fuel consumption, or that which minimizes the travel time. Problems from other areas also may lead to similar mathematical problems, such as in a market economy where a conglomerate of producers has the control variable, price and amount of production at its disposal in order to control the market (state variable) [1]. The most common control problem is stated in the following form:

$$P1. \text{Minimize } \int_{t_0}^{t_f} f^0(x(t), u(t), t) dt + \phi(x(t)) \quad (2)$$

subject to the dynamical system described by the differential (1)

$$\frac{dx(t)}{dt} = f^0(x(t), u(t), t), x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, t \in [t_0, t_f]$$

where $x(t)$ is the state of the system at time t , $u(t)$ is the control applied to the system at time t , t_0 is the given starting time, and t_f is the final time and may or may not be specified in advance. The objective of solving the problem is to find a measurable control function $\hat{U}(\cdot)$ defined on $[t_0, t_f]$ a corresponding trajectory $\hat{x}(\cdot)$, determined by (1-2) and the final time t_f , if it is not specified, which minimize the cost functional $P.2$ where $f^0 : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^l \rightarrow \mathbb{R}^n$, and $f^0 : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \rightarrow \mathbb{R}^l$ are continuously differentiable in x and u , $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^1$, which is the termination time weighting term.

The problem (2) is the continuous optimal control problem. However, digital computers do not compute such functions u , they can only compute a sequence of numbers. Consequently, any numerical method of solving continuous optimal control problems must involve some form of discretization of the problem. Discrete optimal control problems are frequently discretizations of continuous optimal control problems, with the discretization being carried out in such a way as to minimize the mathematical difficulties in dealing with the problem, as well as to minimize the digital computer memory requirements, and to reduce the number of operations required per iteration [15].

One likely question in the mind of a young researcher is whether it is not easier to solve the dynamic equation constraint or integrate the differential equation and plug in the resulting values of the state and control variables in order to optimize the cost functional. However, the need for integrating differential equations in solving optimal control problems results in a number of practical difficulties. It also introduces theoretical difficulties whenever one is honest enough to admit that these integrations can only be performed approximately.

In addition, the dynamically system will generally be nonlinear so that it will not be possible to obtain analytical or closed-form solutions, rather, it will be necessary to seek refuge in numerical and analytical approximation methods.

So far, no satisfactory tests for establishing whether a candidate solution to any of the three central or general optimization problems conceived is optimal or not. It is therefore not surprising that there are no general methods for solving such problems.

The algorithms which we shall describe in this chapter can easily be used to construct a sequence of approximations whose limits satisfy a particular optimality condition. The subject of optimality is rather vast. Here, we shall content ourselves with stating, mostly without proof, the optimality conditions used most frequency in computational methods.

2. Optimality Conditions

The reader should note that if one were to take a completely rigorous approach, there would be three main aspects to be considered in a control problem namely, existence of an optimal control, necessary conditions which it must satisfy, and sufficient conditions to assure that any candidates which have been found are in fact, optimal. In practice, existence is often taken for granted from physical considerations, and extremal quantities (satisfying only necessary conditions) are referred to as optimal.

Now, if an optimal solution can be shown to exist, then the extremal control which yields the lowest cost is, in fact, optimal; and if an extremal pair $(\hat{u}(t), \hat{x}(t))$ is unique then optimality also follows immediately. Necessary conditions for the optimality of a solution

of the control problem are supplied by the Pontryagin maximum principle, which is cited here without proof [3-5]

2.1. Pontryagin Maximum Principle

Pontryagin maximum principle, a far-reaching generalization of Weierstrass's necessary conditions for strongly minima, provides geometric conditions for a (strong) minimum of an integral criterion called the "cost" over the trajectories of a differential control system. These conditions are based on the topological fact that an optimal solution must terminate on the boundary of the extended reachable set formed by the completing curves and their integral costs.

Let $(\hat{u}(t), \hat{x}(t), t)$ be a solution of the boundary-value problem(bvp) defined by the system (1), x_0 and x_f with free terminal time, and minimizing the functional.

$$J(u(t)) = \int_{t_0}^{t_f} f^0(x(t), u(t), t) dt \quad (3)$$

Then there exists continuous piece wise differentiable functions

$$y_0(t), y_1(t), \dots, y_n(t)$$

which satisfy the following conditions:

- The Functions $x_i(t), y_i(t)$ satisfy the differential equations.

$$\dot{x}_i = \frac{\partial H}{\partial y_i} \quad (4)$$

$$\dot{y}_i = -\frac{\partial H}{\partial x_i}, i = 0, 1, 2, \dots, n \quad (5)$$

with

$$H(x_1, \dots, x_n; y_0, \dots, y_n; \hat{u}(t), t) = \sum_{i=0}^n y_i f_i(x_1, \dots, x_n, y_0, \dots, y_n; \hat{u}(t), t)$$

$$H(x_1, \dots, x_n; y_n; \hat{u}(t), t) = \sum_{i=0}^n y_i f_i(x_1, \dots, x_n, y_0, \dots, y_n; \hat{u}(t), t) \quad (6)$$

- the $y_0, y_1, \dots, y_n$ are not all simultaneously zero; and y_0 is a constant ≤ 0 ;
- One has;

$$H(x_1(t), \dots, x_n(t); y_0(t), \dots, y_n(t); \hat{u}(t), t) = \text{Max} H(x_1(t), \dots, x_n(t); y_0(t), \dots, y_n(t); \hat{u}(t); t) \quad (7)$$

$u \in U$ That is, the optimal control function at each instant assumes those values which maximize H_j .

- in terms of the optimal solution, H is a continuous function of the time and has $H(t_0) = 0$. If the functions $f_i, i = 0, 1, 2, \dots, n$ do not depend on t explicitly, then H is a constant and equals to zero.

Clearly, equations (5) and (6) describe a Hamiltonian system where the x_i correspond to the generalized coordinates and where the additionally introduced adjoint variables y_i correspond to the generalized momenta. Note that the Hamiltonian is a first integral for autonomous systems taking on the value zero in the case of an optimal control, as indicated by the fourth assertion above. The optimal controls are determined from equation (7) in their dependence on the variables $x_1, x_2, \dots, x_n; y_1, y_2, \dots, y_n$; and t . Subsequently, one would then attempt to use $u(t)$ as obtained there from t_0 determines the solution of the equations (5) and (6) subject to the given boundary conditions on x . It is immediately apparent from equation (6) that y_0 must be constant, since H is independent of x_0 by definition. Since (6) is linear homogeneous in $y_0, y_1, \dots, y_n$ and since y_0 is constant, there is no loss of generality in choosing $y_0 = -1$ for non-vanishing y_0 [5].

The distinctive character of the optimal control theory can be traced to the mathematical problem of the subject in the middle of the 20th century dealing with inequality constraints, faced with the practical, time-optimal control problems of that period, mathematicians and engineers looked to the calculus of variation for answers, but soon discovered, through the paper of [23] on the bang-bang controls, that the answers to their problems were beyond the scope of the classic theory and would require different mathematical tools. That realization initiated a search for new necessary conditions for optimality suitable for control problems. That search further intensified by the space program and the race to the moon, eventually led to the "maximum principle", which answered the practical needs of that period in [2]. The maximum principle states that the Hamiltonian that corresponds to the optimal trajectory must be maximal relative to the competing directions and that each optimal trajectory is the projection of an integral curve of the corresponding vector field. So strongly was the maximum principle linked to control problems involving bounds on the controls that its significance in a larger context, as an important generalization of Weierstrass's function in the calculus of variations, went unnoticed for a long time after its discovery by L.S Pontryagin.

The treatment here of optimal control problems combines the rich theory of the calculus of variations with control theory innovations. We believe, as did [12], that a control theory point of view is more natural for the calculus of variation. We shall further argue that the calculus of variations is fundamentally concerned with problems of optimal control and that much mathematical clarity is gained through this recognition.

The initial formulation of the maximum principle is followed by a comparison between the maximum principle and other necessary conditions for optimality that emphasize the distinctions between strong and weak maxima: The maximum principle, as a generalization or extension of the Weierstrass's, corresponds to the strong minimum, whereas the Euler-Lagrange equation corresponds to the weak minimum. Continuing with the minor theme that the maximum principle can illustrate even some of the most classic solutions of the calculus of variations, we shall then obtain solutions to the problem of the control of pure inertia plant through control theory perspectives. We first discuss global sufficient conditions for optimality.

Let us ignore the process equations in an optimal control problem. Then it reduces to

the following problem in the case of discrete time:

$$\text{Minimize}_{(x,u) \in E} \sum_{t_f}^{t_0} f^0(x(t), u(t), t) dt \quad (8)$$

Where E is a set of all functions (consequences) defined over the row $A = \overline{t_0, t_f}$ which satisfies the restrictions

$$(x(t), u(t)) \in V(t), \forall t \in A \quad (9)$$

where $V(t)$ is a given family of subsets of the direct product $x \times u$; Problem (8) decomposes into a family of independent minimization problems: for every fixed $t \in (t_0, t_f)$, to find an admissible pair of vectors $x(t)$ and $u(t)$ such that

$$f_0(x(t), u(t), t) = \text{Min}_{x,u \in V(t)} (x, u, t) = v(t), \forall t \text{ (say)}$$

Any solution

$$V \approx (x(t), u(t)) \quad (10)$$

of the equation, and only such a solution, is the solution of problem (8). The minimal value of the functional (8) is

$$\ell = \sum_{t=0}^t v(t) \quad (11)$$

Problem (8) connected with the original optimal control problem by the following fact:

- The value ℓ is a lower bound of the functional (8) over the set of admissible processes.
- Sufficient optimality condition . Let the solution of equation (10), $(x(t), u(t))$ satisfy process equation:

$$x(t+1) = f(x(t), u(t), t), t = t_0, t_0 + 1, t_0 + 2, \dots, t_f - 1 \quad (12)$$

where $f(x, u, t)$ is a given function,

$f : A \times x \times u \rightarrow x$. That is, $(x(t), u(t))$ is admissible . Then it is an optimal process.

