

**FLOW BEHAVIOUR OF NON-DARCIAN MAGNETOHYDRODYNAMIC CASSON
NANOFLUID OVER A STRETCHING OR SHRINKING SHEET WITH SLIPS AND
CONVECTIVE HEATING**

BY

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CERTIFICATION

This is to certify that this thesis has been read and approved as meeting the requirements of the Department of Mathematics, University of Ilorin, Ilorin, Nigeria, for the Award of Doctor of Philosophy (Ph.D.) degree in Mathematics.

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DEDICATION

To Almighty Allah, my father and my mother whose vehicle conveyed me to this world, my brother, Abiola Adeosun, my sister, Omobolanle Adunni and my aunty, Alhaja Muinat.

DECLARATION

I, Adeshina Taofeeq ADEOSUN, a Ph.D. student in the Department of Mathematics, University of Ilorin, Ilorin hereby declare that this thesis entitled "Analysis of Non-Darcian Magnetohydrodynamics Casson Nanofluid Flow Past a Stretching or Shrinking Sheet with Slips and Convective Heating", submitted by me is based on my actual and original work. Any materials obtained from other sources or work done by any other persons or institutions have been duly acknowledged. In addition, the research has been approved by the University of Ilorin Ethical Review Committee.

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NOMENCLATURE

- $a_i, b_k, c_k, i = 0, 2, \dots, 16, k = 1, 2, \dots, 16$ = constants
 b^* = variable Forchheimer inertia coefficient
 $(b^*)_o$ = constant Forchheimer inertia coefficient
 b = thermal slip parameter
 Bi = convective heat transfer parameter
 $B(\bar{x})$ = magnetic field strength
 B_0 = constant magnetic field strength
 c = mass slip parameter
 C = nanoparticle volume fraction
 C_w = wall nanoparticle volume fraction
 C_∞ = ambient nanoparticle volume fraction
 C_p = specific heat at constant pressure, $\frac{kJ}{kg \cdot K}$
 $D_1(x)$ = local thermal slip factor, m
 $(D_1)_o$ = constant thermal slip factor, m
 Da = Darcy number
 D_B = Brownian diffusion coefficient, $\frac{m^2}{s}$
 D_T = thermophoretic diffusion coefficient, $\frac{m^2}{s}$
 E = activation energy parameter
 $E_1(x)$ = local mass slip factor, m
 $(E_1)_o$ = constant mass slip factor, m
 E_a = activation energy, $\frac{kJ}{mol}$
 $f(\eta)$ = dimensionless stream function
 Fs = Forchheimer parameter
 g = acceleration due to gravity, $\frac{m}{s^2}$
 $h_{f(x)}$ = local heat transfer coefficient, m
 $(hf)_o$ = constant heat transfer coefficient
 k = thermal conductivity, $\frac{W}{m \cdot K}$
 k_1 = Rosseland mean absorption coefficient, $\frac{1}{m}$
 k_p = variable permeability of the porous medium, m^2
 $(k_p)_o$ = constant permeability of the porous medium, m^2

K	=	constant that relates to fraction of free versus mixed convection
Kr	=	chemical reaction rate
L	=	characteristic length of the plate, m
Le	=	Lewis number
m	=	unitless exponent fitted rate constant
M	=	magnetic field parameter
N	=	radiation-conduction parameter
$N_1(x)$	=	local first order velocity slip factor, m
$(N_1)_o$	=	constant first order velocity slip factor, m
$(N_2)_o$	=	constant second order velocity slip factor, m
Nb	=	Brownian motion parameter
Nr	=	buoyancy ratio parameter
Nt	=	thermophoresis parameter
Nu_x	=	local Nusselt number
Pp	=	porosity parameter
Pr	=	Prandtl number
q_m	=	wall mass flux, $\frac{kg}{s \cdot m^2}$
q_r	=	radiative heat flux, $\frac{W}{m^2}$
q_w	=	wall heat flux, $\frac{W}{m^2}$
Ra	=	Rayleigh number
Ra_x	=	local Rayleigh number
Re	=	Reynold number
S	=	suction/injection parameter
Sh_x	=	local sherwood number
T	=	nanofluid temperature, K
T_∞	=	ambient fluid temperature, K
Tr	=	temperature ratio parameter
T_w	=	wall temperature, K
$\bar{u}$	=	velocity component along the $\bar{x}$ axis, $\frac{m}{s}$
u	=	dimensionless velocity component along the x axis
$\bar{u}_e$	=	external fluid velocity, $\frac{m}{s}$
u_e	=	dimensionless external fluid velocity
$\bar{u}_{slip}$	=	slip velocity, $\frac{m}{s}$
$\bar{u}_w$	=	velocity of the plate, $\frac{m}{s}$

U_r = reference velocity

$\bar{v}$ = velocity component along the $\bar{y}$ axis, $\frac{m}{s}$

v = dimensionless velocity component along the y axis

v_w = mass transfer velocity, $\frac{m}{s}$

$\bar{x}$ = coordinate along the plate, m

x = dimensionless coordinate along the plate

$\bar{y}$ = coordinate normal to the plate, m

y = dimensionless coordinate normal to the plate

Greek Symbols

α = thermal diffusivity, $\frac{m^2}{s}$

β_0 = volumetric thermal expansion coefficient of the base fluid, $\frac{1}{K}$

κ = Boltzmann constant, $\frac{eV}{k}$

η = similarity independent variable

$\theta(\eta)$ = dimensionless temperature

$\phi(\eta)$ = dimensionless nanoparticle volume fraction

λ = stretching or shrinking parameter

μ_f = viscosity of the base fluid, $\frac{N \cdot s}{m^2}$

ν_f = kinematic viscosity of the base fluid, $\frac{m^2}{s}$

ρ_f = fluid density, $\frac{kg}{m^3}$

ρ_p = nanoparticle mass density, $\frac{kg}{m^3}$

$(\rho C_p)_f$ = volumetric heat capacity of the base fluid, $\frac{J}{m^3 \cdot K}$

$(\rho C_p)_p$ = volumetric heat capacity of the nanoparticle materials, $\frac{J}{m^3 \cdot K}$

σ_1 = Stefan-Boltzman constant, $\frac{W}{m^2 \cdot K^4}$

τ = ratio of effective heat capacity of the nanoparticle material to the fluid heat capacity

σ = variable electric conductivity

σ_0 = constant electric conductivity

$\dot{o}$ = variable viscosity parameter

γ = variable thermal conductivity parameter

δ_1 = first order velocity slip parameter

δ_2 = second order velocity slip parameter

σ^* = chemical reaction rate parameter

ABSTRACT

Investigations of electrically conducting Magnetohydrodynamic (MHD) Casson nanofluid flow are important areas of research due to their applications in industries and engineering for industrial coolant, brake fluid, MHD generator, nanodrug delivery, and others. Several studies concerning heat and mass transfer of MHD nanofluids have been reported in literature. However, most of the existing related works do not consider the combined effects of nonlinear radiation and non-Darcian porous medium. If the fluid velocity is high and the temperature differences within the fluid are sufficiently large, both Darcy and linearized radiative heat flux models may not accurately determine the fluid flow and thermal structure. Therefore, this study was aimed at analysing the effect of nonlinear radiation, binary chemical reaction, Arrhenius activation energy, variable viscosity and thermal conductivity on convective non-Darcian MHD dissipative Casson nanofluid flow past a stretching or shrinking sheet with slip and convective heating. The objectives of the study were to: (i) determine the flow behaviour of a non-Darcian MHD dissipative Casson nanofluid over a stretching or shrinking sheet with multiple slip boundary conditions; (ii) examine the influence of variable viscosity and variable thermal conductivity on natural convective flow of non-Darcian MHD Casson nanofluid flow with velocity slip and convective heating; and (iii) analyse the effect of binary chemical reaction and Arrhenius activation energy on forced convective flow of non-Darcian MHD Casson nanofluid in the presence of non-Navier velocity slip condition.

The mathematical equations governing the flow, heat and mass transfer of Casson nanofluid in a non-Darcian porous medium are:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0,$$

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = \frac{\mu_B}{\rho_f} \left(1 + \frac{1}{\beta} \right) \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\sigma B_0^2(\bar{x})}{\rho_f} \bar{u} - \frac{\mu_B}{\rho_f k_p} \left(1 + \frac{1}{\beta} \right) \bar{u} - \frac{b^*}{k_p} \bar{u}^2,$$

$$\bar{u} \frac{\partial T}{\partial x} + \bar{v} \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \tau \left[D_B \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} + \frac{D_T}{T_\infty} \left(\frac{\partial T}{\partial y} \right)^2 \right] + \frac{\mu_B}{(\rho C_p)_f} \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial \bar{u}}{\partial y} \right)^2 \text{ and}$$

$$\bar{u} \frac{\partial C}{\partial x} + \bar{v} \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty} \frac{\partial^2 T}{\partial y^2}.$$

Where $(\bar{u}, \bar{v})$, μ_B , ρ_f , β , σ , B_0 , k_p , b^* , T , α , τ , D_B , C , C_p , D_T are respectively velocity components in $\bar{x}$ and $\bar{y}$ directions, dynamic viscosity, fluid density, Casson parameter, electric conductivity, magnetic field, porous medium permeability, Forchheimer inertial coefficient, fluid temperature, fluid thermal diffusivity, heat capacity ratio, Brownian diffusion coefficient, nanoparticle concentration, heat capacity and thermophoretic diffusion coefficient. The above governing equations were reduced to nonlinear ordinary differential equations using similarity

transformation and then solved by employing weighted residual method. The computational procedure was implemented by writing codes in MATHEMATICA Software.

The findings of the study were that: [label=()]

1. an increase in non-Darcian porous medium parameter resulted into a reduction in velocity, while it enhanced both temperature and nanoparticle volume fraction;
2. a hike in temperature dependent viscosity and thermal conductivity parameters contributed to a decrease in both temperature and nanoparticle volume fraction profiles and a rise in velocity profile;
3. nanoparticle volume fraction profile was reduced as a result of an increase in chemical reaction parameter, while increasing activation energy parameter led to an increase in nanoparticle volume fraction profile; and
4. an increase in radiation parameter resulted into a decrease in temperature profile.

The study concluded that the thermal radiation, Forchheimer, variable viscosity, variable thermal conductivity, chemical reaction and activation energy parameters have significant effects on the velocity, temperature, nanoparticle volume fraction, Nusselt number and the Sherwood number of Casson nanofluid. It is therefore recommended that those parameters are necessary and should be considered when carrying out a design involving Casson nanofluid flow.

1 GENERAL INTRODUCTION

In this chapter, the background of the study is given. The relevant terms used in this thesis are defined while types of non-Newtonian fluids are discussed. The synthesis of nanofluids and their applications are presented. The basic equations of flow, heat and mass transfer are considered. The statement of the problem, aim, objectives of the study, scope of the research and organization of the thesis are also stated.

1 Background to the Study

Investigation of heat and mass transfer boundary layer flow of non-Newtonian fluid is of importance due to its wide use in engineering and industries such as petroleum industry, aerodynamic heating, paper production, crude oil extraction, electrostatic precipitation and so on. There are some materials such as drilling muds, blood, paint, polymer solutions and shampoo that exhibit the characteristics of non-Newtonian fluid. No single model in literature describes all the properties of non-Newtonian fluid due to the complexity of its physical nature. Non-Newtonian fluids are classified into three broad groups. These are time-dependent, time-independent and viscoelastic fluid. Casson fluid is one of the examples of non-Newtonian (time-independent) fluids. It constitutes a fluid model that exhibits shear thinning characteristics, yield stress and high shear viscosity (Animasaun, 2015b). It is a non-Newtonian fluid that behaves like an elastic solid at low shear strain while at above critical yield stress, it behaves like a Newtonian fluid. Casson fluid is of different types. Some are in plasma form (jelly and blood), while some are like thick viscous form (e.g. tomato, concentrated fruit juice, shampoo and so on) (Najeeb and Zehra, 2015).

The thermal conductivity of fluids (Newtonian or non-Newtonian fluids) has been of concern to scientists and engineers due to their low thermal conductivities when compared to that of metals. Recently, engineers have developed novel technologies based on nanofluids to improve the thermal performance of the working fluids. Nanofluids are made up of base fluids (Newtonian or non-Newtonian) and metallic/nonmetallic nanoparticles to enhance fluid thermal conductivity (Choi, 1995). The examples of nanoparticle materials that are widely used are made from metal (Al, Cu, Fe), non metals (graphite, carbon nanotube), oxide ceramics (Al_2O_3, CuO), Carbides (SiC), nitrides (AlN, SiN).

The physical properties of fluid may change significantly when exposed to an increase in temperature. Examples of these are in fluid lubrication and heat generated by the internal friction, which results in the variation of the viscosity of the fluid. The transport phenomena are also increased by increasing temperature variation thereby reducing the viscosity across the momentum boundary layer and also have a significant effect on the rate of heat transfer at the wall (Animasaun, 2015b).

Non-Newtonian Fluids

There are some fluids such as air and water that exhibit the properties of Newtonian fluids. These fluids describe a linear relationship between shear stress and strain rate. They are modeled by Navier-Stokes equations. On the other hand, there are other categories of fluids in which the relationship between the shear stress and strain rate is non-linear. These types of fluids are classified as non-Newtonian fluids. Common examples of these fluids are toothpaste, tomato paste, liquid soaps, and blood. Non-Newtonian fluids are classified into three broad groups: time-independent, time-dependent and viscoelastic.

Time-independent Non-Newtonian Fluids

Time-independent non-Newtonian fluids are fluids in which the value of shear rate at a

point in the fluid is a function of the corresponding value of shear stress at that point only (Animasaun, 2015a). These fluids are subdivided into three groups.

Shear-thinning or Pseudoplastic Fluids

These fluids are characterized by fluid viscosity which gradually decreases with increasing shear rate. A familiar example of shear-thinning fluid is wall paint. The paint flows readily off the brush when it is being applied to a surface but not drip excessively. Other examples of fluids that behave like shear-thinning fluid are nail polish, whipped cream, ketchup and syrups. (Rao, 2007 and Schramm and Laurier, 2005).