This optimality condition has some strong properties. It is a global optimality condition. This makes it possible for us to decompose the original optimal control problem into a family of independent elementary minimization problem (10)

3. Global Bounds for The Cost Functional

Let us introduce a class of $\mathbb{D}$ the real differentiable functions $\phi(t, x) : A \times x \rightarrow \mathbb{R}'$ and the following constructions:

$$R(t, x, u) = \frac{\partial \phi}{\partial x} f(t, x, u) - f^0(t, x, u) + \frac{\partial \phi}{\partial t} \quad (13)$$

$$G_0(t) = F_0(x) + \phi(t, x) : G(t_f, x) = F(t_f, x) + \phi(t_f, x) \quad (14)$$

$$L(\hat{v}, \phi) = G(t_f, x(t_f)) - \int_A^{\mathbb{R}} (t, x(t), u(t)) dt - G_0(x(t)) \quad (15)$$

where A is a segment of $[t_0, t_f]$, $f(t, x, u) = f_i(t, x, u)$, $i=1$ right hand side of the process equation (1), $i = \overline{1, n}$, $f^0(t, x, u)$ is the integrand (3), $F_0(x)$ and $F(t_f, x)$ are the initial and terminal addends of the optimality criterion (3) and $(\frac{\partial \phi}{\partial x})f(x, t, u) = (\frac{\partial \phi}{\partial x})f^0(x, t, u)$ is an Euclidean product.

Lemma 3.1. *The functionals $L(\hat{u}, \phi)$ and $J(u)$ coincide for any admissible process $u \in \mathbb{D}$ and functions $\phi(t, x) \in \mathbb{D}$ where $\mathbb{D}$ is the set of admissible processes.*

$$L(\hat{u}, \phi) = J(u), \forall u \in \mathbb{D}, \phi \in \mathbb{D} \quad (16)$$

Indeed, using (13) and (15), we get

$$R(t, x(t), u(t)) = \frac{d\phi[t, x(t)]}{dt} - f_1^0(x(t)u(t)) \in \mathbb{D} \quad (17)$$

inserting (17) in (15) and remembering that functions $x(t)$ are continuous, we obtain $L = J$.

Equation (15) defines a family of representations of the functional $J(u)$ on set D with parameter function $\phi(x, t)$ and the corresponding family of equivalent optimal control problems (L, D) (say). We can use this representation to solve the improvement problem of a process $u \in D$ such that

$$L(u, \phi) = J(u) < J(u_0) = L(u_0, \phi) \quad [9] \quad (18)$$

Details about representations for multistage processes and multi argument processes are succinctly contained in . We next introduce lower bounds, bounding functions, solving functions and normalized functions. In what follows, we describe some useful constructions to the function $\phi \in \mathbb{D}$ namely,

$$\mu(t) = \max_{(x, u) \in V(t)} R(t, x, u), \forall t \in (t_0, t_f) \quad (19)$$

$$m = \min G(t_f, x), x \in W, m_0 = \max G_0(x), x \in W_0 \quad (20)$$

$$\ell(\phi) = m - \int_A \mu(t) dt - m_0 \quad (21)$$

where $A = [t_0, t_f]$, the terms W and W_0 are sets of admissible terminal and initial states, respectively, $R(t, x, u)$, $G(t_f, x)$ and $G_0(x)$ have been defined by (13) and (14). The function ϕ will be called a bounding function when the functional $\ell(\phi)$ is defined. The functional $\ell(\phi)$ is a lower bound of the functional $L(u, \phi)$ on the set of piecewise-continuous function pairs $(x(t), u(t))$ for any function $\phi(t, x) \in \mathbb{D}$ such that

$$L(u, \phi) \geq \ell(\phi), \forall u \in \text{set of piecewise continuous function pairs } \phi \in \mathbb{D} \quad (22)$$

We have by definitions

$$\begin{aligned} R(x, t, u) &\leq \mu(t), \forall t, x : G_0(x) \leq m_0, \forall x_j \\ G(t_f, x) &\geq m_1 \forall t_f, x \end{aligned}$$

Lemma 3.2. *The functional $\ell(\phi)$ is a lower bound of the functional $J(u)$ on the set D of admissible processes $(x(t), u(t))$ for any function $\phi(t, x) \in D$.*

$$J(u) \geq \ell(\phi), \forall u \in \mathbb{D}, \phi \in \mathbb{D} \quad (23)$$

Generally speaking, the function ϕ is essentially non-unique, and when specified for different subclasses of $\mathbb{D}$, leads to different method of solution. We next discuss the different methods of solution and of control improvement. The function $\phi(t, x) \in \mathbb{D}$ is called a solving function and the set of all these functions is denoted $\mathbb{D}$. Having found a pair $u \in \mathbb{D}, \phi \in \mathbb{D}$ means that the pair of dual problems

$$J(u) = \min_{u \in \mathbb{D}} J(u) = \ell\phi = \max_{\phi \in \mathbb{D}} \ell(\phi) \quad (24)$$

has been solved [1].

4. Methods of Control Improvement

In this section, we discuss computer methods of iterative improvement in a control program, for control problems with a large dimension of the state vector. We consider the following:

$$\min_{\hat{u} \in \mathbb{D}} J(\hat{u}) = \int_{t_0}^{t_f} f^0(t, x(t), u(t)) dt + F(x(t_f)) \quad (25)$$

subject to

$$\frac{dx}{dt} = f(t, x, u); x(t_0) = x_0; x \in \mathbb{R}^n; u \in u$$

where the vector function $f(t, x, u)$ and the function $F(x)$ are twice differentiable for all values of arguments; u is a compact set. The discrete variant of this problem is:

$$\begin{aligned} \min_{\hat{u} \in \mathbb{D}} J &= \sum_{t=t_0}^{t_f-1} f^0(t, x(t), u(t)) + F(x(t_0)) \\ x(t+1) &= f(t, x, u), t = 1, 2, \dots, t_f - 1 \\ x(t_0) &= x_0; x \in \mathbb{R}^n; u \in u \end{aligned} \quad (26)$$

We separate out from these problems the improvement subproblem. Let us assume that we know an admissible non-optional pair $(x_0(t), u_0(t)) \in D$. Our intention is to improve this pair by finding a pair $(\hat{x}(t), \hat{u}(t))$ such that

$$J(u) < J(u_0)$$

Repeating this process, an improving sequence $u_s \in \mathbb{D}$ such that $J(u_{s+1}) < J(u_s)$ will be constructed and such that

$$\lim_{s \rightarrow \infty} J(u_s) = \inf_{u \in \mathbb{D}} J(u) \quad (27)$$

A basic operation of the group of methods considered is a solution of the shuttle pair of Cauchy problems:

1. integrating the process equation (25) or (26) closed by an assigned control $u_0(t)$ forward in time to obtain the corresponding trajectory $x_0(t)$;
2. integrating the adjoint equation closed by the assigned process $x_0(t), u_0(t)$ backward in time with the initial transversality condition to obtain the adjoint vector function $\psi(t)$, say. The latter gives information for synthesis of an improved control program $u(t)$ that is infinitesimally close to $u_0(t)$.

The process of generating an optimal feedback control from a known optimal open-loop control is called the synthesis of optimal control. Successive repetition of this operation replaces the boundary value problem for equations of the maximum principle. Development of these methods is due to [16–19] and others.