Shear-thickening or Dilatant Fluids

The viscosity of a shear thickening or dilatant fluids appears to increase when the shear rate increases (Garay and Paul, 1996). Corn starch, when dissolved in water, is a common example, when it is stirred slowly it looks milky, and when stirred vigorously it feels like a very viscous liquid.

Visco-plastic Fluids

These fluids are characterized by the presence of yield stress which must be overcome before they can begin to deform. They behave like an elastic solid when the external stress is less than the yield stress. On the other hand, they behave like a shear-thinning fluid when the applied stress is greater than the yield stress. The common fluid models for visco-plastic fluid models are Bingham and Casson model (Byron et al., 1983). The examples of fluids that exhibit visco-plastic properties are blood, yogurt, tomato sauce, cosmetic and nail polishes (Chhabra, 2010).

Time dependent non-Newtonian fluids

In time-dependent non-Newtonian fluids, the apparent viscosity does not only depend on shear rate, but also the time for which the fluids are subjected to shear stress. The materials that belong to these categories include crude-oil, red mud suspension (waste stream from aluminum industry), bentonite-water suspension, certain foodstuff, gypsum paste, coal-water slurries and protein solutions (Chhabra, 2010). Time-dependent non-Newtonian fluids are further sub-divided into two categories: thixotropy and rheopexy.

Thixotropy Fluids

A fluid is said to exhibit thixotropy properties if when it is subjected to a constant shear rate, its apparent viscosity (or corresponding shear stress) decreases with the time of shearing (Nguyen and Uhlherr, 1983). Examples of materials that exhibit thixotropy behaviour include Yogurt, peanut butter, xanthan gum solutions, aqueous iron oxide gels, gelatin gels, pectin gels, hydrogenated castor oil and some drilling muds (Tropea et al., 2007).

Rheopexy Fluids

Rheopexy fluids are the opposite of thixotropy fluids. When these are subjected to a constant shear rate, their apparent viscosity (or corresponding shear stress) increase with time of shearing. This behaviour can be found in gypsum paste, coal-water slurries, protein solution, Synovial fluid and printer ink.

Viscoelastic Fluids

Viscoelastic fluids are fluids that exhibit elastic solid properties under some circumstances and/or the response of a viscous liquid under another circumstance. The fluids that possess these characteristics are macromolecular (that is, they have high molecular weight). Examples of these fluids are polymeric fluids (melts and solutions) used in making plastic articles, some foods such as dough used in making bread and pasta and some biological fluids such as synovial fluids found in joint (Baird, 2011). Viscoelastic fluids behave like Newtonian fluid under a sudden change in rates of deformation. Some models that are found in literature exhibiting viscoelastic properties are Maxwell fluid model, Jeffreys fluid model, Oldroyd fluid model and so on (Ildebrando et al., 2018).

Magnetohydrodynamics (MHD)

Magnetohydrodynamics (MHD) is the study of the flow of electrically conducting fluids in electromagnetic fields. Examples of such fluids include plasma, liquid metals (such as mercury, gallium, molten iron) and saltwater. The word "magnetohydrodynamics" is derived from a magneto-meaning magnetic field, hydro-meaning water, and dynamics-meaning movement. Another name for MHD that is less commonly used is hydromagnetics. The realm of MHD was first discovered by Swedish physicist, Hannes Alfvén (1908-1995), and he was awarded the Nobel prize in physics in 1970 for tremendous work and discovery in magnetohydrodynamics. The basic concept behind MHD is that the magnetic field induces an electric current in the moving conductive fluid, which in turn creates a force known as Lorentz force on the liquid and as well changes the magnetic field itself. MHD equations are described by combining Navier-Stokes equations of the fluid dynamics and Maxwell's equations of electro-magnetism. Both equations are to be solved simultaneously by employing an analytical or numerical method.

Viscosity

Viscosity is a measure of fluid resistance to deformation when it is being subjected to shear stress. However, the rate of deformation for different fluids varies. For example, water flows freely and easily when compared to honey. It has been shown experimentally that temperature has a major impact when considering fluid viscosity. For instance, shea butter is thick in the cold environment and becomes thin in the hot environment.

Porous Medium

A porous medium can be defined as a solid structure which comprises of void spaces or pores that are randomly distributed. The skeletal part of the structure is frequently referred to as a matrix or frame. If the voids are very small, they are known as molecular interstices, and the very large voids are called caverns. The intermediate between caverns and molecular interstices are what are known as pores. Fluid flow can only take place in the interconnected pore space of the porous media. Application of porous media can be found in the petroleum industry where petroleum reservoirs serve as porous media and storage. The flow of hydrocarbon takes place in the pore spaces which serve as a conduit to the flow of oil, gas, and water.

Heat Transfer

Heat transfer occurs whenever there is a temperature difference. The heat flow from the region of high temperature to the region of low temperature. It is the exchanging of thermal energy and heat between physical systems which occur as a result of temperature gradients. Heat transfer is commonly encountered in our day to day activities. There are three modes of heat transfer: conduction, convection, and radiation.

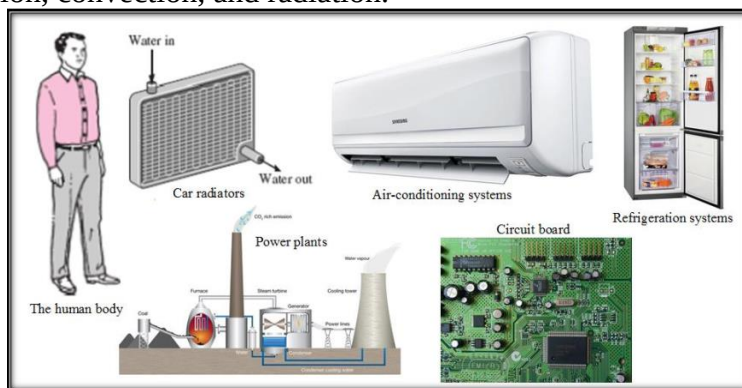


Figure 1: Energy transfer in daily life (Mutuku, 2014)

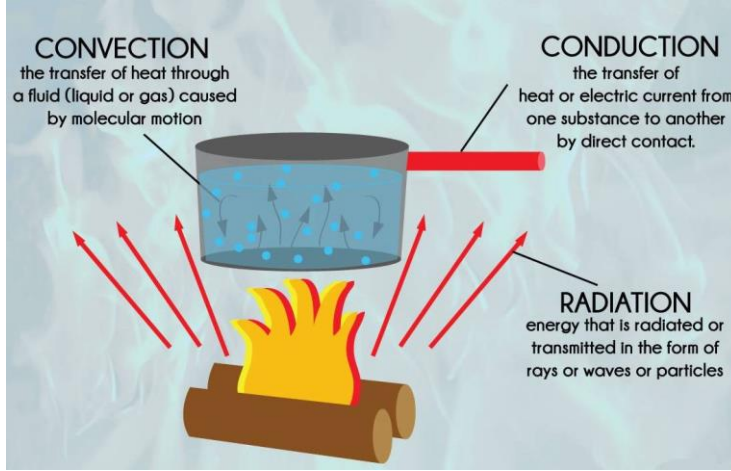


Figure 2: Modes of heat transfer

Conduction

Conduction is the transfer of energy from more energetic to less energetic particles of a substance due to the interaction between the particles. This energy transfer is related to the random translational motion, internal rotation and vibrational motions of the molecules. The rate at which heat is transferred by conduction is based on Fourier's law of heat conduction. For one dimensional plane in the x-direction, the rate equation is expressed as

$$q_x'' = -k \frac{dT}{dx} \quad (1)$$

The heat flux q_x'' ($\frac{W}{m^2}$) is the heat transfer rate in the x direction per unit area perpendicular to the direction of heat transfer, and it is proportional to the temperature gradient, $\frac{dT}{dx}$, in this direction. The parameter k is a transport property known as thermal conductivity ($\frac{W}{m.K}$) (Incropera *et al.*, 2007).

Convection

Convection: is the transfer of energy from one region to another due to the random molecular and bulk motion of a fluid. The contribution due to random molecular movement dominates near the surface where the fluid velocity is low, while the contribution due to bulk fluid movement originates as the fluid flow progress which in turn leads to an increase in boundary layer thickness. Convection heat transfer occurs between a surface and a moving fluid when they are at different temperatures. The appropriate rate equation is of the form

$$q'' = h(T_s - T_\infty) \quad (2)$$

where q'' , the convective heat transfer flux ($\frac{W}{m^2}$), is proportional to the difference between the surface and the fluid temperature, T_s and T_∞ respectively. This expression is known as Newton's law of cooling and the parameter h ($\frac{W}{m^2.K}$) is termed the convection heat transfer coefficient. It

depends on the conditions in the boundary layer, which are influenced by surface geometry, the nature of the fluid motion, and an assortment of the fluid thermodynamic and transport properties (Incropera *et al.*, 2007).

Convection heat transfer can be classified into two: forced convection and free or natural convection. When a flow is caused by external means, such as by a fan, a pump, or atmospheric wind, this type of flow is known as forced convection (see Figure 1.3a). For example, the use of a fan to provide air for cooling of hot electrical components on a stack of printed circuit boards. On the other hand, free or natural convection is caused by buoyancy forces which are due to density differences as a result of temperature gradients in the fluid (see Figure 1.3b). An example of natural convection is the exposure of hot components on the vertical array of a circuit board to the surrounding air. Air that makes contact with hot components experiences an increase in temperature and it becomes lighter in density. Since it is lighter than the surrounding air, the warm air ascends from the board and replaced by an inflow of cooler surrounding air.

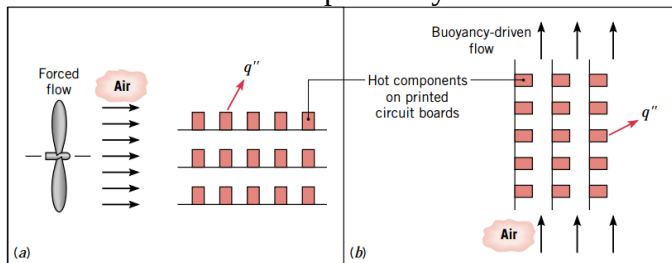


Figure 3: (a) Forced convection and (b) natural convection (Incropera *et al.*, 2007)

Radiation

Radiation is the emission of energy by an object in the form of an electromagnetic wave due to vibrational and rotational movement of the molecules and atoms which make up the object.

The rate at which energy is released per unit area ($\frac{W}{m^2}$) is known as surface emissive power E which is prescribed by

$$E_b = \sigma T_s^4 \quad (3)$$

where T_s is the absolute temperature (K) of the surface and σ is the Stefan-Boltzmann constant ($\sigma = 5.67 \times 10^{-8} \frac{W.K^4}{m^2}$). Such a surface is called an ideal radiator or black body. The heat emitted by a real surface is less than that of a blackbody at the same temperature and is given by

$$E = \varepsilon \sigma T_s^4 \quad (4)$$

where ε is the radiative property of a surface known as emissivity with value in the range of $0 \leq \varepsilon \leq 1$ (Incropera *et al.*, 2007)

Mass Transfer

Just as temperature gradient establishes the driving potential for heat transfer, species concentration gradient in a mixture constitutes the driving potential for a mass transfer (see Figure 1.4). Mass transfer encompasses both mass diffusion (due to molecular vibration of the species concentration) and the bulk mass transport of the species. Mass transfer occurs in solid, liquid and gas. Mass transfer is strongly affected by molecular spacing, as a result of that, mass transfer occurs more readily in gas than liquid and occurs more readily in liquid than solid. Examples of mass transfer in solids, liquids, and gases respectively can be found in nitrous oxide from an automobile exhaust in the air, dissolved oxygen in water and helium in Pyrex.

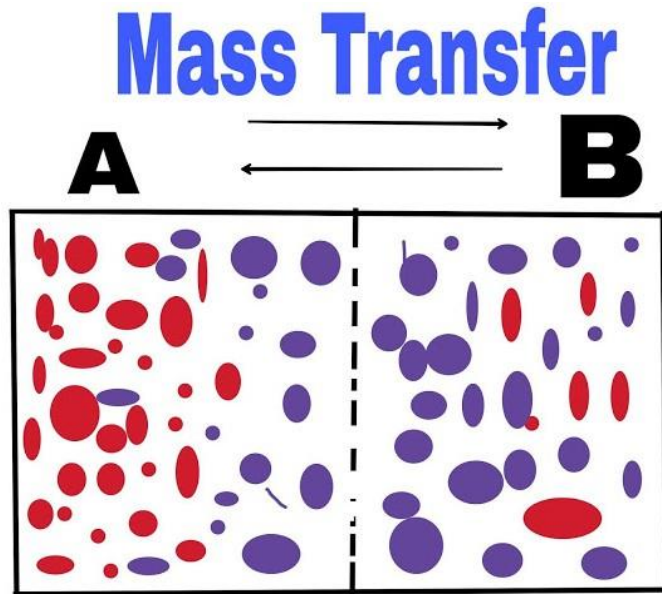


Figure 4: Transfer of mass species from higher concentration to lower one

Boundary Layer

The boundary layer concept in fluid flow over a surface is attributed to Ludwig Prandtl (1874–1953). His research work on the boundary layer flow formed the foundation for future works on skin friction, heat and mass transfer rates. The concept of boundary layer is centered on the understanding of convection heat and mass transfer between the surface and a fluid flowing past it. There are three types of boundary layers: velocity, heat, and mass boundary layers.