4.1. Infinitesimal Methods of Control Improvement

This section considers methods of local improvement of control in terms of improving function ϕ . We shall look for a solution of the improvement problem u that is sufficiently close to the assigned process u_0 in such a way that the sign of $\Delta J = J(u) - J(u_0)$ is the same as its main linear part.

$$\delta J = \delta L = G(x_0(t_f))\delta x(t_f) - \int_{t_0}^{t_f} dt \left(\sum_{t_0}^{t_f-1} \right) (R_x \delta x(t) + R_u \delta u(t)) \quad (28)$$

$$\delta x = x - x_0, \delta u = u - u_0,$$

functions $R(t, x, u)$ and $G(x)$ are differentiable. Formula (28) is given to within the function $\phi(t, x)$ in the points of the trajectory $x_0(t)$ and required that it comply with the equalities.

$$R_x(t, x_0(t), u_0(t)) = 0, \quad G_x(x(t_f)) = 0 \quad (29)$$

These equations contain only the gradient of function $\phi(t, x)$ in the points of the trajectory $x_0(t)$. The vector function $\psi(t) = \phi_x(t, x_0(t))$ is determined by the equations

$$R_x[t, x_0(t), u_0(t)] = \frac{d}{dt} \psi(t) + H_x[t, \psi(t), x_0(t), u_0(t)] = 0 \quad (30)$$

$$G_x(x(t_f)) = \psi(t_f) + F_x(x(t_f)) = 0 \quad (31)$$

$$H(t, \psi, x, u) = \psi f(t, x, u) - f^0(t, x, u) \quad (32)$$

In the discrete variant,

$$R_x[t, x_0(t), u_0(t)] = \psi(t) - \frac{\delta}{\delta x} H[t, \psi(t+1), x_0(t), u_0(t)] = 0 \quad (33)$$

which implies that those equations are satisfied by functions of the form $\phi(t, x) = \psi(t)x$, where the vector $\psi(t) = \psi_i(t)$ is determined by equation (30) or (33) and initial condition (31). This function is described as locally improving for control. Then,

$$\delta J(u_0) = \delta L(u_0, \phi) = \int_{t_0}^{t_f} dt \left(\sum_{t_0}^{t_f-1} \right) R_u(t, x_0(t), u_0(t)) \delta u(t)$$

$$R_u(t, x_0(t), u_0(t)) = H_u(t, \psi(t+1), x_0(t), u_0(t)), \phi = \psi(x)x \quad (34)$$

where $R_u(t, x_0(t), u_0(t))$ equals $H_u(t, \psi(t), x_0(t), u_0(t))$ or $H_u(t, \psi(t+1), x_0(t), u_0(t))$ for the continuous and discrete formulations of the problem respectively.

4.1.1. Summary of the Improvement Algorithm

Let $\delta u_0(t)$ be a given function and $\varepsilon > 0$ an arbitrarily small parameter such that the right hand side of (34) is positive, $u(t, \varepsilon) = u_0 + \varepsilon \delta u_0 \in u, t \in A$, the trajectory $x(t, \varepsilon)$ determined by the control $u(t, \varepsilon)$, the process equations, and the initial conditions (25) and (26) is admissible [i.e, $u(\varepsilon) = (x(t, \varepsilon), u(t, \varepsilon)) \in \mathbb{D}$], then $\exists > 0$ such that

$$J(u) < J(u_0), u = u(\varepsilon) \quad (35)$$

Therefore, the algorithm for improvement of control reduces to the following steps:

- Find the trajectory $x_0(t)$ by solving the Cauchy problem (25) or (26) with $u = u_0(t)$.
- Find $\psi(t)$ and $R_u(t, x_0(t), u_0(t))$ by solving the Cauchy problem (26) with the initial condition (27) which determines a local improving function $\phi = \psi(t)x$.
- Set a variation of the control u_0 written as $\delta u_0(t)$, which makes the right hand side of (34) positive.
- For different $\varepsilon > 0$, find the trajectory $x(t, \varepsilon)$ by solving the Cauchy problem (25) or (26) with $u = u + \varepsilon \delta u_0$. The value of ε should be chosen such that (35) holds.

Remark 4.1. The basic part of this algorithm is the sweeping solution of the pair of Cauchy problems: the equation of motion from t_0 to t_f and the adjoint equation from t_f to t_0 . The consecutive repetition of these operations makes it possible to find the improving sequence $u_s \subset \mathbb{D}$.

The expression in (34) gives the gradient of the functional $J(u)$ in the space of control function $u(t)$. Hence, the method can be considered as an application of the gradient techniques to the class of problems under consideration. The limitations of this method include the local character of improvement guaranteed only for small variations of the control $u(t)$, the convergence is also slow because the small variations can be unrealisable when the set u is finite. These deficiencies can be avoided when globally improving functions are used.

4.2. The Needle Method of Control Improvement

This method also uses the linear form of the improving function $\phi(t, x) = \psi(t)x$, where the vector $\psi(t) = \psi_i(t)$ is determined by equation (30) and (31). Let us introduce an infinitesimal parameter $\theta \rightarrow 0$, a number $\eta \in [t_0, t_f]$ and a vector $u \in u$. We construct the following family of control variations:

$$u(t, \theta, \eta, u) = \begin{cases} u, & \eta \leq t < \eta + \theta \\ u_0(t), & t_0 \leq t < \eta \\ u_0(t), & \eta + \theta \leq t \leq t_f \end{cases} \quad (36)$$

The admissible processes family $u(\theta, \eta, u) \in D$ corresponds to these controls and for this family, in terms of (24) and (25).

$$\begin{aligned}\Delta L &= L(\phi, u(\theta), \eta(u)) - L(\phi, u_0) \\ &= - \int_{\eta}^{\eta+\theta} [H(t, \psi(t), x(t), u) - H(t, \psi(t), x_0(t), u_0(t))] dt + O(\theta^2)\end{aligned}$$

The difference is a piecewise continuous function. Therefore, $\theta > 0$ exists such that it is a continuous limited function over the segment $[\eta, \eta + \theta]$. Therefore we can write

$$\begin{aligned}\Delta L &= -[H(\eta, \psi(\eta), x_0(\eta), u) - H[\eta, \psi(\eta), x_0(\eta), u_0(\eta)]\theta + O(\theta^2) \\ \text{i.e.} \\ \Delta L &= -[H(\eta, \psi(\eta), x_0(\eta), u) - H[\eta, \psi(\eta), x_0(\eta), u_0(\eta)]\theta + O(\theta^2)\end{aligned}\tag{37}$$

Let there be a pair η and u such that

$$[H(\eta, \psi(\eta), x_0(\eta), u) > H[\eta, \psi(\eta), x_0(\eta), u_0(\eta)]$$

Then $\Delta L < 0$ for small enough θ . Thus we can offer the following control improvement algorithm.

1. Initialization: Choose an admissible control $u_0(t)$.
2. Find the trajectory $x_0(t)$ by solving the Cauchy problem (25) or (26) with $u = u_0(t)$ and $x(0) = x_0$.
3. Find $\psi(t)$ and the function $H(t, u) = H(t, \psi(t), x_0(t), u)$ by solving the linear Cauchy problem (31) with the initial condition (33).
4. Verify the equality

$$H(t, u_0(t)) = \max_u H(t, u)\tag{38}$$

for different values $t \in [t_0, t_f]$. Fix a value $t = \eta$ where (37) is not satisfied. Also fix a value u such that

$$H(t, u) > H(t, u_0(t))$$

5. For different $\theta > 0$, assign the control $u(t, \theta, \eta, u)$. Find the trajectory $x_0(t, \theta, \eta, u)$ by solving the Cauchy problem (24) or (25) with $u = (t, \theta, \eta, u)$, $x(0) = x_0$.

The value of θ should be taken in such a way that $\Delta L < 0$.

5. Global Method of Successive Improvement Algorithm of Control

We have described an iterative method of constructing a sequence $u_s \subset D$ that at each step improves the functional being minimized. The improvement algorithm reduces to the following sequence of operations [1].

1. Initialization: Choose an admissible control program $u_0(t)$.
2. use process equation (24) or (25) with $u = u_0(t)$ to determine the trajectory $x_0(t)$. For the process $v_0 = (x_0(t), u_0(t))$, calculate the value of the functional $J(v_0)$.
3. use the equation (31) or (33) with the initial condition (32) to determine the vector function $\psi(t)$.
4. For definiteness, set all the functions $\delta_i(t)$ equal to a nonnegative constant δ . Analogously, set all the α_i equal to a nonnegative number α . use equations

$$\begin{aligned}\frac{\partial^2 R_0}{\partial x^i \partial x^j} &= 0, & i \neq j, & \quad i, j = 1, 2, \dots, n \\ \frac{\partial^2 R_0}{\partial x^i \partial x^j} &= \delta_i(t), & i &= 1, 2, \dots, n\end{aligned}\tag{39}$$

where $\delta = \delta_i(t)$ denotes a positive vector function:

$$R(t, x_0(t), u_0(t)) = \min R(t, x, u_0(t)), \forall t \neq t_f\tag{40}$$

$$G(x_0(t_f)) = \max_x G(x)\tag{41}$$

with initial conditions

$$\begin{aligned}\frac{\partial^2 G}{\partial x^i \partial x^j} &= 0, & i \neq j, & \quad i, j = 1, 2, \dots, n \\ -\frac{\partial^2 G}{\partial x^i \partial x^j} &= \alpha_i, & i &= 1, 2, \dots, n\end{aligned}\tag{42}$$

where the $\alpha = \alpha_i$ is a given positive vector to determine the matrix Σ and form the function $\phi(t, x)$ in accordance with

$$\phi(t, x) = \psi_i(t)x^j + 0.5\sigma_{ij}(t)\Delta x^i \Delta x^j, i, j = 1, 2, \dots, n\tag{43}$$

where $\sigma_{ij}(t)$ are the entries of the matrix Σ .

5. For this function $\phi(t, x)$, we find the control $\tilde{u}(t, x)$ by maximization (39) of the function $R(t, x, u)$. Find also the process $V = (x(t), u(t)) = \tilde{u}(t, x)$
6. Find the trajectory $x(t)$ using the process equation (24) or (25) with $u = \tilde{u}(x, t)$ and $J(v)$.

If the functional has not improved, increase δ and x and return to steps (3) – (5). This completes one iteration.