Velocity Boundary Layer

When a fluid viscosity is sheared, the fluid velocity immediately adjacent to the wall surface is zero and it becomes attached to the surface (i.e. it wets the surface). This is known as a no-slip condition. The fluid layers above the surface are moving, therefore shear stress occurs between the fluids' layer and the wall. The shear stress between the wall and the first moving fluid layers is referred to as wall shear stress (τ_w). The thin layer of the fluid adjacent to the wall surface in which the fluid motion is significantly retarded by the wall surface due to the viscous effect is called the velocity boundary layer. The boundary layer thickness (δ) is the distance from the boundary to the height above the surface at which velocity becomes 0.99 of the free stream velocity (u_∞) (see Figure 1.5). The boundary layer thickness is a function of Reynolds number (the ratio of inertial force to viscous force). At low Reynolds number, the viscous force dominates the boundary layer and the flow is laminar, while the fluid flow is turbulent for high Reynolds number where the inertial force occupies the boundary layer.

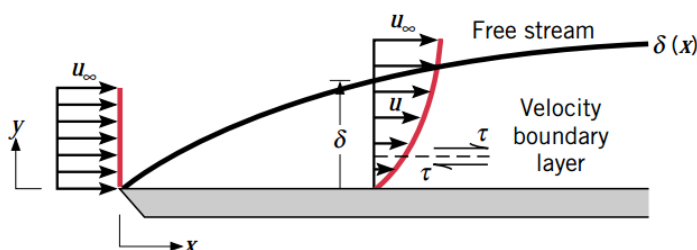


Figure 5: Velocity boundary layer development on a flat plate (Incropera *et al.*, 2007)

Thermal Boundary Layer

The thermal boundary layer develops if the fluid free stream and surface temperatures differ. For flow over a surface with constant temperature (see Figure 1.6), at the leading edge, the temperature profile is uniform. The fluid particles that are in contact with the plate achieve thermal equilibrium at the plate surface and in turn exchange energy with those particles in the adjacent fluid layers. The fluid region where temperature gradient exists is called the thermal boundary layer. The effect of heat transfer penetrates further into the free stream as the distance from the leading-edge increases, and the thermal boundary layer grows. Thermal boundary layer thickness (δ_t) is the value of y for which the ratio $[(T_s - T) / T_s - T_\infty] = 0.99$.

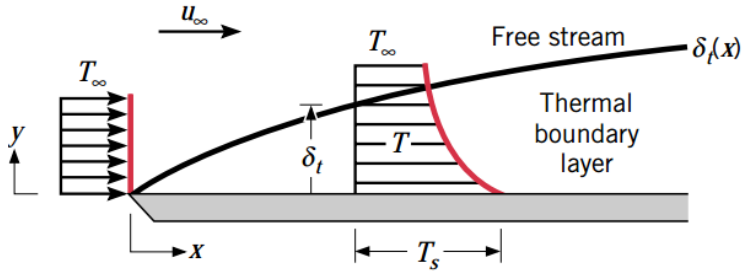


Figure 6: Thermal boundary layer development on a flat plate (Incropera *et al.*, 2007)

Concentration Boundary Layer

The concentration boundary layer behaves the same way as the thermal boundary layer (see Figure 1.7). It is the mass in transit as a result of species concentration difference in a mixture. In other words, it is the region of the fluid in which the concentration gradient exists. The species concentration penetrates further into the free stream as the distance from the leading-edge increases. As a result of that, the concentration boundary layer grows. The concentration boundary layer thickness (δ_c) is defined as the value of y at which the ratio $[(C_{A,s} - C_A) / C_{A,s} - C_{A,\infty}] = 0.99$.

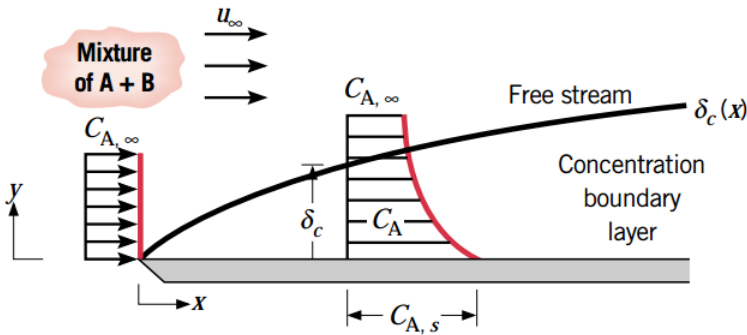


Figure 7: Concentration boundary layer development on a flat plate (Incropera *et al.*, 2007)

Nanofluids

Heat carrier fluids such as water, mineral oil, ethylene glycol and so on are very useful in industrial sectors which include power generation, chemical production, microelectronics and air conditions to mention a few. The performance of these fluids is limited due to their low thermal

conductivity which obstructs their applications in heat exchangers. In these modern days, there is a high demand for thermal conductivity fluid by the engineers and scientists for intensification and miniaturization of various electronic devices. For this to be achieved, it is suggested to disperse nanoparticle (with higher thermal conductivity (see Figure 1.9)) to a base fluid to have what is known as nanofluid (see Figure 1.8). The commonly used nanoparticle materials are made from metals (Al, Cu, Ag, Au, Fe), non-metals (graphite, carbon nanotube), oxide ceramics ($Al_2O_3, CuO, TiO_2, SiO_2$) and carbides (SiC), nitrides (AlN, SiN).

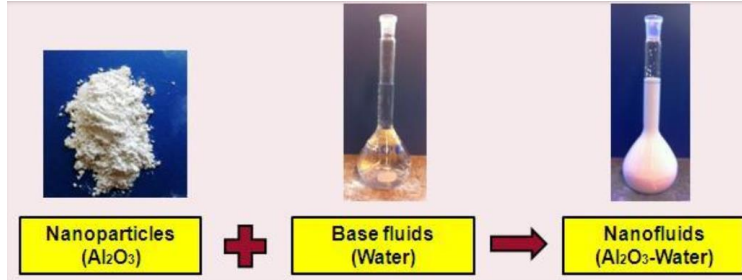


Figure 8: Al_2O_3 -water nanofluid

The commonly used base fluids are mostly a conductive fluid such as water, oil, polymer solution and biofluids. The base fluid model to be considered in this work is a Casson fluid model. Casson fluid model has its origin in modeling blood flow. As a result of that this model is widely used in biomedical applications. Casson fluid model can also find a good approximation for many other materials (Barnes et al., 1989 and Bird et al., 1983). Nanofluids have enhanced thermophysical properties such as thermal conductivity, thermal diffusivity, viscosity, and convective heat transfer compare to base fluids (Choi, 2009, France et al., 2008, Tyler et al., 2006, Das et al., 2006, Liu et al., 2005 and Choi et al., 2004).

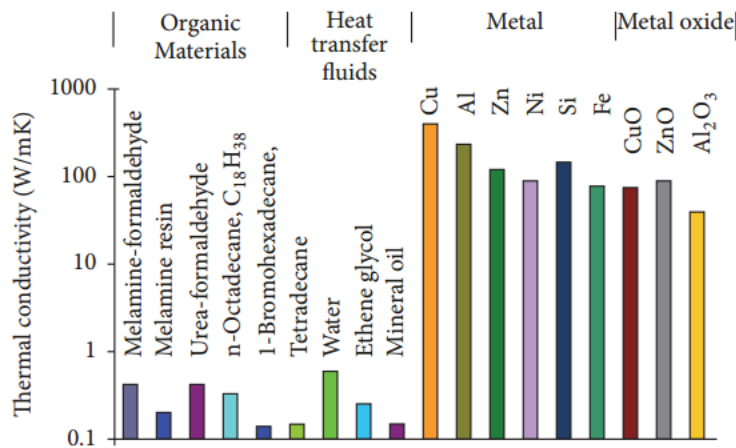


Figure 9: Thermal conductivity comparison of common polymers, liquids and solids (Naser et al., 2018)

Preparation Method of Nanofluid

Nanofluid can be synthesized using two ways: single-step method and two-step method.

Two-Step Method

The two-step method is the most widely used for preparing nanofluids. Nanoparticles, nanofibers, nanotubes, or other nanomaterials used in this method are first produced as dry powders by physical or chemical process in the first step, and then dispersed into a fluid with the help of intensive magnetic force agitation, ultrasonic agitation, high-shear mixing, homogenizing

and ball milling in the second step (see Figure 1.10). This method (two-step method) is the most economical to produce nanofluid on a large scale because nanopowder production techniques have already been scaled up to industrial production levels. In this method, nanoparticles tend to aggregate due to high surface area and surface activity thereby constituting a problem. To enhance the stability of nanoparticles in the fluid, surfactants are added to the fluids. However, the functionality of the surfactant under high temperatures brings a big concern. Due to the difficulty encountered in synthesizing stable nanofluids by the two-step method, another technique was developed (single-step method).

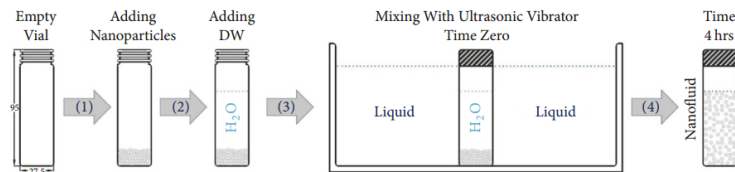


Figure 10: Schematic procedure of two-step nanofluids preparation (Naser *et al.*, 2018)

Single-Step Method

Nanofluid one-step physical vapor condensation method was developed by Eastman *et al.* (2001) to prepare Cu/ethylene glycol nanofluids to reduce the agglomeration of nanoparticles. This process (single-step method) consists of simultaneously making the particles and dispersing the particles in the fluid. Several processes such as drying, storage, transportation, and dispersion of nanoparticles are avoided thereby minimizing the number of steps involved in the agglomeration of the nanoparticles and in turn increases the stability of the nanofluids (Zhou *et al.*, 2009). Some of the disadvantages of the one-step method are that the cost is high, therefore the nanofluid might not be synthesized in large scales, and also, the residual reactants are left in the nanofluids due to incomplete reaction or stabilization. This method is mostly applicable for low vapor pressure fluids.

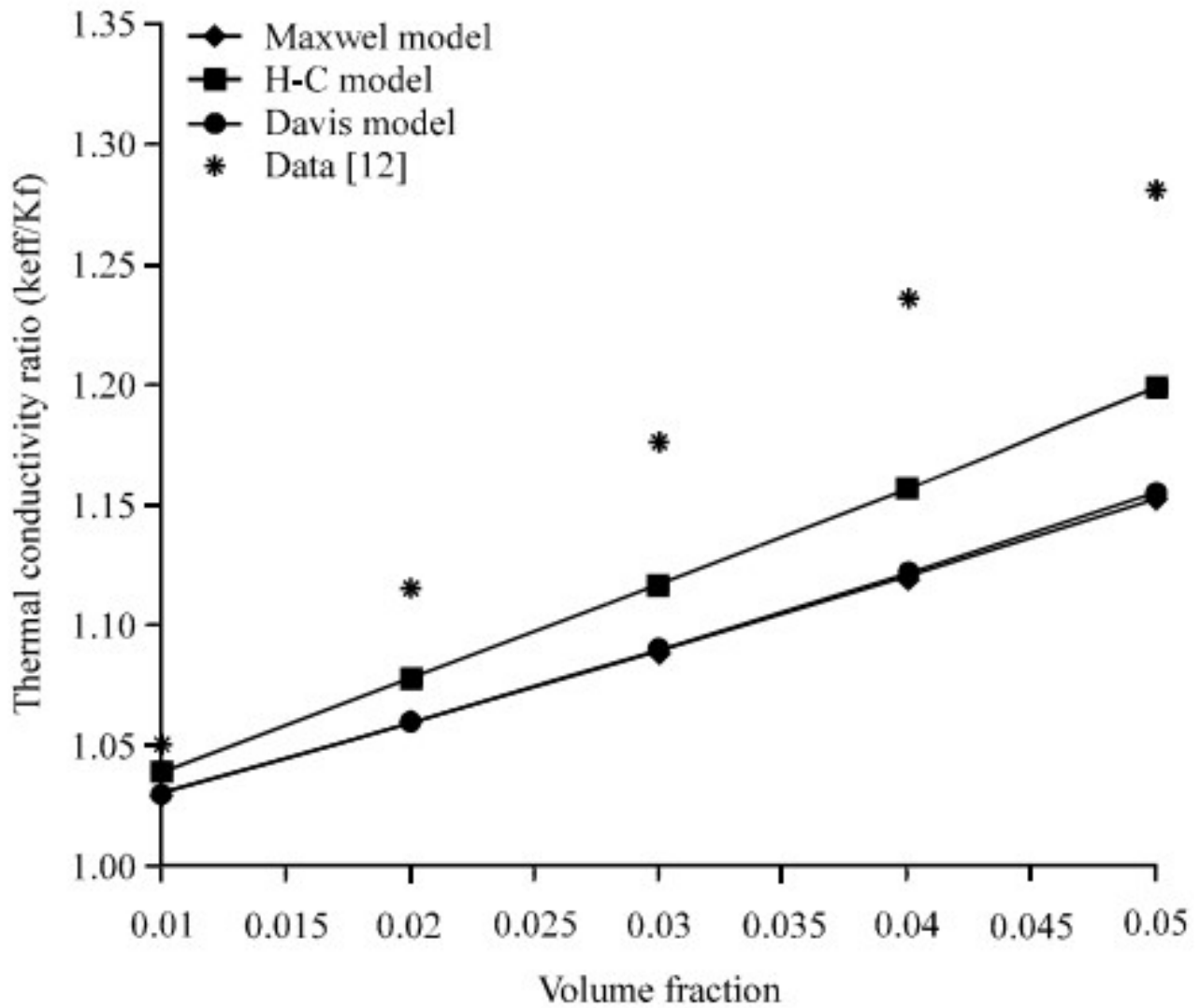
Effect of Particle Volume Fraction, Size, Material Type and Shape of Nanoparticle on Nanofluid

Thermal Properties

It has been shown in the literature that the particle material, volume fraction (volumetric concentration of nanoparticle in the nanofluid), shape and size of nanoparticles have significant effects on thermal properties of nanofluid. Baheta and Woldeyoham (2013) used existing models (Maxwell model, H-C model, and Davis model) to predict effective thermal conductivity of Al_2O_3 water nanofluid. Their results were compared with data obtained from literature (Eastman *et al.*, 1996). It is found that the thermal conductivity of the fluid was enhanced as the nanoparticle volume fraction increased (see Figure 1.11(a)). The effectiveness of thermal conductivity of nanofluid also depends on the particle size. The effect of thermal conductivity at different spherical nanoparticle diameters (6, 8, 10 and 20nm) for a wide range of volume fraction was also studied by Baheta and Woldeyoham (2013). It is observed that the thermal conductivity ratio enhancement decreased as the nanoparticle size increased (see 1.11(b)). The thermal conductivity enhancement of different nanoparticle materials (Al_2O_3 and CuO) suspension in water were studied by Senthilraja *et al.* (2015). They observed that CuO-water has thermal conductivity enhancement of 8 percent compared to Al_2O_3 -water with 6.1 percent thermal conductivity enhancement at a given nanoparticle volume fraction of 0.2 percent. Nanofluid is also influenced by nanoparticle shape. Xie *et al.* (2002) examined the thermal conductivity of SiC/distilled water nanofluid using the spherical particles with 26nm average diameter and cylindrical particles with 600nm average

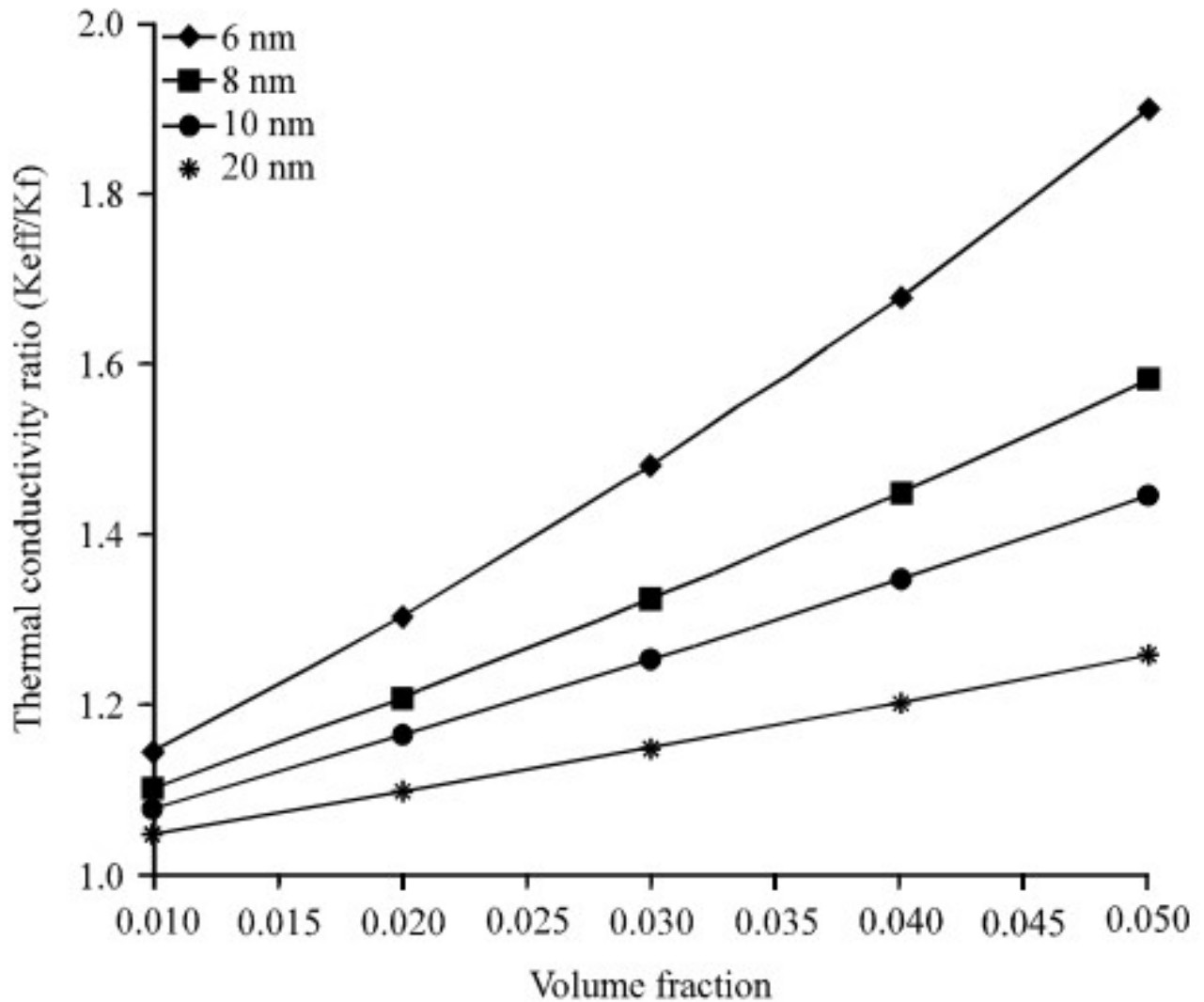
diameter. It was noticed that 4.2 percent volume fraction water-based nanofluid with spherical particles had a thermal conductivity enhancement of 15.8 percent while 4 percent of volume fraction nanofluid with cylindrical nanoparticles has a thermal conductivity enhancement of 22.9 percent. It can be concluded that cylindrical shape nanoparticles have higher thermal conductivity than spherical shape nanoparticles. Hence, the effectiveness of the thermal conductivity of nanofluid depends not only on the volume fraction, size and material type of the nanoparticle alone but also on the shape of nanoparticles.

[b]0.4tw



Nanofluid thermal conductivity increases by an increase in nanoparticle volume fraction (Baheta and Woldeyoham, 2013)

[b]0.4tw



Nanofluid thermal conductivity reduces by an increase in nanoparticle size (Baheta and Woldeyoham, 2013)

Figure 11: Effect of nanoparticle volume fraction and nanoparticle size on nanofluid thermal conductivity

Nanofluid Stability

The challenges which engineers and scientists are facing in synthesizing nanofluid are the poor stability as a result of the interaction between the particles themselves as well as between the particles and the surrounding fluid (Naser *et al.*, 2018). This behavior can be attributed to two opposing forces: (1) Van der Waals attractive forces on the surface which cause the particles to be attracted to each other and thereby forming clusters or agglomeration of particles and later separating from the base fluid by settling at the bottom due to gravitational force. (2) The electrical double layer repulsive force which leads to separation of the particles from each other through steric and electrostatic repulsion (Bushehri *et al.*, 2016, Hong and Kim, 2012, Arthur and Karim, 2016). The steric and electrostatic repulsions are shown in Figures 1.12a and 1.2b respectively. Stability is very important in nanofluid because it extends its shelf life and preserves its thermophysical properties. To have a stable nanofluid, the electrical double layer repulsive force

should surpass the van der Waals attractive force. Stability can be enhanced by applying any of the following methods (Bushehri *et al.*, 2016, Hong and Kim, 2012, Arthur and Karim, 2016): (i) Addition of surfactants (ii) Surface modification technique (iii) Ultrasonic Agitation (iv) PH control of nanofluid.

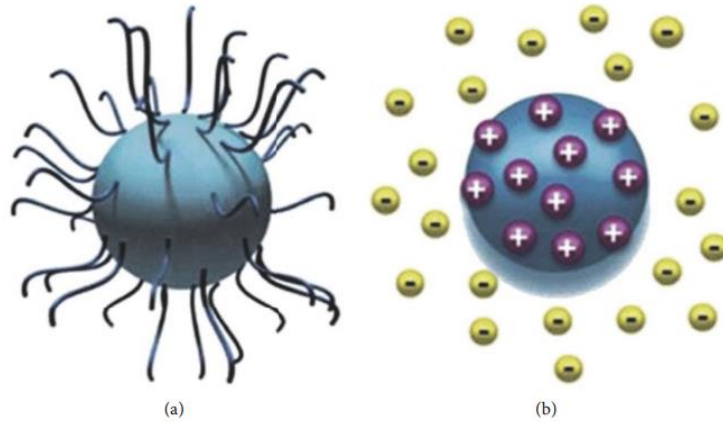


Figure 12: Electrical double layer repulsive forces (a) Steric repulsion and (b) electrostatic repulsion (Naser *et al.*, 2018)

Application of Nanofluids

Nanofluids offer fascinating heat transfer characteristics when compared to conventional heat transfer fluid. A lot of researches due to the superiority of heat transfer properties of nanofluids has been carried out. Nanofluids are applicable in industrial and engineering areas which include industrial coolant, smart fluids, nanofluid in automobile fuel, brake fluids, car radiator coolant, microelectronic cooling, bio and pharmaceutical industries (Tian and Zhao, 2013, Yu and Xie, 2012 and Wong and Leon, 2010).

Cooling of Microchips

Nanofluids are useful in the cooling of a microchip (see Figure 1.13) in the computer and elsewhere due to its higher thermal conductivity and boiling point compared to the conventional base fluid. The major limitation of developing smaller microchips and more compact component is the rapid heat dissipation. The heat generated by the personal computer becomes much more severe with an increase in the thermal dissipation of central processing unit (CPU). One of the applicable solutions to the heat generated by electronics is the use of some particular class of nanofluid as a smart material working as a heat valve to control the heat generated. In the work of Donzelli *et al.* (2009), it was opined that nanofluid can readily be configured either in a low state where it conducts heat poorly or in a high state where the dissipation is more efficient.



Figure 13: Microchip (Extracted from learn.sparkfun.com)

Cooling of Electronic Voice Coil

Nanofluid with suspended magnetic nanoparticles (ferrofluid) is commonly used in loudspeakers to remove heat from the voice coil. Ferrofluids are normally reside at the air gap around the voice coil, held in place by the heat generating magnet in the speaker. Sound is produced in the loudspeaker when the voice coil vibrates. Ferrofluids are paramagnetic thus they lose some of their magnetic properties at high temperature. The magnetic field placed near the voice coil will attract cold ferrofluid more than hot ferrofluid thus forcing the heated ferrofluid away from the electric voice coil towards a heated sink and thereby cools it (electric voice coil) (Rlums and Elmars, 1995).

Nanofluid as Engine Coolant

Nanofluids have great potentials to improve automotive and heavy-duty engine cooling rates by increasing efficiency, lowering the weight and reducing the complexity of thermal management systems (Yu and Xie, 2012). The use of nanofluid as coolant would allow for smaller size and better positioning of a radiator, shrunk coolant pump, engine operating at high temperature with more horsepower while still meeting stringent emission standard (Wong and Leon, 2010). According to Singh *et al.* (2006), the use of high thermal conductivity nanofluid in radiators can lead to a reduction in the frontal area of the radiator by up to 10 percent, which leads to a reduction in aerodynamic drag and reduce fuel consumption of up to 5 percent. A practical example of an engine cooling system is shown in Figure 1.14.

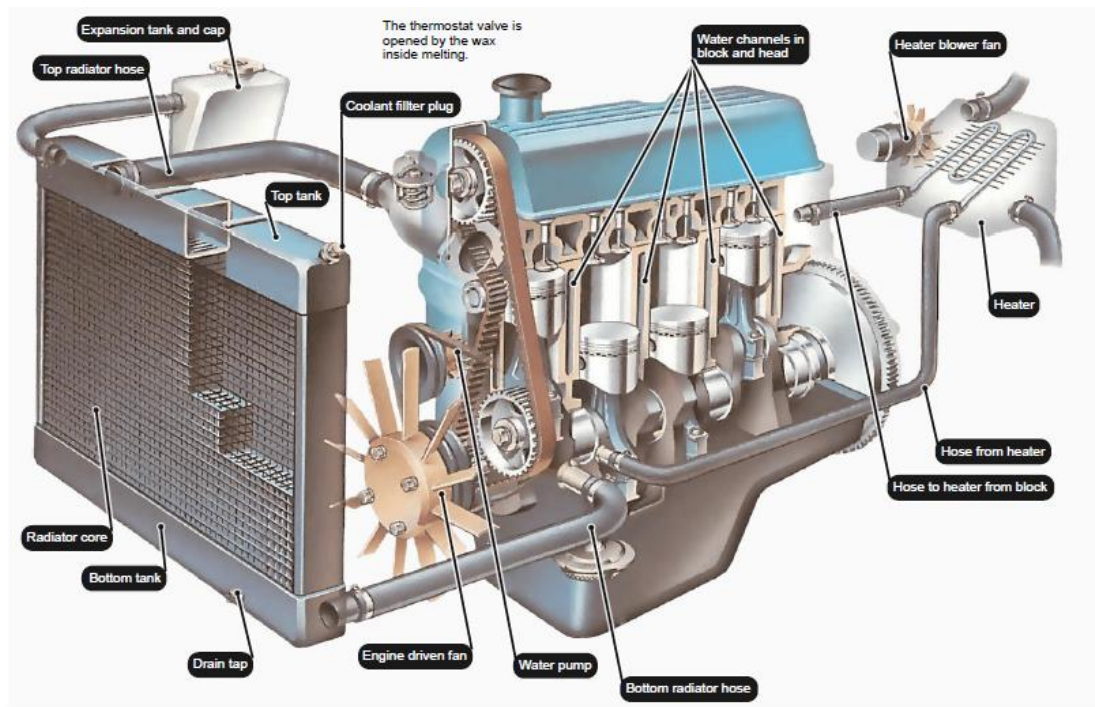


Figure 14: Engine Cooling System (Extracted from howcarworks.com)

Nanofluid as Brake Oil

An improvement in modern vehicle aerodynamics results in a decrease in drag force. Therefore, there is a higher demand for a braking system with more efficient heat dissipation mechanics. The vehicle kinematic energy is dispersed through the heat produced when a brake is applied, and it (the heat) is transmitted throughout the brake fluid in the hydraulic braking system (Popa *et al.*, 2010). This heat might cause the brake fluid to reach its boiling point, and a vapor lock is created that retards the hydraulic system from dispersing the heat caused by braking. This situation usually leads to brake malfunction and constitutes a safety hazard in a vehicle. Two types of brake nanofluid (Copper-oxide brake nanofluid (CBN) and aluminum-oxide brake nanofluid (AOBN)) with higher thermal conductivity, higher viscosity and higher boiling point than that of traditional brake fluid were manufactured using the arc submerge nanoparticle synthesis system and plasma charging arc system respectively (Kao *et al.*, 2006 and Kao *et al.*, 2007). It is discovered that CBN has a thermal conductivity 1.6 times higher than of brake fluid designated DOT3 and AOBN's thermal conductivity is 1.5 times higher than DOT3.