5.1. Gradient-based Method of Control Improvement

5.2. Theoretical Background

In what follows, we employ the explicit knowledge of the initial costate vector to improve the control. We note that there could be an improvement in the initial costate vector determined in the equations (74) to (103) and we set to use this in the variation of extremals

so that if the terminal or the resulting final costate $\lambda(t_f) = 0$ obviously our guess of initial costate vector was correct and we have solved the problem. We started by considering the problem of minimizing

$$J = \int_{t_0}^{t_f} g(x, u, t) dt \quad (44)$$

subject to the dynamical constraints

$$\dot{x} = f(x, u, t) \quad (45)$$

where the states and costates are not constrained by any boundaries, the final time t_f is fixed and the final state $x(t_f)$ is free. To write the necessary conditions for optimality, let us define the Hamiltonian functional as usual [6].

$$H = g(x, u, t) + \lambda^T(t) f(x, u, t) \quad (46)$$

so that the necessary conditions become

$$\dot{x}(t) = \frac{\partial H}{\partial \lambda} = g(x, u, t) \quad (47)$$

$$\dot{\lambda} = -\frac{\partial H}{\partial \lambda} = -\left[\frac{\partial g(x, u, t)}{\partial x}\right]^T \lambda(t) - \frac{\partial f(x, u, t)}{\partial \lambda} \quad (48)$$

$$\frac{\partial H}{\partial u} = 0 \text{ or } \left[\frac{\partial g(x, u, t)}{\partial u}\right]^T \lambda(t) + \frac{\partial f(x, u, t)}{\partial u} = 0 \quad (49)$$

$$x(t_0) = x_0, \quad \lambda(t_f) = 0 \quad (50)$$

Solving (49) for u and substituting u in terms of x, λ into equations (47) and (48) yields the reduced differential equations of the form

$$\dot{x} = \phi_1(x(t), \lambda(t), t) \quad (51)$$

$$\dot{\lambda} = \phi_2(x(t), \lambda(t), t) \quad (52)$$

Therefore, to solve the optimization problem, we need to solve equations (47) and (48) subject to the split boundary conditions given by equation (50). We recall that in the previous method, we determine the initial costate vector explicitly so that in any intermediate iteration, the nonlinear differential equation (51) and (52) were satisfied by the initial state and the final costate vector that will make the pair $(x(t), u(t))$ admissible at every iteration and by some iterative technique due to Newton-Raphson method. The intention is to improve upon $\lambda(t_0)$ determined previously and employ this in integrating the equations (47) and (48). We shall pretend that the final costate vector $\lambda(t_f)$ is unknown and that it is a function of $\lambda(t_0)$ and t , and use $\lambda(t_0)$ determined beforehand, namely $\lambda(t_0) = (0, \frac{1}{3})^T$ as initial guess to compute the tangent to the unknown function $\lambda(t_f)$ versus $\lambda(t_0)$ relationship and use as the next guess for $\lambda(t_0)$ the extrapolated point where the tangent intersects the $\lambda(t_f)$ -axis, that is, where $\lambda(t_f)$ is zero. The equation of the tangent for the scalar case, by taking two arbitrary points $(\lambda_0(t_0), \lambda_0(t_f))$ and $(\lambda(t_0), \lambda(t_f))$ is given by

$$\lambda(t_f) = \beta \lambda(t_0) + [\lambda_k(t_f) - \beta \lambda(t_0)] \quad (53)$$

So, for the scalar case the equation

$$\lambda_{k+1}(t_0) = \lambda_k(t_0) - \frac{\partial \lambda_k(t_f)}{\partial \lambda(t_f)}^{-1} \lambda_k(t_0) \quad (54)$$

holds. The equation for the multivariable case (n-state and n-costate) can be written as

$$\lambda_{k+1}(t_0) = \lambda_k(t_0) - [[P_\lambda(\lambda_k(t_0), t_f)]]^{-1} \lambda_k(t_f) \quad (55)$$

where the matrix

$$P_\lambda(\lambda_k(t_0), t) = \left(\frac{\partial \lambda_i(t)}{\partial \lambda_j(t_0)} \right)_{\lambda(t_0)}, i, j = 1, 2, \dots, n \quad (56)$$

where $\frac{\partial^2 h}{\partial^2 x}(x(t_f))$ is the matrix whose p^{th} element is

$$[\partial^2 h(x(t_f))]_{2p} = \frac{\partial^2 h(x(t_f))}{\partial x_p \partial x_p} \quad (57)$$

using $\lambda(t_0) = (\lambda_{10}, \lambda_{20}) = (0, \frac{1}{3})$ as initial guess is the iterative scheme defined by equation.

$$\lambda_{k+1}(t_0) = \lambda_k(t_0) - [[\frac{\delta^2 h}{\delta x^2}(x(t_f))]P_x(\lambda(t_f)) - P_\lambda \lambda(t_0), t_f]^{-1} \lambda(t_f) - \frac{\delta h}{\delta x}(x(t_f))_k \quad (58)$$

Where $[\cdot]_k$ implies that the enclosed terms are evaluated on the k^{th} trajectories and $P_x(\lambda_k(t_0), t_f)$ is then the $n \times n$ state influence function matrix defined by

$$P_\lambda(\lambda_k(t_0), t) = \left(\frac{\delta x_i(t)}{\delta \lambda_j(t_0)} \right), i, j = 1, 2, \dots, n \quad (59)$$

equation(60) is converted into a four dimensional problem and employ an integrator , convergence was obtained to:

$$\underline{\lambda}(t_0) = (4.05 \times 10^{-6}, 4.3 \times 10^{-9})^T$$

$$\underline{\lambda}(t_0) = (4.05 \times 10^{-6}, 4.3 \times 10^{-9})^T$$

in seven iterations. This is expected to be $(0, 0)^T$

5.3. Algorithm

The method solves the problem

$$\text{Minimize } J = h(x(t)) + \int_{t_0}^{t_f} g(x(t), u(t), t) dt$$

subject to the dynamical constraints

$$\dot{x}(t) = f(x(t), u(t), t)$$

The algorithm is as follows

Step 0: Form the Hamiltonian functional

Step 1: Form the reduced differential equations by solving the equation $\frac{\partial H}{\partial y} = 0$ for $u(t)$ in terms of $x(t), \lambda(t)$ and substituting in the state-costate equations.

Step 2: use as initial guess the value of the costate vector $\lambda_0(t_0)$ obtained before and set the iteration counter $t_0 = 0$.

Step 3: using $\lambda(t_0) = \lambda_k(t_0)$ and $x(t_0) = \lambda_0$ as initial conditions, integrate the reduced state-costate equations and the influence function equations. Store only the value $\lambda_k(t_f), x_k(t_f)$ and the $n \times n$ matrices $P_\lambda(\lambda_k(t_0), t_f), P_x(\lambda_k(t_0), t_f)$.

Step 4: Check to see if the termination criterion $\|\lambda_k(t_f) - \frac{\delta h}{\delta x}(x(t_f))\| < \gamma$ is satisfied. where γ is a small prechosen positive constant. If it is, use the final iterates $\lambda_0^*(t_0)$ to integrate the states and controls. Otherwise, find new values for $\lambda_0(t_0)$ i.e $\lambda_{k+1}(t_0)$. Set $k = k + 1$ and return to step 3.

Remark: Steps 1 and 2 are performed off-line by the user whilst step 3 and 4 are performed on a digital computer.

6. Applications

6.1. Global Control of Pure Inertia Plant

In this section, we shall proceed by considering applications that illustrate the theory already developed, namely, linear plant and the nonlinear plant. For the linear plant consider the system governed by the plant equation

$$\ddot{x} + \dot{x} = m \quad (60)$$

We shall take the system objective as transferring the initial state point of the plant to a prescribed terminal state in such a manner as to minimize the performance index,

$$IP = \int_{t_0}^{t_f} \frac{1}{2} [x^2 + \dot{x} + \mu^2 m^2] dt \quad (61)$$

It can be shown that this system is totally controllable [1], so that an arbitrary initial point in the state space may be chosen. Put in standard form, let $x_1 = x, x_2 = \dot{x}$ then equation (60) is equivalent to the set of first-order equations.

$$\begin{aligned} \dot{x}_1 &= x_2 = f \\ \dot{x}_2 &= -x_2 + M = f_2 \end{aligned} \quad (62)$$

That is if the quantity x is taken to be a system error, then x_1 and x_2 are the error and rate respectively. For the end conditions, we have $t = t_1 = 0, t = t_f$ (specified).

$$\begin{aligned} x_1(t_1) &= x_1, & x_1(t_f) &= 0 \\ x_2(t_1) &= x_2, & x_2(t_f) &= 0 \end{aligned} \quad (63)$$

We need to determine the control m required to drive the error and error rate to zero while minimizing the cost functional (performance index), IP of equation (61) which is obviously dependent on the amount of control m used. Let us define a new variable $x_3(t)$ by

$$\dot{x}_3 = \frac{1}{2} [x_1^2 + x_2^2 + \mu^2 m^2] = f_3 \quad (64)$$

$$(x_3(t_f) = \phi) \quad x_3(t_1) = 0$$

Equation (61) gives

$$IP = x_3(t_f) \quad (65)$$

The problem now becomes

$$\begin{aligned} &\text{Minimize } \int_{t_0}^{t_f} \frac{1}{2} [x_1^2 + x_2^2 + \mu^2 m^2] dt \\ &\text{subject to } \dot{x}_1 = x_2; \dot{x}_2 = -x_2 + m \end{aligned} \quad (66)$$