Nanodrug Delivery

The advancement in technology has resulted in the recent development of the microelectromechanical system (MEMS). Examples include an electronically activated drug delivery microchip (Shawgo *et al.*, 2002), a MEMS-based DNA sequencer developed by Cepheid (2009), arrays of in-plane and out-of-plane hollow micro-needles for dermal/transdermal drug delivery (Ovsianikov *et al.*, 2007 and Kim and Lee, 2007) and nanomedicine applications of nanogels or gold-coated nanoparticles (Sefiane *et al.*, 2008). The major interest of developing integrated micro or nano-drug delivery is to easily monitor and control target-cell responses to pharmaceutical stimuli, understand biological cell activities or enable drug developing process, unlike conventional drug delivery system which is characterized by the high-and-low phenomenon. By employing nanodrugs delivery system, the drug release that takes place over an extended period will be controlled.

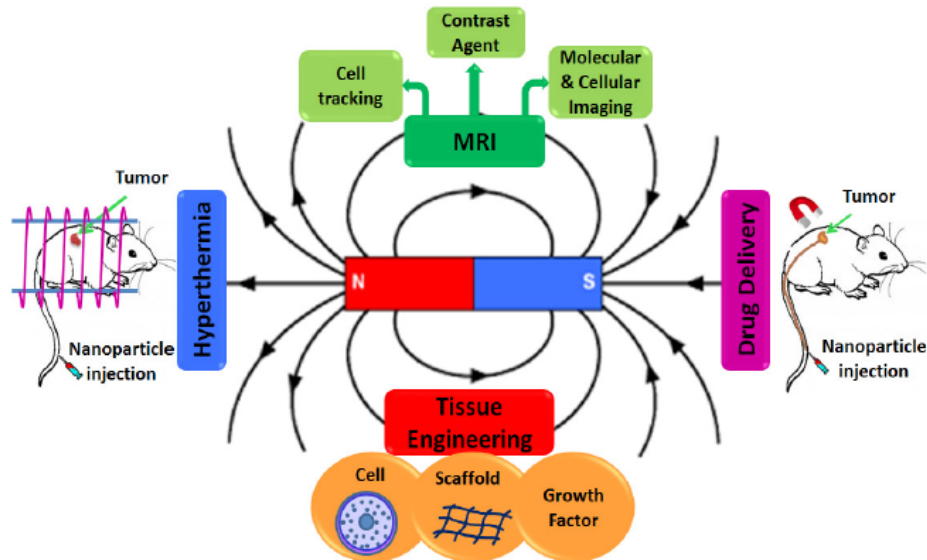


Figure 15: Schematic representation of the different applications of magnetic nanoparticles in medicine (Zare and Sattarahmady, 2015)

Iron-based nanoparticles can be used in cancer imaging as a delivery vehicle for drugs or radiation in cancer patients (see Figure 1.15). Magnetic nanoparticles are to be used compared to other metal-type nanoparticles to guide the drug components up to the bloodstream to a tumor with a magnet. This will also allow the doctor to deliver high local doses of drugs or radiation to the target without affecting surrounding healthy tissue (see Figure 1.16). Magnetic nanofluid also acts as a super-pragmatics fluid which in an alternating electromagnetic field absorbs energy that is producing controllable hyperthermia. The hyperthermia can produce a preferential radiation effect on malignant cells by enhancing chemotherapeutic efficacy (Bica *et al.*, 2007).

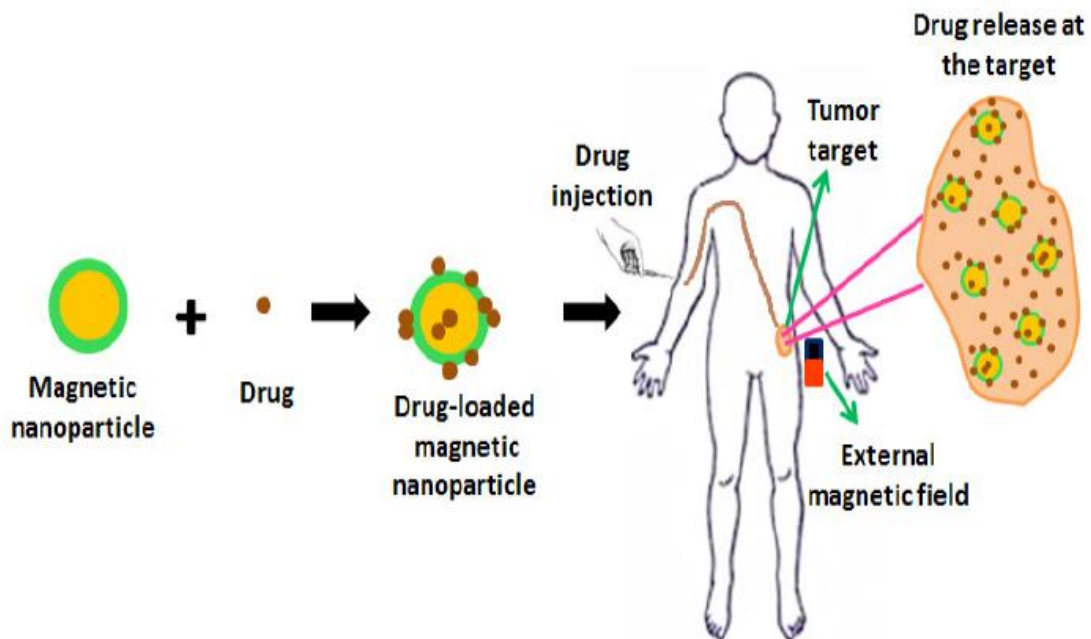


Figure 16: Diagram of targeted drug delivery using magnetic nanoparticles (Zare and Sattarahmady, 2015)

Solar Absorption

Solar thermal collectors are designed to absorb and convert solar irradiation into thermal

energy and then transfer to a working fluid (air, water or oil) contained within the collector structure. The circulating working fluid carries the energy away from the collector in the form of heat to be directly used or used to charge up a thermal storage tank (Tian and Zhao, 2013). Solar absorption systems are used in many industrial, commercial and domestic applications to supply energy for heating water, space heating, and cooling, and at high temperatures to drive power-generating turbine (Wisut *et al.*, 2017).

There are two methods for collecting solar thermal energy. The first is the use of a non-concentrating collector configuration where the absorbing panel has the same surface area as the collector, while the second method is the use of several reflecting surfaces to gather, concentrate and focus the solar irradiation onto a collector (Kalogirou, 2004). In these traditional solar absorptions, the efficiency of the collectors is limited by the absorption properties of the working fluid. The use of nanofluid as the working fluid in the solar collector enhances the absorption properties of working fluid (Yu and Xie, 2012). Otanical *et al.* (2010) gave a report on the experimental results on solar collectors based on nanofluids from a variety of nanoparticles. It was concluded that the efficiency improvement was up to five percent in solar thermal collectors by using nanofluid as absorption media. Concentrating solar collector styles are shown in Figure 1.17.

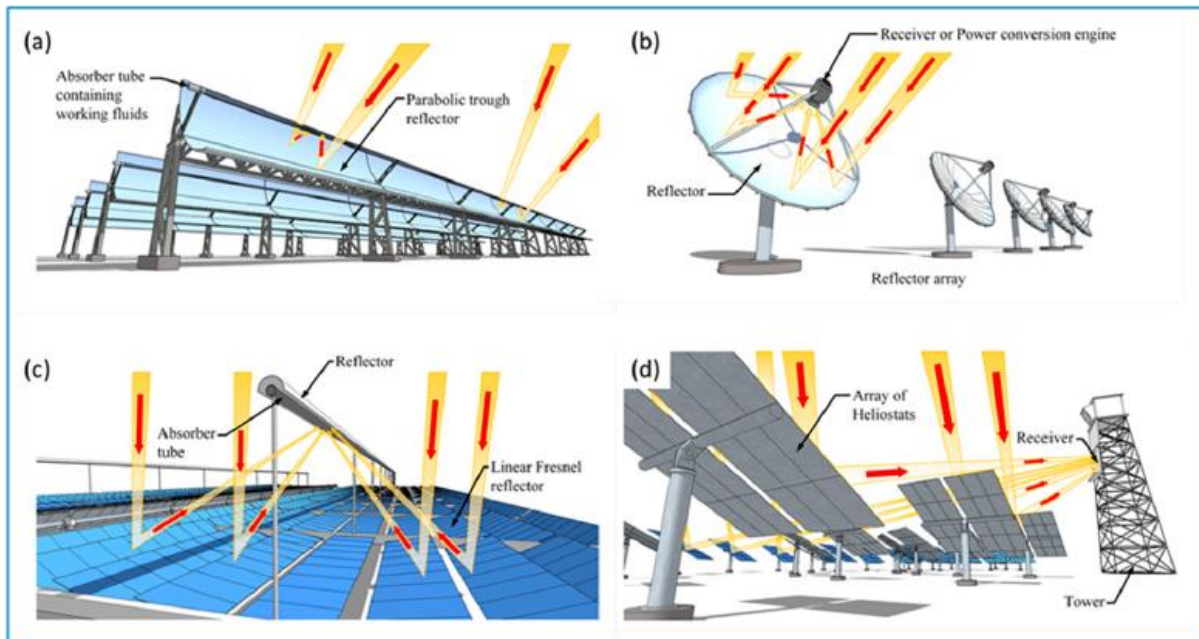


Figure 17: Concentrating solar collector styles: (a) parabolic trough collector (b) parabolic dish collector (c) linear Fresnel collector (d) heliostat field collector (Wisut *et al.*, 2017).

2 Basic Equations of Flow, Heat and Mass Transfer

The purpose of this section is to derive the fundamental laws that govern the fluid dynamics which are law of conservation of mass (continuity equation), Newton's second law of motion (momentum equation), first law of thermodynamics (Energy equation) and Maxwell's Equations. These fundamental equations are derived by considering an infinitesimally small fixed volume of a fluid element with dimensions dx, dy, dz moving in a stream.

Continuity Equation

The law of mass conservation states that the rate of mass accumulation in the control volume is equal to the rate of mass entering the control volume minus the rate of mass leaving the

control volume. A small fluid element (in control volume) with density ρ and velocity u, v, w in x, y, z direction is considered (see Fig.1.18).

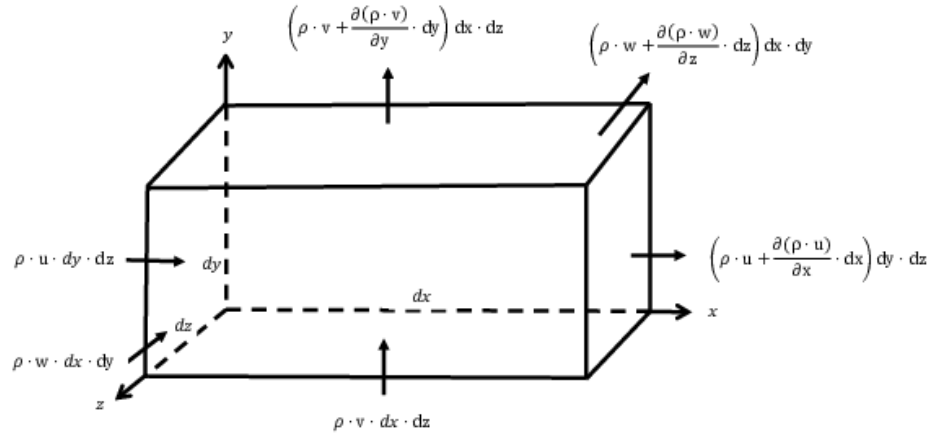


Figure 18: Mass flow rate entering and exiting the control volume
The rate of mass accumulation within the control volume is

$$\frac{\partial}{\partial t}(\rho \cdot dx \cdot dy \cdot dz) = \frac{\partial \rho}{\partial t} \cdot dx \cdot dy \cdot dz \quad (5)$$

The net mass flow rate through the control volume in x direction is

$$\rho \cdot u \cdot dy \cdot dz - (\rho \cdot u + \frac{\partial(\rho \cdot u)}{\partial x} \cdot dx) \cdot dy \cdot dz = -\frac{\partial}{\partial x}(\rho \cdot u) \cdot dx \cdot dy \cdot dz \quad (6)$$

Similarly, the net mass flow rate in y and z directions are

$$-\frac{\partial}{\partial y}(\rho \cdot v) \cdot dx \cdot dy \cdot dz \text{ and } -\frac{\partial}{\partial z}(\rho \cdot w) \cdot dx \cdot dy \cdot dz, \text{ respectively.} \quad (7)$$

Applying conservation mass law and dividing through by $dx \cdot dy \cdot dz$ to obtain

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho \cdot u) + \frac{\partial}{\partial y}(\rho \cdot v) + \frac{\partial}{\partial z}(\rho \cdot w) = 0 \quad (8)$$

This equation is known as continuity equation for a compressible flow. For the incompressible flow, the continuity equation reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (9)$$

Momentum equation

Momentum equation is derived from Newton's second law which states that in a control volume (CV)

$$\left\{ \begin{array}{l} \text{The rate of change} \\ \text{of momentum in CV} \end{array} \right\} = \left\{ \begin{array}{l} \text{The net momentum} \\ \text{flow rate} \\ \text{through CV} \end{array} \right\} + \left\{ \begin{array}{l} \text{The net forces} \\ \text{acting on the CV} \end{array} \right\} \quad (10)$$

The rate of change of momentum in the CV in x-direction

$$\frac{\partial(\rho \cdot dx \cdot dy \cdot dz \cdot u)}{\partial t} = \frac{\partial(\rho \cdot u)}{\partial t} \cdot dx \cdot dy \cdot dz \quad (11)$$

The net momentum flow rate $\dot{M}_{net}$ through CV in x-direction

$$\dot{M}_{net} = -\left(\frac{\partial(\rho \cdot u \cdot u)}{\partial x} dx + \frac{\partial(\rho \cdot u \cdot v)}{\partial y} dy + \frac{\partial(\rho \cdot u \cdot w)}{\partial z} dz \right) \cdot dx \cdot dy \cdot dz \quad (12)$$

The net forces acting on the control volume are

(a) Volume forces (body forces) act on the mass of the volume element. They include gravity as well as the electric and magnetic forces acting on the flow. The only body force that is considered in this work is the gravity.