The Hamiltonian is written as

$$H = \frac{1}{2} [x_1^2 + x_2^2 + \mu^2 m^2] + \lambda_1 x_2 + \lambda_2 (m - x_2) \quad (67)$$

Then the conditions of optimality yield the two-point boundary-value problem:

$$\left. \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + \frac{\lambda_2}{\mu^2} \\ \dot{\lambda}_1 &= x_1 \\ \dot{\lambda}_2 &= -\lambda_1 + \lambda_2 + x_2 \end{aligned} \right\} \quad (68)$$

One method of solving (68) is provided in [8]. Alternatively, a solution to equations (68) may be found by first eliminating variables until a single equation in one of the variables is obtained. Differentiating the $\dot{x}_1$ equation three times, one finds on using the remaining relations that

$$\ddot{x} - (1 - \frac{1}{\mu^2})\ddot{x}_1 + \frac{1}{\mu^2}x_1 = 0 \quad (69)$$

The variables $x_2(t)$ can then be obtained by differentiating the solution for $x_1(t)$. With $x_2(t)$ known, the equation for $\dot{x}_2$ provides $\lambda_2(t)$. If $\mu^2 = 1$, by superposition, the $z_i, i = 1, 2, 3, 4$ are linear combinations of the functions e^t, e^{-t}, te^t and te^{-t} , where z_i is one of $x_1, x_2, \lambda_1, \lambda_2$ [10]. The case of simple linear plant is thus easy to handle. The reader is referred to [10] for some other methods. We shall therefore proceed by considering the nonlinear case. For a specific example, we shall consider the problem of the sliding mass system [10] described by the second-order differential equation:

$$m\ddot{x} + c\dot{x}^2 \text{sgn}\dot{x} = M \quad (70)$$

where

$$\text{sgn}x = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \quad (71)$$

m is the mass of the sliding solid, c is the coefficient of friction, M is the controlled force applied and, x is the displacement. This example was previously solved by [7] using a hierarchical technique and subsequently by [8] by quasilinearization technique. The early attention accorded this system is due to the simple nature of the plant. The current interest in the control of spacecraft outside the earth's atmosphere now gives new and practical

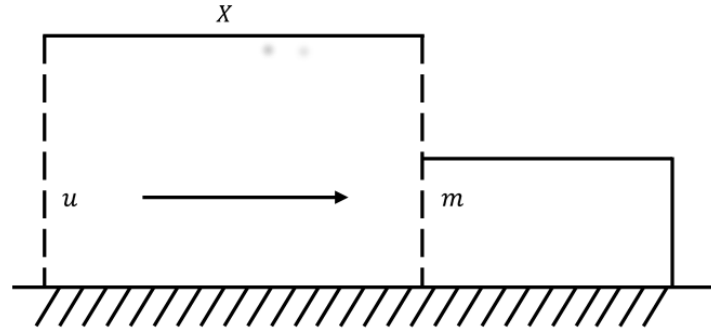


Figure 1. Sliding mass system

meaning to this type of problem. the technique adopted in this section shall show that although the Pontryagon's maximum principle does indeed quickly yield the form of the expected solution as often stated, the complete solution is somewhat more evasive. The system equations of (70) and (71) can be written in standard form by letting the $x_1 = x, x_2 = \dot{x}$. Then equation (70) is equivalent to the set of equations

$$\dot{x}_1 = x_2 = f_1 \quad (72)$$

$$m\dot{x}_2 + cx_2^2 \operatorname{sgn} x_2 = m \text{ or } x_2 = \frac{1}{m}(m - cx_2^2 \operatorname{sgn} x_2) = f_2 \quad (73)$$

The problem now reduces to that of minimizing the time required to transfer the system between initial and final states, where at $t = t_1 = 0$, $x_1(0) = x_{1_0}, x_2(0) = x_{2_0}$ and at $t = t_f$ (unspecified), $x_1(t_f) = 0, x_2(t_f) = 0$. The system cost functional then becomes.

$$IP = J(u) = \underset{m}{\text{minimize}} \int_0^1 \frac{1}{2} [x_1^2 + x_2^2 + M^2] dt$$

subject to:

$$\left. \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m} (m - cx_2^2 \operatorname{sgn} x_2), m > 0, c > 0 \\ \text{with } x_1(0) &= 2, x_2(0) = 2 \end{aligned} \right\} \quad (74)$$

The Hamiltonian can be written as:

$$H = \frac{1}{2} [x_1^2 + x_2^2 + m^2] + \lambda x_2 + \lambda_2 \left[\frac{1}{m} (m - cx_2^2 \operatorname{sgn} x_2) \right]$$

Then the conditions of optimality yield the two-point BVP

$$\left. \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= m - cx_2^2 \operatorname{sgn} x_2 \\ \dot{\lambda}_1 &= x_1 \\ \dot{\lambda}_2 &= x_2 - \lambda_1 + \frac{2c\lambda_2 x_2 \operatorname{sgn} x_2}{m} \end{aligned} \right\} \quad (75)$$

substituting $m = \frac{\lambda_2}{m}$ yields the boundary value problem (by taking $m=1, c=1$)

$$\begin{aligned}
 \dot{x}_1 &= x_2 \\
 \dot{x}_2 &= \lambda_2 - x_2^2 \operatorname{sgn} x_2 \\
 \dot{\lambda}_1 &= x_1 \\
 \dot{\lambda}_2 &= x_2 - \lambda_1 + \frac{2\lambda_2 x_2 \operatorname{sgn} x_2}{m} \\
 \text{since } m=1 &\Rightarrow m = \lambda_2, x_1(0) = 2, x_2(0) = -2, \lambda_1(1) = 0 \\
 \lambda_2(1) &= 0
 \end{aligned} \tag{76}$$

The fifth equation of (76) shows that, provided $\lambda_2(t) \neq 0$, the control takes on its maximum magnitude at all times, that is, the control is bang-bang. The available control will be limited by upper and lower bounds so that

$$|M| < U \tag{77}$$

$$\Omega(t, x, M) = (M - U)(M + U) \leq 0 \tag{78}$$

Either maximum positive control or maximum negative control is to be employed to maximize H in order to minimize $IP = J(M)$. Also, it can be expected that the control will switch from the upper value to the lower value at the zero crossings of the multiplier $\lambda_2(t)$ on which M depends. Hence, $\lambda_2(t)$ is called the *switching function* for the control. The presence of the quantity $\operatorname{sgn} x_2(t)$ destroys the linearity of equation (74). In terms of our notation, we can now take refuge under the imbedding procedure to solve (74), then the reduced Hamiltonian H^* is given by

$$H^* = \frac{1}{2}[x_1^2 + x_2^2 + \lambda_2^2] + \lambda_1 x_2 + \lambda_2[\lambda_2 - x_2^2 \operatorname{sgn} x_2] \tag{79}$$

using $x(t_0) = D, \lambda(t_0) = \psi(D, t_0)$ the invariant imbedding equation is given by

$$\frac{\partial \psi}{\partial t_0} + \frac{\partial \psi}{\partial D_1} \phi_1(D, \psi, t_0) = \phi_2(D, \psi, t_0) \tag{80}$$

where we need to determine $\lambda(t_0) = \psi(D, t_0)$ from equation (80). Equation (80) is a system of partial differential equations:

$$(1 + \lambda_2 + x_2^2 \operatorname{sgn} x_2) \frac{\partial \psi}{\partial t} + x_2 \frac{\partial \psi}{\partial D_1} = x_1 \tag{81}$$

$$(1 + \lambda_2 + x_2^2 \operatorname{sgn} x_2) \frac{\partial \psi_2}{\partial t} + x_2 \frac{\partial \psi_2}{\partial D_2} = x_1 - \lambda_1 + 2\lambda_2 x_2 \operatorname{sgn} x_2 \tag{82}$$

whence, we can solve for ψ_1 and ψ_2 and hence, determine $\psi(t_0)$. Now, from (81) by Charpit's method:

$$\frac{dt}{1 + \lambda_2 - x_2^2 \operatorname{sgn} x_2} = \frac{dD_1}{x_2} = \frac{d\psi_1}{x_1} \tag{83}$$

Integrating, we obtain

$$x_2 t - (1 + \lambda_2 - x_2^2 \operatorname{sgn} x_2) D = K_1 \tag{84}$$

$$x_1 D_1 + \psi_1 x_2 = k_2 \quad (85)$$

at $t = 0, x_1 = 2, x_3 = -2$

$$K_1 = -(1 + \lambda_2 - 4(\text{sgn}(-2)))D_1 = -(1 + \lambda_2 + 4) = -(\lambda_2 + 5) \quad (86)$$

$$2D_1 + 2\psi_1 = k_2$$

$$(2D_1 + 2\psi_1) = 2(\psi_1 + D_1) \quad (87)$$

That is:

$$x_1 t - (1 + \psi_2 - x_2^2 \text{sgn} x_2)D_1 + \psi_2 + 5 = 0 \quad (88)$$

$$x_1 D_1 - \psi_1(x_2) = 2(\psi_1 + D_1) \Rightarrow$$

$$(x_1 - 2)D_1 = 2\psi_1 + 2D_1 \Rightarrow \psi_1 = \frac{(x_1 - 2)D_1}{x_2 + 2} \quad (89)$$

$$\text{From (88)} D_1 = \frac{(x_2 t + \lambda_2 + 5)}{1 + \lambda_2 - x_2^2 \text{sgn} x_2} \quad (90)$$

substituting this in (88) yields

$$\psi_1 = \frac{(x_2 - 2)(x_2 t + \lambda_2 + 5)}{1 + \lambda_2 - x_2^2 \text{sgn} x_2} \quad (91)$$

Also, from (82)

$$\frac{dt}{1 + \lambda_2 - x_2^2 \text{sgn} x_2} = \frac{dD_2}{x_2} = \frac{d\psi_2}{x_1} \quad (92)$$

whence by integrating, we obtain

$$x_2 t - (1 + \lambda_2 - x_2^2 \text{sgn} x_2)D_2 = k_3 \quad (93)$$

$$x_2 \psi_2 - (x_2 + \lambda_2 - 2\lambda_2 x_2^2 \text{sgn} x_2)D_2 = k_4 \quad (94)$$

At $t = 0, x_1 = 2, x_2 = -2$

$$k_3 = -(1 + \lambda_2 - 4(-1))D_2 = -1(1 + \lambda_2 + 4) = -(\lambda_2 + 5)$$

$$k_4 = -2\psi_2 - (-2 - \lambda_1 - 4\lambda_2 \text{sgn}(-2))D_2$$

$$= -2\psi_2 - (-2 - \lambda_1 - 4\lambda_2)D_2 = (4\lambda_2 - \lambda_1 - 2)D_2 - 2\psi_2$$

From (94)

$$k_2 \psi_2 - (x_2 + \lambda_2 - 2\lambda_2 x_2^2 \text{sgn} x_2)D_2 + 2\psi_2 + (\lambda_1 + 2 - 4\lambda_2)D_2 = 0 \quad (95)$$

$$(x_2 + 2)\psi_2 = (x_2 + \lambda_2 - 2\lambda_2 x_2^2 \text{sgn} x_2)D_2 + (4\lambda_2 - \lambda_1 - 2)D_2 x_2 + 2$$

$$\psi_2 = \frac{(x_2 + 4\lambda_2 - 2\lambda_2 + 2\lambda_2 x_2 \text{sgn} x_2 - 2)D_2}{x_2 + 2}$$

From (93)

$$D_2 = \frac{x_2 t + (\lambda_2 + 5)}{1 + \lambda_2 - x_2^2 \text{sgn} x_2}$$

$$\psi_2 = \frac{(x_2 + 4\lambda_2 + 2\lambda_2 x_2 \operatorname{sgn} x_2 - 2\lambda_2 - 2)(x_2 t + (\lambda_2 + 5))}{1 + \lambda_2 - x_2^2 \operatorname{sgn} x_2} \quad (96)$$

Hence, we can write down $\lambda_2(t_0)$ as $\lambda(t_2) = \psi(x(t_0), t_0)$ as expected, Equation (96) implies that

$$\begin{aligned} \lambda_1(t_0) &= \frac{(x_1(t_0) - 2)(x_2(t_0)t_0 + \lambda_2(t_0) + 5) + 5}{1 + \lambda_2(t_0) - x_2^2(t_0) \operatorname{sgn} x_2(t_0)} \\ &= \frac{(x_1(0) - 2)(x_2(0) + \lambda_2(0) + 5)}{1 + \lambda_2(0) - x_2^2(0) \operatorname{sgn} x_2(0)}, \text{ at } t=0 \\ &= \frac{(2 - 2)(0.0 + \lambda_2(0) + 5)}{1 + \lambda_2(0) - (-2)^2 \operatorname{sgn}(-2)}, x_1(0) = 2, x_2(0) = -2 \end{aligned} \quad (97)$$

That is, for this problem, $\lambda_1(0) = 0$. Also,

$$\lambda_2(t_0) = \frac{x_2(t_0)t_0 + \lambda_2(t_0 + 5)(x_2(t_0)t_0 + 4\lambda_2(t_0 + 2\lambda_0(t_0)x_2(t_0) \operatorname{sgn} x_2(t_0)) - 2\lambda_1(t_0 - 2) \operatorname{sgn} x_2(0)))}{1 + \lambda_2(t_0) - x_2^2(t_0) \operatorname{sgn} x_2(t_0)} \text{ at } t_0 = 0 \quad (98)$$

$$\begin{aligned} &= \frac{(-2.0 + \lambda_2(0) + 5)(-2 \cdot 0) + 4\lambda_2(0)(-2) \operatorname{sgn}(-2) - 2\lambda_1(0) - 2}{1 + \lambda_2(0) - 4 \operatorname{sgn}(-2)} \\ &= 8\lambda_2(0) - 2\lambda_1(0) - 2 \end{aligned} \quad (99)$$

i.e.

$$\lambda_2(0) = 8\lambda_2(0) - 2\lambda_1(0) - 2 \quad (100)$$

$$-6\lambda_2(0) = -2\lambda_1(0) - 2 \Rightarrow 3\lambda_2(0) = \lambda_1(0) + 1 \quad (101)$$

$$\text{or } \lambda_2(0) = \frac{1}{3}(\lambda_1(0) + 1) \quad (102)$$

Hence, $\lambda_2(0) = \frac{1}{3}$ since $\lambda_1(0) = 0$.

Therefore, the initial condition vector is

$$\lambda(0) = \left(0, \frac{1}{3}\right)^T \quad (103)$$

We shall next improve upon this using the flowchart in 2

The results are presented in Tables 1 and 2

6.2. Control of Electric power generation

The electric power generating system can be described as follows: Considering the i^{th} generator with the rate of change in generation at time (t) depending on invested capital $C_i(t)$ and the generator output $G_i(t)$ in turn depending on the power input, generator capacity, running cost and transmission losses. Also, the investment on capital $C_i(t)$ at time t is known to be dependent on labour cost s_i , maintenance cost y_i , fuel cost rate r_i , actual capacity rate x_i , and the rate of cost of transmission to the grid centre γ_i . Because of the physical law that governs power generation and the operating limitation on the equipment, since the operation limitation on the equipment cannot be neglected because of its lifespan, we choose

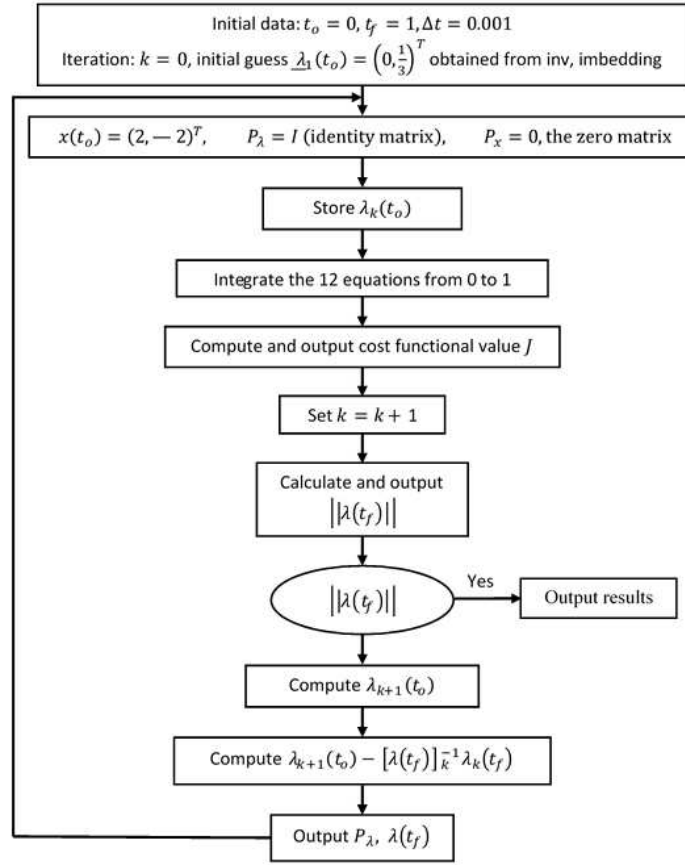


Figure 2. Flowchart

the generator capacity as our control $u_i(t)$ for which the upper bound is chosen to be 1 to represent the total capability of the generator.

In the above setting, an important objective is to minimize the total operating cost incurred in the process of generating the required quantity of electric power G at time t , and the components of the total operating cost are $C(t)$ and $u(t)$.