(b) Surface forces act on the surface of the fluid element which are normal stress ($\tau_{x,x}$) and shear stress ($\tau_{x,y}$)

The body force acting on the fluid element in x-direction is

$$\left\{ \begin{array}{l} \text{Body force} \\ \text{in x direction} \end{array} \right\} = \rho \cdot g_x \cdot dx \cdot dy \cdot dz \quad (13)$$

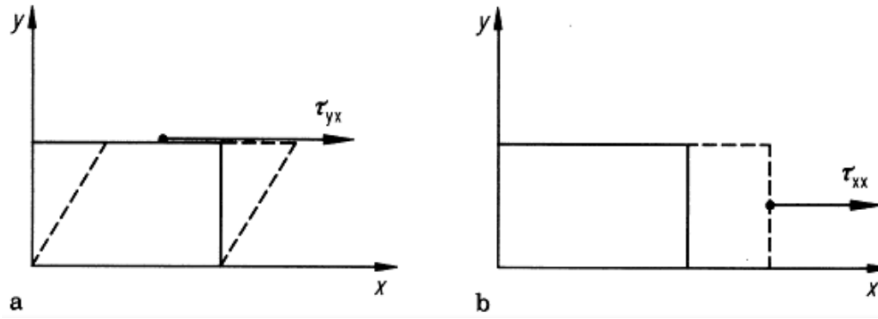


Figure 19: Shear stress and normal stress

The surface forces exerted on the fluid element are shown in Figure 1.19. The net surface forces in x -direction is given as (see *Figure1.20*)

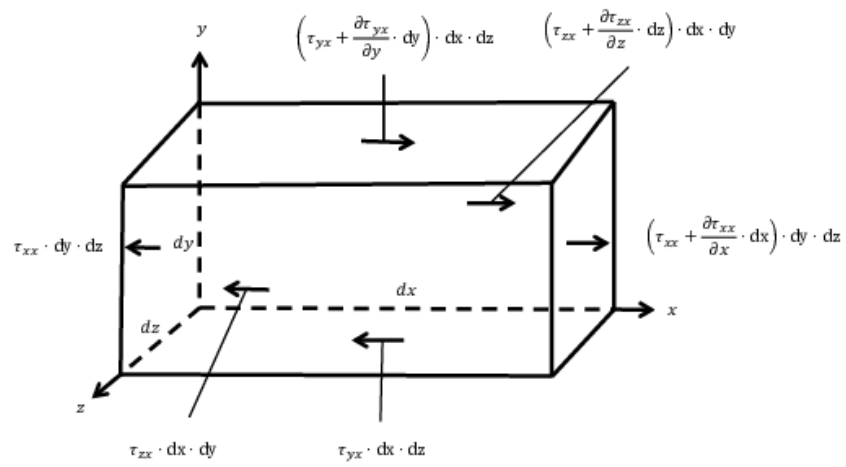


Figure 20: Normal stresses and shear stresses acting on the fluid surface in x direction

$$\left\{ \begin{array}{c} \text{Net} \\ \text{surface forces} \\ \text{in } x \text{ direction} \end{array} \right\} = + \left\{ \begin{array}{l} \left(\tau_{x,x} + \frac{\partial \tau_{x,x}}{\partial x} \cdot dx \right) \cdot dy \cdot dz \\ \left(\tau_{y,x} + \frac{\partial \tau_{y,x}}{\partial xy} \cdot dy \right) \cdot dx \cdot dz \\ + \left(\tau_{z,x} + \frac{\partial \tau_{z,x}}{\partial z} \cdot dz \right) \cdot dx \cdot dy \end{array} \right\} \quad (14)$$

The total force F_x acting in x-direction is given by summing up the body force in Eq.(1.13) and net surface force in Eq.(1.14)

$$F_x = \left(\frac{\partial \tau_{x,x}}{\partial x} + \frac{\partial \tau_{y,x}}{\partial y} + \frac{\partial \tau_{z,x}}{\partial z} \right) \cdot dx \cdot dy \cdot dz + \rho \cdot g_x \cdot dx \cdot dy \cdot dz \quad (15)$$

The normal stress ($\tau_{x,x}$) can be splitted up into two parts, the pressure (P) and the contribution due to friction of the fluid ($\sigma_{x,x}$) (Herbert, 2004).

$$\tau_{x,x} = \sigma_{x,x} - P \quad (16)$$

Substituting Eq.(1.16) into Eq.(1.15) to obtain

$$F_x = \left(-\frac{\partial P}{\partial x} + \frac{\partial \sigma_{x,x}}{\partial x} + \frac{\partial \tau_{y,x}}{\partial y} + \frac{\partial \tau_{z,x}}{\partial z} \right) \cdot dx \cdot dy \cdot dz + \rho \cdot g_x \cdot dx \cdot dy \cdot dz \quad (17)$$

Substituting Eqs.(1.11),(1.12) and (1.17) into Eq.(1.10) and dividing through by $dx \cdot dy \cdot dz$ we have momentum equation in x directions as

$$\begin{aligned} & \left(\frac{\partial(\rho \cdot u)}{\partial t} + \frac{\partial(\rho \cdot u \cdot u)}{\partial x} + \frac{\partial(\rho \cdot u \cdot v)}{\partial y} + \frac{\partial(\rho \cdot u \cdot w)}{\partial z} \right) \\ &= -\frac{\partial P}{\partial x} + \frac{\partial \sigma_{x,x}}{\partial x} + \frac{\partial \tau_{y,x}}{\partial y} + \frac{\partial \tau_{z,x}}{\partial z} + \rho \cdot g_x \end{aligned} \quad (18)$$

Similarly, in y and z directions respectively as

$$\begin{aligned} & \left(\frac{\partial(\rho \cdot v)}{\partial t} + \frac{\partial(\rho \cdot v \cdot u)}{\partial x} + \frac{\partial(\rho \cdot v \cdot v)}{\partial y} + \frac{\partial(\rho \cdot v \cdot w)}{\partial z} \right) \\ &= -\frac{\partial P}{\partial y} + \frac{\partial \tau_{x,y}}{\partial x} + \frac{\partial \sigma_{y,y}}{\partial y} + \frac{\partial \tau_{z,y}}{\partial z} + \rho \cdot g_y \end{aligned} \quad (19)$$

and

$$\begin{aligned} & \left(\frac{\partial(\rho \cdot w)}{\partial t} + \frac{\partial(\rho \cdot w \cdot u)}{\partial x} + \frac{\partial(\rho \cdot w \cdot v)}{\partial y} + \frac{\partial(\rho \cdot w \cdot w)}{\partial z} \right) \\ &= -\frac{\partial P}{\partial z} + \frac{\partial \tau_{x,z}}{\partial x} + \frac{\partial \tau_{y,z}}{\partial y} + \frac{\partial \sigma_{z,z}}{\partial z} + \rho \cdot g_z \end{aligned} \quad (20)$$

Applying continuity Eq.(1.8) in Eqs.(1.18)–(1.20) to obtain Navier-Stoke equations in non conservative form:

$$\rho \cdot \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \frac{\partial \sigma_{x,x}}{\partial x} + \frac{\partial \tau_{y,x}}{\partial y} + \frac{\partial \tau_{z,x}}{\partial z} + \rho \cdot g_x, \quad (21)$$

$$\rho \cdot \frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{x,y}}{\partial x} + \frac{\partial \sigma_{y,y}}{\partial y} + \frac{\partial \tau_{z,y}}{\partial z} + \rho \cdot g_y, \quad (22)$$

and

$$\rho \cdot \frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \frac{\partial \tau_{x,z}}{\partial x} + \frac{\partial \tau_{y,z}}{\partial y} + \frac{\partial \sigma_{z,z}}{\partial z} + \rho \cdot g_z, \quad (23)$$

where

$$\left. \begin{aligned} \frac{Du}{Dt} &= u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial u}{\partial y} + w \cdot \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \\ \frac{Dv}{Dt} &= u \cdot \frac{\partial v}{\partial x} + v \cdot \frac{\partial v}{\partial y} + w \cdot \frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \\ \frac{Dw}{Dt} &= u \cdot \frac{\partial w}{\partial x} + v \cdot \frac{\partial w}{\partial y} + w \cdot \frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} \end{aligned} \right\} \quad (24)$$

are known as total derivative in x, y and z directions

Energy Equation

The energy equation is derived from the first law of thermodynamics which states that in a closed system, the amount of heat added to a system Q is equal to a change in internal energy E plus the amount of energy lost due to work done on the system W . Considering an infinitesimally small fluid element with dimensions dx, dy and dz in a control volume CV (see Figure 1.21) we have (Herbert, 2004):

$$\left\{ \begin{array}{l} \text{Rate of} \\ \text{Change of} \\ \text{Energy} \\ \text{in the CV} \end{array} \right\} = \left\{ \begin{array}{l} \text{Net energy} \\ \text{flow rate} \\ \text{through} \\ \text{the CV} \end{array} \right\} + \left\{ \begin{array}{l} \text{Net rate} \\ \text{of heat} \\ \text{transfer} \\ \text{through} \\ \text{the CV} \end{array} \right\} + \left\{ \begin{array}{l} \text{work done rate} \\ \text{on the CV due} \\ \text{to body and} \\ \text{surface forces} \end{array} \right\} \quad (25)$$

The total energy E within the control volume is made up of the internal energy $\rho \cdot e \cdot dx \cdot dy \cdot dz$ and kinetic energy $\rho \left(\frac{V^2}{2} \right) \cdot dx \cdot dy \cdot dz$. The rate of change of energy in the control volume is

$$\frac{\partial \left[\rho \cdot \left(e + \frac{V^2}{2} \right) \cdot dx \cdot dy \cdot dz \right]}{\partial t} = \frac{\partial \left[\rho \cdot \left(e + \frac{V^2}{2} \right) \right]}{\partial t} \cdot dx \cdot dy \cdot dz \quad (26)$$

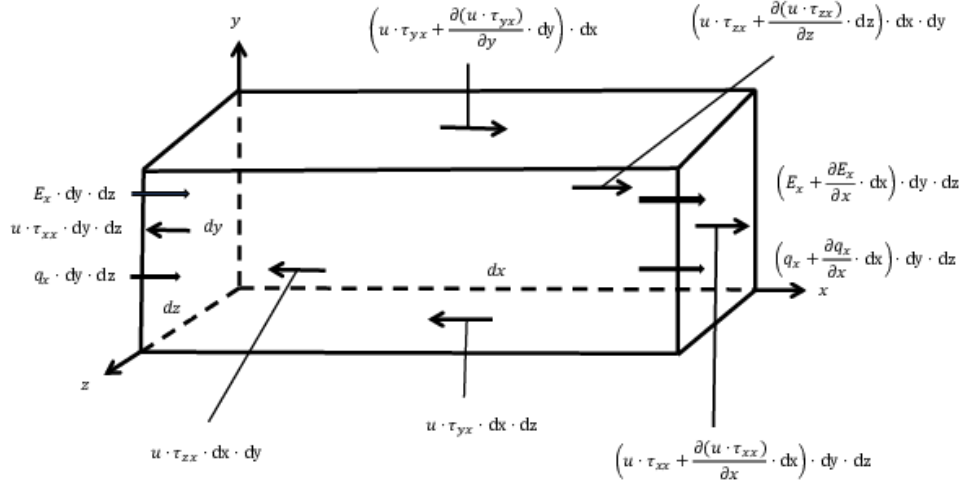


Figure 21: Energy flow rate entering and exiting CV in x direction
The net energy flow rate through the control volume is

$$\begin{aligned} \dot{E} = & -\left(\frac{\partial(\rho \cdot \left(e + \frac{V^2}{2} \right) \cdot u)}{\partial x} + \frac{\partial(\rho \cdot \left(e + \frac{V^2}{2} \right) \cdot v)}{\partial y} \right. \\ & \left. + \frac{\partial(\rho \cdot \left(e + \frac{V^2}{2} \right) \cdot w)}{\partial z} \right) \cdot dx \cdot dy \cdot dz \end{aligned} \quad (27)$$

The net heat fluxes due conduction and volumetric heating:

$$\dot{q} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) \cdot dx \cdot dy \cdot dz + \rho \cdot q'' \cdot dx \cdot dy \cdot dz \quad (28)$$

where q_x, q_y , and q_z is heat transfer due to conduction in x, y and z direction respectively and q'' is the heat generation. According to Fourier's law of heat conduction, heat transfer by thermal conduction is proportional to local temperature gradient;

$$q_x = -k \cdot \frac{\partial T}{\partial x}, q_y = -k \cdot \frac{\partial T}{\partial y} \text{ and } q_z = -k \cdot \frac{\partial T}{\partial z} \quad (29)$$

where k is the fluid thermal conductivity. Substituting Eq.(1.29) into Eq.(1.28) we obtained

$$\dot{q} = \left(\frac{\partial}{\partial x}(k \cdot \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k \cdot \frac{\partial T}{\partial y}) + \frac{\partial}{\partial z}(k \cdot \frac{\partial T}{\partial z}) + \rho \cdot q''\right) \cdot dx \cdot dy \cdot dz \quad (30)$$

The forces resulting from the stresses, pressure and body force perform work on the fluid element. This work per unit time is computed from the product of the velocity and the force that act in the direction of the velocity component. The work per unit time dA_x in the x-direction is given as (Herbert, 2004)

$$\dot{A}_x = \left\{ \begin{aligned} &P \cdot dy \cdot dz \cdot u - \left(P \cdot dy \cdot dz \cdot u + \frac{\partial(P \cdot dy \cdot dz \cdot u)}{\partial x} \cdot dx \right) \\ &-\sigma_{xx} \cdot dy \cdot dz \cdot u + \left(\sigma_{xx} \cdot dy \cdot dz \cdot u + \frac{\partial(\sigma_{xx} \cdot dy \cdot dz \cdot u)}{\partial x} \cdot dx \right) \\ &-\tau_{xy} \cdot dy \cdot dz \cdot v + \left(\tau_{xy} \cdot dy \cdot dz \cdot v + \frac{\partial(\tau_{xy} \cdot dy \cdot dz \cdot v)}{\partial x} \cdot dx \right) \\ &-\tau_{xz} \cdot dy \cdot dz \cdot w + \left(\tau_{xz} \cdot dy \cdot dz \cdot w + \frac{\partial(\tau_{xz} \cdot dy \cdot dz \cdot w)}{\partial x} \cdot dx \right) \\ &+ \rho \cdot g_x \cdot u \cdot dx \cdot dy \cdot dz \end{aligned} \right\} \quad (31)$$

$$\dot{A}_x = \left(-\frac{\partial(P \cdot u)}{\partial x} + \frac{\partial(\sigma_{xx} \cdot u)}{\partial x} + \frac{\partial(\tau_{xy} \cdot v)}{\partial x} + \frac{\partial(\tau_{xz} \cdot w)}{\partial x} + \rho \cdot g_x \cdot u \right) \cdot dx \cdot dy \cdot dz \quad (32)$$

Similarly in y and z directions, we obtained the expression for dA_y and dA_z

$$\dot{A}_y = \left(-\frac{\partial(P \cdot v)}{\partial y} + \frac{\partial(\tau_{yx} \cdot u)}{\partial y} + \frac{\partial(\sigma_{yy} \cdot v)}{\partial y} + \frac{\partial(\tau_{yz} \cdot w)}{\partial y} + \rho \cdot g_y \cdot v \right) \cdot dx \cdot dy \cdot dz \quad (33)$$

$$\dot{A}_z = \left(-\frac{\partial(P \cdot w)}{\partial z} + \frac{\partial(\tau_{zx} \cdot u)}{\partial z} + \frac{\partial(\tau_{zy} \cdot v)}{\partial z} + \frac{\partial(\sigma_{zz} \cdot w)}{\partial z} + \rho \cdot g_z \cdot w \right) \cdot dx \cdot dy \cdot dz \quad (34)$$

Inserting Eqs.(1.26),(1.27),(1.30),(1.32),(1.33) and (3.34) into Eq.(1.25) and apply

continuity Eq.(1.8), we obtained a balanced energy equation;

$$\begin{aligned}
 \rho \cdot \frac{D\left(e + \frac{V^2}{2}\right)}{Dt} = & \frac{\partial}{\partial x} \left(k \cdot \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \cdot \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \cdot \frac{\partial T}{\partial z} \right) + \rho \cdot q'' \\
 & - \left(\frac{\partial(P \cdot u)}{\partial x} + \frac{\partial(P \cdot v)}{\partial y} + \frac{\partial(P \cdot w)}{\partial z} \right) + \rho \cdot \vec{g} \cdot \vec{V} \\
 & + \left(\frac{\partial(\sigma_{x,x} \cdot u)}{\partial x} + \frac{\partial(\tau_{x,y} \cdot v)}{\partial x} + \frac{\partial(\tau_{x,z} \cdot w)}{\partial x} \right) \\
 & + \left(\frac{\partial(\tau_{y,x} \cdot u)}{\partial y} + \frac{\partial(\sigma_{y,y} \cdot v)}{\partial y} + \frac{\partial(\tau_{y,z} \cdot w)}{\partial y} \right) \\
 & + \left(\frac{\partial(\tau_{z,x} \cdot u)}{\partial z} + \frac{\partial(\tau_{z,y} \cdot v)}{\partial z} + \frac{\partial(\sigma_{z,z} \cdot w)}{\partial z} \right)
 \end{aligned} \tag{35}$$

This is the non-conservative form of the energy equation and it involves the total energy $(e + \frac{V^2}{2})$. The energy equation is mostly written in a form that involves only the internal energy e . To achieve this, multiplying Eqs.(1.21)–(1.23) by u, v and w respectively, we have

$$\rho \cdot \frac{D\left(\frac{u^2}{2}\right)}{Dt} = -u \cdot \frac{\partial P}{\partial x} + u \cdot \frac{\partial \sigma_{x,x}}{\partial x} + u \cdot \frac{\partial \tau_{y,x}}{\partial y} + u \cdot \frac{\partial \tau_{z,x}}{\partial z} + u \cdot \rho \cdot g_x \tag{36}$$

$$\rho \cdot \frac{D\left(\frac{v^2}{2}\right)}{Dt} = -v \cdot \frac{\partial P}{\partial y} + v \cdot \frac{\partial \tau_{x,y}}{\partial x} + v \cdot \frac{\partial \sigma_{y,y}}{\partial y} + v \cdot \frac{\partial \tau_{z,y}}{\partial z} + v \cdot \rho \cdot g_y \tag{37}$$

$$\rho \cdot \frac{D\left(\frac{w^2}{2}\right)}{Dt} = -w \cdot \frac{\partial P}{\partial z} + w \cdot \frac{\partial \tau_{x,z}}{\partial x} + w \cdot \frac{\partial \tau_{y,z}}{\partial y} + w \cdot \frac{\partial \sigma_{z,z}}{\partial z} + w \cdot \rho \cdot g_z \tag{38}$$

Adding Eqs.(1.36)–(1.38), we obtain

$$\begin{aligned}
 \rho \cdot \frac{D\left(\frac{V^2}{2}\right)}{Dt} = & -(u \cdot \frac{\partial P}{\partial x} + v \cdot \frac{\partial P}{\partial y} + w \cdot \frac{\partial P}{\partial z}) + u \cdot \left(\frac{\partial \sigma_{x,x}}{\partial x} + \frac{\partial \tau_{y,x}}{\partial y} \right. \\
 & \left. + \frac{\partial \tau_{z,x}}{\partial z} \right) + v \cdot \left(\frac{\partial \tau_{x,y}}{\partial x} + \frac{\partial \sigma_{y,y}}{\partial y} + \frac{\partial \tau_{z,y}}{\partial z} \right) \\
 & + w \cdot \left(\frac{\partial \tau_{x,z}}{\partial x} + \frac{\partial \tau_{y,z}}{\partial y} + \frac{\partial \sigma_{z,z}}{\partial z} \right) + \rho \cdot \vec{g} \cdot \vec{V}
 \end{aligned} \tag{39}$$

where $V^2 = u^2 + v^2 + w^2$ and $\vec{g} \cdot \vec{V} = ug_x + vg_y + wg_z$. Subtracting Eq.(1.39) from Eq.(1.35)

to obtain

$$\begin{aligned} \rho \cdot \frac{De}{Dt} &= \frac{\partial}{\partial x} \left(k \cdot \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \cdot \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \cdot \frac{\partial T}{\partial z} \right) + \rho \cdot q'' \\ &\quad - P \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \sigma_{x,x} \frac{\partial u}{\partial x} + \tau_{x,y} \frac{\partial v}{\partial x} + \tau_{x,z} \frac{\partial w}{\partial x} \\ &\quad + \tau_{y,x} \frac{\partial u}{\partial y} + \sigma_{y,y} \frac{\partial v}{\partial y} + \tau_{y,z} \frac{\partial w}{\partial y} + \tau_{z,x} \frac{\partial u}{\partial z} + \tau_{z,y} \frac{\partial v}{\partial z} + \sigma_{z,z} \frac{\partial w}{\partial z} \end{aligned} \quad (40)$$

For an ideal incompressible fluid, The following relationship holds for temperature T and specific heat capacity C_p (Herbert, 2004)

$$e = C_p \cdot T - \frac{P}{\rho} \quad (41)$$

Also, for Newtonian fluid (water, air and oil) the following relationships hold

$$\begin{aligned} \sigma_{xx} &= 2 \cdot \mu \cdot \frac{\partial u}{\partial x} - \frac{2}{3} \cdot \mu \cdot \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right), \\ \sigma_{yy} &= 2 \cdot \mu \cdot \frac{\partial v}{\partial y} - \frac{2}{3} \cdot \mu \cdot \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right), \\ \sigma_{zz} &= 2 \cdot \mu \cdot \frac{\partial w}{\partial z} - \frac{2}{3} \cdot \mu \cdot \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right), \\ \tau_{yx} &= \tau_{xy} = \mu \cdot \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right), \quad \tau_{yz} = \tau_{zy} = \mu \cdot \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right), \\ \tau_{zx} &= \tau_{xz} = \mu \cdot \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \end{aligned} \quad (42)$$

Inserting Eqs.(1.41) and (1.42) into Eq.(1.40) and using the continuity Eq.(1.9), we obtain energy equation for incompressible three-dimensional flow in Cartesian coordinate as;

$$\begin{aligned} \rho C_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) &= \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) \\ &\quad + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \rho q'' + \mu \phi' \end{aligned} \quad (43)$$

where

$$\begin{aligned} \phi &= 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 \\ &\quad + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2 \end{aligned} \quad (44)$$

Mass transfer equation

In a viscous fluid consist of binary mixture for which there are species concentration

gradients, there will be relative transport of species and the law of species conservation must be satisfied. Considering a species of mass concentration C_A in control volume CV (see Figure 1.22), then (Incropera *et al.*, 2007)

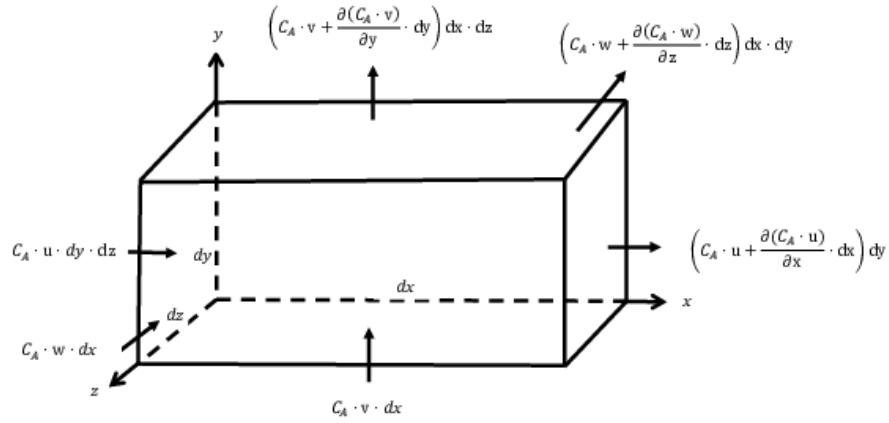


Figure 22: Mass concentration flow rate entering and exiting the control volume

$$C_A \left\{ \begin{array}{l} \text{Rate of} \\ \text{change of} \end{array} \right\} \quad (45)$$

in the CV = {Net rate of}

C_A

enters the CV due to convection + { Net rate of }

C_A

enters the CV due to diffusion + { Rate of }

C_A

in the CV

The rate of change of species (C_A) in the control volume is

$$\frac{\partial (C_A \cdot dx \cdot dy \cdot dz)}{\partial t} = \frac{\partial C_A}{\partial t} \cdot dx \cdot dy \cdot dz \quad (46)$$

The net rate at which species (C_A) enters the control volume due to convection is given as (Incropera et al., 2007)

$$\dot{M}_c = - \left(\frac{\partial (C_A \cdot u)}{\partial x} + \frac{\partial (C_A \cdot v)}{\partial y} + \frac{\partial (C_A \cdot w)}{\partial z} \right) \cdot dx \cdot dy \cdot dz \quad (47)$$

The net rate at which species (C_A) enters the control volume due to diffusion and species generation is (Incropera et al., 2007)

$$\dot{M}_{d,g} = - \left(\frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y} + \frac{\partial m_z}{\partial z} \right) \cdot dx \cdot dy \cdot dz + R_k \cdot dx \cdot dy \cdot dz \quad (48)$$

where m_x, m_y and m_z are the mass concentration transfer due to diffusion in x, y and z respectively and R_k is the species generation. According to Fick's law,

$$m_x = -D_B \frac{\partial C_A}{\partial x}, m_y = -D_B \frac{\partial C_A}{\partial y} \text{ and } m_z = -D_B \frac{\partial C_A}{\partial z} \quad (49)$$

where D_B is the mass concentration diffusion coefficient. Substituting Eq.(1.49) into Eq.(1.48) to have

$$\dot{M}_{d,g} = \left(\frac{\partial}{\partial x} (D_B \cdot \frac{\partial C_A}{\partial x}) + \frac{\partial}{\partial y} (D_B \cdot \frac{\partial C_A}{\partial y}) + \frac{\partial}{\partial z} (D_B \cdot \frac{\partial C_A}{\partial z}) + R_k \right) \cdot dx \cdot dy \cdot dz \quad (50)$$

Inserting Eqs.(1.46), (1.47) and (1.50) into Eq.(1.45), we have conservation of mass transfer equation

$$\begin{aligned} & \left(\frac{\partial C_A}{\partial t} + \frac{\partial (C_A \cdot u)}{\partial x} + \frac{\partial (C_A \cdot v)}{\partial y} + \frac{\partial (C_A \cdot w)}{\partial z} \right) \\ &= \frac{\partial}{\partial x} (D_B \cdot \frac{\partial C_A}{\partial x}) + \frac{\partial}{\partial y} (D_B \cdot \frac{\partial C_A}{\partial y}) + \frac{\partial}{\partial z} (D_B \cdot \frac{\partial C_A}{\partial z}) + R_k \end{aligned} \quad (51)$$

For an incompressible flow, Eq.(1.51) reduces to

$$\frac{DC_A}{Dt} = \frac{\partial}{\partial x} (D_B \cdot \frac{\partial C_A}{\partial x}) + \frac{\partial}{\partial y} (D_B \cdot \frac{\partial C_A}{\partial y}) + \frac{\partial}{\partial z} (D_B \cdot \frac{\partial C_A}{\partial z}) + R_k \quad (52)$$

where

$$\frac{DC_A}{Dt} = \frac{\partial C_A}{\partial t} + u \cdot \frac{\partial C_A}{\partial x} + v \cdot \frac{\partial C_A}{\partial y} + w \cdot \frac{\partial C_A}{\partial z} \quad (53)$$

Maxwell's Equations

The Maxwell's equations governing the electric and magnetic fields are;

$$\nabla \cdot B = \mu_o j \text{ (Ampere's Law)} \quad (54)$$

$$\frac{\partial B}{\partial t} = -\nabla \times E \text{ (Faraday's Law)} \quad (55)$$

$$j = \sigma(E + v \times B) \text{ (Ohm's Law)} \quad (56)$$

where μ_o is the magnetic permeability, B is the magnetic field, j is the electric density, E is the electric field and σ is the electrical conductivity.