The electric power generating system described above has been modelled by [13], as an optimal control problem of the form: Find the control u that minimizes the cost function

$$J(u) = \int_{t_0}^{t_f} [\delta^T C(t) + \eta u^T u] dt \quad (104)$$

subject to:

$$\frac{d}{dt} G_i(t) = \alpha_i + q_i C_i(t) G_i(t) - k_i G_i(t) \quad (105)$$

$$\frac{d}{dt} C_i(t) = (s_i + y_i) + r_i C_i(t) G_i(t) + x_i u_i(t) C_i(t) + \gamma_i C_i(t) \quad (106)$$

and

$$G_i(t_0) = G_0, C_i(t_0) = C_0, \quad a_i \leq u \leq b_i, i = 1, 2, \dots, m \quad (107)$$

Table 1. The sliding mass problem: Component trajectories

T_k	$x(T)$	$Y_1(I)$	$Z(I)$	$W(I)$
.001	2.001998	-1.995676	0	0.333
.002	2.003991	-1.991369	1.999167×10^{-3}	0.3366961
.003	2.005980	-1.982803	4.000335×10^{-3}	0.3343371
.004	2.007965	1.987077	8.00865×10^{-2}	0.3350025
.005	2.009946	-1.978544	1.001579×10^{-2}	0.33698271
.006	2.011923	-1.974302	1.202491×10^{-2}	0.3369864
.007	2.0113895	-1.970076	1.403601×10^{-2}	0.3376437
.008	2.015863	-1.961671	1.604909×10^{-2}	0.3382991
.009	2.017827	-1.965866	1.804410×10^{-2}	0.3389525
.010	2.019786	-1.957493	2.008114×10^{-2}	0.3396040
.011	2.021742	-1.949783	2.400110×10^{-2}	0.3402537
.012	2.023693	-1.945051	2.614392×10^{-2}	0.3340901
.013	2.025640	-1.940934	2.816875×10^{-2}	0.3415474
.014	2.027583	-1.936833	3.019553×10^{-2}	0.3421915
.015	2.029522	-1.932742	3.222424×10^{-2}	0.3428339
.016	2.031466	-1.928676	3.425486×10^{-2}	0.3434745
.017	2.033387	-1.924621	3.628748×10^{-2}	0.3441134
.018	2.035314	-1.920580	3.832199×10^{-2}	0.3447506
.019	2.037236	-1.916554	4.035843×10^{-2}	0.3453861
.020	2.039155	-1.912543	4.239678×10^{-2}	0.3460199

Table 2. Minimization of Cost Functional

Iterative Step	Cost Functional ($J \times 10^{-11}$)
1	3.714764
2	3.713646
3	3.691509
4	3.668900
5	3.522146
6	3.264780
7	3.186122
8	3.118502
9	1569398
10	1.496917

where $G_i(t)$ represents the amount of power generated by the i^{th} generator at the time t , $C_i(t)$ the capital investment on the i^{th} generator at time t , s_i labour cost, y_i maintenance cost, x_i actual capacity rate, γ rate of cost of transmitting from generating station, r_i fuel cost

rate and $u(t)$ generator capacity while α_i , q_i and k_i are respectively the actual mechanical / electrical energy from the high pressure turbine and low pressure turbine (capacity of generator i), the corresponding running cost and the transmission loss rate, which depends on the distance from the grid centre, $\delta^T = (\delta_1, \delta_2, \delta_3, \dots, \delta_m)$ is the unit expenditure on the generators, η is a parameter to balance the size of the control.

The solution for the optimal control in (104)-(107) can be obtained using Pontryagin's Maximum principle as follows: The adjoint system can be written as

$$\lambda'_{1i} = -\frac{\partial H}{\partial G_i} = -\lambda_{1i}[q_i C_i(t) - k_i] - \lambda_{2i}[r_i C_i(t)] \quad (108)$$

$$\lambda'_{2i} = -\frac{\partial H}{\partial C_i} = -\delta_i^T - \lambda_{1i} q_i G_i(t) - \lambda_{2i}[\gamma_i - x_i u_i(t) + r_i G(t)] \quad (109)$$

The optimality equations are

$$\frac{\partial H}{\partial u_i} = 2\eta u_i - \lambda_{2i} x_i C_i(t) - M_i + N_i = 0 \text{ at } u_i^* \quad (110)$$

Hence, we obtain

$$u_i^*(t) = \frac{1}{2\eta} [\lambda_{2i} x_i C_i(t) + M_i - N_i] \quad (111)$$

There are three cases for $u_i^*(t)$ at any time t , using standard control theory involving the bounds on the controls.

Case 1

$$a_i = u_i^*(t) ; \text{ since } u_i^*(t) \neq b_i, \quad M_i = 0$$

and (111) becomes

$$\Rightarrow a_i = u_i^*(t) = \frac{\eta^{-1}}{2} [\lambda_{2i} x_i C_i(t) - N_i] \quad (112)$$

By (104), solving for N_i gives

$$\begin{aligned} 0 \leq N_i &= -2\eta a_i + \lambda_{2i} x_i C_i(t) \\ &\Rightarrow 2\eta a_i \leq \lambda_{2i} x_i C_i(t) \end{aligned}$$

$$\text{Therefore, } a_i \leq \frac{\eta^{-1}}{2} \lambda_{2i} x_i C_i^*(t)$$

Case 2

$$a_i < u_i^*(t) < b_i$$

By definition of the penalty multipliers, $M_i = N_i = 0$ and (108) gives

$$u_i^*(t) = \frac{\eta^{-1}}{2} \lambda_{2i} x_i C_i(t)$$

Case 3

$$u_i^*(t) = b_i$$

Since, $u_i^*(t) \neq a_i, N_i = 0$ and (108) gives

$$\begin{aligned} b_i &= u_i^*(t) = \frac{1}{2\eta} [\lambda_{2i} x_i C_i + M_i] \\ \Rightarrow b_i &\geq \frac{\eta^{-1}}{2} \lambda_{2i} x_i C_i(t) \end{aligned}$$

Hence, we conclude that

$$u_i^*(t) = \begin{cases} \frac{1}{2\eta} \lambda_{2i} x_i C_i(t) & \text{if } \frac{1}{2\eta} \lambda_{2i} x_i C_i(t) < b_i \\ a_i & \text{if } \frac{1}{2\eta} \lambda_{2i} x_i C_i(t) \geq a_i \\ b_i & \text{if } \frac{1}{2\eta} \lambda_{2i} x_i C_i(t) \leq b_i \end{cases} \quad (113)$$

In compact notation,

$$u_i^*(t) = \text{Min} \left\{ \text{Max} \left\{ a_i, \frac{1}{2\eta} \lambda_{2i} x_i C_i(t) \right\}, b_i \right\} \quad (114)$$

The optimal system consists of the state system coupled with adjoint system together with the initial and transversality conditions. utilizing (114), we have the following optimality systems which characterize the optimal control model;

$$\begin{aligned} \frac{dG_i(t)}{dt} &= \alpha_i + q_i C_i(t) - k_i G_i(t) \\ \frac{dC_i(t)}{dt} &= (s_i + y_i) - \text{Min} \left\{ \text{Max} \left\{ a_i, \frac{1}{2\eta} \lambda_{2i} x_i C_i^*(t) \right\}, b_i \right\} x_i C_i(t) + r_i C_i(t) G_i(t) + \gamma_i C_i(t) \end{aligned} \quad (115)$$

$$\lambda'_{1i} = -\lambda_{1i} [q_i C_i(t) - k_i] - \lambda_{2i} [r_i C_i(t)] \quad (116)$$

$$\lambda'_{2i} = -\delta^T - \lambda_{1i} q_i G_i(t) - \lambda_{2i} [\gamma_i \text{Min} \left\{ \text{Max} \left\{ a_i, \frac{1}{2\eta} \lambda_{2i} x_i C_i^*(t) \right\}, b_i \right\} x_i + r_i G_i(t)]$$

where

$$\lambda_{1i}(t_f) = \lambda_{2i}(t_f) = 0, \quad G_i(t_0) = G_0, \quad C_i(t_0) = C_0$$

Employing the Runge-Kutta method of order 4 together with real data obtained from the National Grid Centre, Osogbo, Nigeria as presented in Table 3, the numerical results obtained are shown in Table 4.

From the numerical results, we have that $u_1^* = 0.8$, $u_2^* = 0.7$, $u_3^* = 0.7$ with $J^* = 574.84430$ at $t_f = 1$

Table 3. Parameter for a three-generator station (National Grid Centre, Osogbo, Nigeria)

Parameter	Meaning	Value
a	actual mechanical / electrical energy available from the turbine	100MW, 100MW, 80MW G_1 100MW, G_2 100MW, G_3 80MW
q	total running cost	#0.3217, #0.3112, #0.312 per unit
r	fuel cost rate	0.3478 each per unit
x	actual capacity rate	0.606, 0.502, 0.402
k	rate of energy loss during transmission	0.002 MW each per unit
s	labour cost	70 per h (assumed)
y	maintenance cost	50 per h (assumed)
γ	Cost of transmitting from generating station	0.3421 per unit
u	generator actual capacity rate (control)	$a \leq u \leq b, 0.3 \leq u \leq 0.9,$ $a = 0.0, b = 1.0$
δ	unit of power generating station	1 each
η	Parameters to balance the size of the control. (Numbers of hours for which the machines can be on)	3

Table 4. Numerical Solutions for a three-generator electric power model

u	C_1	C_2	C_3	J at $t_f = 1$	J at $t_f = 6$
0.2	186.95889	186.63370	186.32130	560.27389	3361.64336
0.3	187.90813	187.56171	186.38327	562.66311	3375.97864
0.4	188.86003	188.20629	187.57888	566.08521	3396.51124
0.5	189.81461	189.33645	188.20943	569.63048	3417.78290
0.6	190.63340	189.78621	188.84115	572.50077	3435.00463
0.7	191.73181	190.57893	189.47405	576.19479	3457.16875
0.8	191.91594	191.37349	190.10813	579.15756	3474.94538
0.9	193.11363	192.16990	190.74339	583.23692	3499.42155
G_1	99.900				
G_2		99.850			
G_3			77.920		

6.3. Global Control of Spacecraft Orbital Manouvres: Cauchy Problems

In this section we derive a set first-order quasilinear partial differential equations with the associated initial conditions, starting from the Hamilton-Jacobi equation for generat-

ing functions representing a two-point boundary value problem (TPBVP), which forms the well-known Cauchy problem. These equations can also be derived by applying the invariant imbedding method to the TPBVP as demonstrated in [20]. The solution to this Cauchy problem is utilized for determining the optimal control logic of spacecraft orbital manoeuvres with hard and soft constraint boundary conditions. Since [21] initially introduced primer vector theory, optimal control problems for orbital manoeuvres using continuous thrust have been a topic of continual interest. While some early works still remain meaningful and noteworthy, many significant results have appeared in literature since 1980s.

Again, we consider minimization of the following performance index.

$$J = \Phi(x(t_f), t_f) + \int_t^{t_f} L(x(t), u(t), t) dt$$

Subject to the following system with terminal boundary conditions

$$\dot{x} = f(x, u, t); \Psi(x(t_f), t_f) = 0 \quad (117)$$

where

$$x \in \mathbb{R}^n, u \in \mathbb{R}^m, t \in \mathbb{R}$$

$\Phi(x(t_f), t_f) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}; L(x(t), u(t), t) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$,
 $f(x(t), u(t), t) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^n$ and $\Psi(x(t_f), t_f) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^{p < n}$. The control $u[u_1, u_2, \dots, u_m]^T$ is bounded by the following inequality by component $|u_i| \leq u_{i0} = \text{constant}$. The unconstrained problem can be dealt with by letting

$$u_{i0} \rightarrow \infty, i = 1, 2, \dots, m.$$

Given the problem statement above, we desire to find the optimal control logic in the feedback sense for a given domain in $(x, t) \in \mathbb{R}^n \times \mathbb{R}$. Then from any initial point, we can evaluate the optimal trajectory satisfying the terminal constraints by simple forward integration of the system (117), updating the control as new state measurements are made.

Instead of resorting to dynamic programming and solving the Hamilton-Jacobi-Bellman equation (HJBE), we formulate the given problem as a Hamiltonian system. First we define the pre-Hamiltonian $\tilde{H}$ as

$$\tilde{H}(x, \lambda, u, t) = L(x, u, t) + \lambda^T f(x, u, t) \quad (118)$$

Then, by Pontryagin's principle we write down the necessary conditions for optimality and define the Hamiltonian system for states and costates only as

$$H(x, \lambda, t) = \tilde{H}(x, \lambda, u^*(x, \lambda, t), t) \quad (119)$$

$$\dot{x} = H_\lambda(x, \lambda, t) \quad (120)$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x}(x, \lambda, t) \quad (121)$$

$$u^*(x, \lambda, t) = \arg \min_u \tilde{H}(x, \lambda, u, t) \quad (122)$$

As noted in the problem definition, the initial states are chosen explicitly on a given domain. For the terminal condition, suppose we have an explicit condition for

$$\Psi : \Psi(x(t_f), t_f) = x(t_f) - x_f = 0 \quad (123)$$

Where $x_f \in \mathbb{R}^n$ is a constant vector. Then the terminal states are completely specified, which form the hard constraint problem. Otherwise if $\Psi(x(t_f), t_f) = 0$ is given by an implicit equation or does not exist, then the following transversality condition determines the terminal boundary conditions [20].

$$\lambda(t_f) = \frac{\partial[\phi(x(t_f), t_f) + v^T \Psi(x(t_f), t_f)]}{\partial x(t_f)} \quad (124)$$

In the sense that the terminal states are not directly specified but indirectly affected by ϕ and Ψ , we recall this type of boundary condition, the soft constraint problem. In either case, we have $2n$ split boundary conditions equally divided between the initial and terminal time. Therefore, the optimal control problem is reduced to a two point boundary value problem (TPBVP). There are diverse numerical techniques for solving this TPBVP, which usually yield the open loop optimal trajectory.

6.4. Optimal Control of a Spacecraft Orbital Manouvres

As a main application we now study optimal control of spacecraft orbital manouvres. Specifically we deal with fuel-optimal manouvres of a spacecraft in a central gravity field:

$$\text{Minimize } J = \frac{1}{2} x^T(t_f) Q_f x(t_f) + T \int_{t_0}^{t_f} |u(t)| dt \quad (125)$$

$$\dot{x} = v \quad (126)$$

$$\dot{v} = Ax + Bv + u \quad (127)$$

where $x = [x_1, x_2]^T$, $v = [v_1, v_2]^T$, $u = [u_1, u_2]^T$, and t_f is assumed to be fixed. A and B are defined respectively as.

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

This system is called the Clohessy-Wiltshire equation and represents the linearized planar motion of a particle in an inverse-square gravity field with its coordinates expressed in a circular rotating coordinate frame. The reader should consult the work of [22] for more detailed description of this system.

Defining the pre-Hamiltonian as $\tilde{H} = |u| + p^T v + q^T (Ax + Bv + bu)$, we can obtain the costate equations:

$$\dot{p} = -A^T q \quad (128)$$

$$\dot{q} = -p - B^T q \quad (129)$$

and the optimal control logic by Pontryagin's principle:

$$u^* = \underset{u}{\operatorname{argmin}} H \begin{cases} 0, & |q| < 1 \\ -q/|q|, & |q| > 1 \\ \text{singular}, & |q| \equiv 1 \text{ on } [t_1, t_2] \end{cases} \quad (130)$$

According to [22], the singular arc occurs only when $q = [\pm 1, 0]^T$. For the sake of simplicity, we do not consider this special case. Then introducing the regular optimal control logic into the pre-Hamiltonian $\tilde{H}$ yields the Hamiltonian as a function of states (r, v) and costates (p, q) only.

6.5. Hard Constraint Problem

We consider first the rendezvous-type boundary conditions

$$x(t_0) = x_0, x(t_f) + x_f = x_f \quad (131)$$

$$v(t_0) = v_0, v(t_f) + x_f = v_f \quad (132)$$

Without loss of generality, we can set $Q_j = 0$. Then we can construct the Cauchy problem for this hard constraint problem

$$\frac{\partial \bar{x}}{\partial t} = A_1 \frac{\partial \bar{x}}{\partial \lambda_1} + A_2 \frac{\partial \bar{x}}{\partial \lambda_2} + A_3 \frac{\partial \bar{x}}{\partial \lambda_3} + A_4 \frac{\partial \bar{x}}{\partial \lambda_4} + C_0 \bar{x} + c_1, \bar{x}(t_f, \lambda) = [x(t_f, \lambda)^T v(t_f, \lambda)^T]^T \quad (133)$$

where

$$\tilde{x} = \begin{bmatrix} x \\ v \end{bmatrix}; \quad \tilde{\lambda} = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \tilde{c}_0 = \begin{bmatrix} 0 & I \\ A & B \end{bmatrix}, \quad \tilde{x} = \begin{bmatrix} o \\ u \end{bmatrix}$$

$$A_1 = 3\lambda_3, A_2 = 0, A_3 = \lambda_1 - \lambda_4, A_4 = \lambda_2 + \lambda_3$$

We now launch the 1st order upwind finite difference scheme to properly interpret the non-smoothness of control logic [19]. As a specific example, we locate the terminal boundary condition at the origin, and consider time, span $[0, 0.1]$.

Temporal grid and spatial grids have been chosen as $\Delta t = 0.01$ and $\Delta \lambda_i = 0.02, i = 1 \dots 4$ respectively. As this yields a numerical solution in the form of $\bar{x} = f(t, \bar{\lambda})$, it provides numerical charts with which to compute the initial costate for the desired initial state by the implicit relation

$$x(t_0, \lambda_0) = x_0 \quad (134)$$

Then the optimal strategy is obtained from the Pontryagin's principle.

$$u^*(x(t_0, \lambda(t, x), t)) = \arg \min_u \tilde{H}(x, \lambda(t, x), u, t) \quad (135)$$

It must be emphasized that we do not need to guess the initial costate in contrast to the direct numerical solution to the TPBVP for the Hamiltonian system. Also, we do not need to solve the Cauchy problem again as long as our desired initial states implicitly remains in the numerical solution chart. To test the validity of our method, we have chosen some arbitrary initial states around the origin, obtain the initial costates for them by (135), and solve the initial value problem for the Hamiltonian system. As Δt and $\Delta \lambda_i$ become smaller, the terminal states come closer to the origin. It appears as if our method captures non-smooth control profiles as well as smooth ones. One formidable challenge of this method is memory location for our variables and solution ingredient.

We have applied the Hamilton-Jacobi theory to present a technique for solving the optimal control problem in feedback sense. Formulating the optimal control problem as a

Hamiltonian system, we have derived a set of partial differential equations with their associated boundary conditions to form Cauchy problems, which, in turn, can be used to solve constraint problems. In the future, we shall focus more on computational issues; more effective numerical scheme for solving Cauchy problems, more efficient storage for vast Cauchy problem data in high dimensional problems, enhancement of the implicit/explicit interpolation process, etc. These issues will accompany researches on more realistic non-linear spacecraft orbital manouvres formulations.

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