General Equations Governing MHD Casson Nanofluid Flow

Casson nanofluid will hereby be treated as a two-component mixture (base fluid+nanoparticles) by considering the following assumptions. (i) The flow is incompressible (ii) the mixture is in a dilute form ($\phi \ll 1$, ϕ is the nanoparticle volume fraction) (iii) nanoparticle and base fluid (Casson fluid) are assumed to be in thermal equilibrium. Under the above assumptions and Boussinesq's approximation, the equations governing Casson nanofluid are:

Continuity equation of MHD Casson Nanofluid

Continuity equation of MHD Casson nanofluid is given as

$$\nabla \cdot v = 0 \quad (57)$$

Momentum Equation of MHD Casson Nanofluid

The rheological equations for an isotropic incompressible flow of a non-Newtonian Casson fluid can be written as (Animasaun, 2015b)

$$\left. \begin{aligned} \tau &= \left(\mu_B + \frac{P_y}{\sqrt{2\pi}} \right) e \text{ when } \pi > \pi_c \\ \tau &= \left(\mu_B + \frac{P_y}{\sqrt{2\pi c}} \right) e \text{ when } \pi < \pi_c \end{aligned} \right\} \quad (58)$$

$$P_y = \frac{\mu_B \sqrt{2\pi}}{\beta}, e = \nabla v + (\nabla v)^T \quad (59)$$

where v is the velocity of nanofluid, τ is the component of stress tensor, P_y is the fluid yield stress, μ_B is the plastic dynamic viscosity of the non Newtonian fluid, β is Casson parameter, e denotes component of the deformation rate, $\pi = ee$ is the product of component of deformation rate with itself, π_c is the critical value of π .

The fluid dynamic viscosity μ in the case of non-Newtonian Casson fluid can be written as

$$\mu = \mu_B + \frac{P_y}{\sqrt{2\pi}} = \mu_B \left(1 + \frac{1}{\beta}\right), \pi > \pi_c \quad (60)$$

and

$$\tau = \mu_B \left(1 + \frac{1}{\beta}\right) e \quad (61)$$

The momentum equation in the presence of nanoparticle volume fraction can be written as (Kuznetsov and Nield, 2010, Oyelakin *et al.*, 2015)

$$\begin{aligned} \rho_f \left(\frac{\partial v}{\partial t} + v \cdot \nabla v \right) = & -\nabla P + \nabla \cdot \tau - \frac{\mu}{k_p} v - \frac{b^* \rho_f}{k_p} v^2 + [\phi \rho_p \\ & + (1-\phi) \{ \rho_f (1-\beta_0 (T-T_\infty)) \}] g + j \times B \end{aligned} \quad (62)$$

where ρ_f is the Casson fluid density, ρ_p is the nanoparticle density, P is the pressure, μ is the dynamic viscosity, v is the velocity, j is the electric current density, B is the magnetic field, g is the acceleration due to gravity, β_0 is the fluid volumetric expansion coefficient, ϕ is the nanoparticle volume fraction, T is the nanofluid temperature, k_p is the permeability of the porosity medium and b^* is the Forchheimer's inertial coefficient.

Energy Equation of MHD Casson Nanofluid

Energy equation involving nanoparticle concentration is given as (Buongiorno, 2006)

$$(\rho C_p)_f \left(\frac{\partial T}{\partial t} + v \cdot \nabla T \right) = -\nabla \cdot q + h_p \nabla \cdot j_p + \tau \cdot e \quad (63)$$

where $(\rho C_p)_f$ is the heat capacity of the base fluid, T is the temperature of the nanofluid, h_p is the specific enthalpy of the nanoparticle material, q is the energy flux relative to the nanofluid velocity v and j_p is the nanoparticle diffusion mass flux. q can be calculated as the sum of the conduction, radiation heat flux and heat flux due to nanoparticle diffusion such that

$$q = -k \nabla T + q'' + h_p j_p \quad (64)$$

where k is the Casson nanofluid thermal conductivity and q'' is the radiation heat flux. Inserting Eq.(1.64) into Eq.(1.63) and on noting that $\nabla \cdot (h_p j_p) \equiv h_p \nabla \cdot j_p + j_p \cdot \nabla h_p$, we get

$$(\rho C_p)_f \left(\frac{\partial T}{\partial t} + v \cdot \nabla T \right) = \nabla \cdot k \nabla T - \nabla q'' - j_p \cdot \nabla h_p + \tau \cdot e \quad (65)$$

by following assumption (iii) above, ∇h_p equals to $(C_p)_p \nabla T$, $(C_p)_p$ is the nanoparticle specific heat capacity.

j_p can be written as the sum of two diffusion terms, Brownian and thermophoretic diffusion, and it is defined as

$$j_p = j_{p,B} + j_{p,T} = -\rho_p D_B \nabla \phi - \rho_p D_T \frac{\nabla T}{T} \quad (66)$$

where D_B and D_T are the Brownian motion and thermophoretic diffusion coefficients respectively. Substituting Eq.(1.66) into Eq.(1.65) the energy equation for Casson nanofluid is found to be

$$\begin{aligned}
(\rho C_p)_f \left(\frac{\partial T}{\partial t} + v \cdot \nabla T \right) = \nabla \cdot k \nabla T - \nabla q''' + (\rho C_p)_p [D_B \nabla \phi \cdot \nabla T \\
+ D_T \frac{\nabla T \cdot \nabla T}{T}] + \tau \cdot e
\end{aligned} \quad (67)$$

Mass Conservation Equation of MHD Casson Nanofluid

The mass conservation equation for the nanoparticle concentration is given as

$$\frac{\partial \phi}{\partial t} + v \cdot \nabla \phi = - \frac{1}{\rho_p} \nabla \cdot j_p \quad (68)$$

Substituting Eq.(1.66) into Eq.(1.68) we arrive at the final form of nanoparticle concentration equation:

$$\frac{\partial \phi}{\partial t} + v \cdot \nabla \phi = \nabla \cdot \left[D_B \nabla \phi + D_T \frac{\nabla T}{T} \right] \quad (69)$$

3 Statement of the Problem

Several studies on electrically conducting nanofluid have been investigated in various fluid flow such as second grade nanofluid (Zuhra *et al.*, 2018), third grade nanofluid (Shehzad *et al.*, 2015, Shah *et al.*, 2018), couple stress nanofluid (Umavathi and Sheremet, 2019, Sithole *et al.*, 2018), Casson nanofluid (Abolbashari *et al.*, 2015, Oyelakin *et al.*, 2016, Khan *et al.*, 2019), etc. It is worth pointing out that few recent works in this area investigated the effect of linear radiation, velocity slip, viscous dissipation, Biot number, Prandtl number and Darcy number. MHD Casson nanofluid flow in the presence of Arrhenius activation energy, binary chemical reaction, non-Darcian porous medium as well as nonlinear radiation has not been given due attention. However, if the fluid velocity is high and the temperature differences within the fluid are sufficiently large, both Darcy and linearized Rosseland radiative heat flux models may not accurately determine the fluid flow and thermal structure.

Due to the aforementioned facts, this work investigates heat and mass transfer of nonlinear radiative MHD Casson nanofluid flow over a stretching, shrinking or static sheet embedded in a non-Darcian porous medium. Moreover, the analysis is carried out under the following configurations: mixed convective, natural convective and forced convective flows. Furthermore, the effect of other parameters on MHD Casson nanofluid flow such as first and second-order velocity slip, thermal slip, mass concentration slip, Activation energy and chemical reaction rate are investigated. Weighted residual method (WRM) is adopted in this work to carry out our analysis.

4 Aim and Objectives

This work aims to study the effect of nonlinear thermal radiation, binary chemical reaction, Arrhenius activation energy, variable viscosity and thermal conductivity on different models of convective heat and mass transfer of non-Darcian MHD dissipative Casson nanofluid flow with slips and convective boundary conditions

The objectives of this study are to:

1. study the flow of a non-Darcian MHD dissipative Casson nanofluid over a stretching or shrinking sheet with multiple slip boundary conditions.

2. investigate the influences of variable viscosity and variable thermal conductivity on a natural convective flow of non-Darcian MHD Casson nanofluid flow with velocity slip and convective boundary conditions.

3. analyze the effects of a binary chemical reaction and Arrhenius activation energy on a forced convective flow of non-Darcian MHD Casson nanofluid in the presence of first-order and second-order velocity slip boundary condition.

5 Scope of the Research

This fluid model considered two-dimensional steady viscous incompressible laminar convective boundary-layer flow of Casson nanofluid over a stretching, shrinking or static sheet in the presence of viscous dissipation, nonlinear radiation and non-Darcian porous medium subjected to various boundary conditions. Temperature dependent thermal conductivity and viscosity are also put into consideration.

6 Significance of the Study

MHD nanofluid (Newtonian or non-Newtonian fluid) manufacturing process is increasing in industries due to their diverse applications in biomaterials (wound treatment, gastric medication and sterilized devices) and engineering (MHD flow meters, generators, pumps, cooling of nuclear reactors, geothermal energy extractors and liquid metal flow control). Casson fluid flow model being a model for blood flow will be very useful in biomedical applications when combined with nanoparticle (magnetic nanoparticle) to form what is known as ferrofluid for the treatment of cancer.

This research will also give insight into the enhancement of fluid thermal conductivity for the importance of heat transfer applications, Automotive applications, electronic applications and so on.

7 Organization of the Thesis

The detail outline of the thesis is as follows:

Chapter 1 starts with the background to the study and followed by the statement of the problem, aim, objectives of the study, significance of the study, the scope of the research, relevant definitions and concept used in the study. In chapter two, the relevant studies related to this work and the methodology used to handle the governing equations of the formulated problems are appropriately reviewed in the literature. Chapter three deals with detail mathematical formulation of the following models: analysis of non-Darcian MHD dissipative Casson nanofluid flow over a stretching or shrinking sheet with multiple slip boundary conditions; Effect of variable thermal conductivity and viscosity on a Casson nanofluid flow with convective heating and velocity slip; and Arrhenius activation energy and binary chemical reaction effects on non-Darcian MHD dissipative Casson nanofluid flow with second order-slip condition. Also, in this chapter, the detail explanations of the method of solution and its advantages over other methods are discussed. Chapter four aims at the application of the method of the solution on the governing equations of the aforementioned models. The obtained results are presented graphically. The various flow, heat, and nanoparticle controlling parameters on velocity profile, temperature profile, nanoparticle volume fraction profile, Nusselt number, and Sherwood number are discussed. Chapter five which is the concluding part of this thesis outlines the summary of the work, contribution to knowledge and recommendation for further studies.

2 REVIEW OF RELATED LITERATURE

1 Introduction

In the past several decades, the study of the non-Newtonian fluids model has gained a lot of attention due to its applications in industries and engineering such as petroleum industry, aerodynamic heating, paper production, coating and polymer processing, hot rolling and so forth. Some materials such as mud, blood, paint, polymers solutions exhibit non-Newtonian fluid characteristics. There is no single model in literature that describes all the properties of non-Newtonian fluids due to the complexity of its physical nature. Casson fluid is one of the examples of non-Newtonian fluid with distinct features. It is a fluid that behaves like an elastic solid as a result of yield stress existing in the constitutive equation (Najeeb and Zehra, 2015). There are different types of Casson fluid. Some behave like plasma such as jelly and blood while some are in a thick viscous form like tomato paste, concentrated fruit juices and shampoo (Najeeb and Zehra, 2015). Despite various application of Casson fluid, it has a low thermal conductivity. However, Casson fluid thermal conductivity can be enhanced by introducing nanoparticles obtained from metals.

Conventional heat and mass transfer fluids (Newtonian or non-Newtonian fluids) are poor heat transfer fluid due to their low thermal conductivity. Fluid thermal conductivity can be enhanced by suspending nanoparticles ($< 100\text{nm}$) obtained from metallic oxides ($\text{Al}_2\text{O}_3, \text{CuO}$), nitride ceramics (AlN, SiN), Carbide ceramics (SiC, TiC), metals ($\text{Cu}, \text{Ag}, \text{Au}$), semiconductors (TiO_2, SiC) or alloyed nanoparticles ($\text{Al}_{70}\text{Cu}_{30}$) into conventional fluid (Mahatha *et al.*, 2015). Nanofluids are very useful in several engineering (Industrial cooling smart fluid, nuclear reactor, brake fluid, cooling of microchips), biomedical applications (nanodrug delivery, cancer therapeutics, nanosurgery, cryopreservation, sensing and imaging) and process industries (paper, chemical, textile and detergent manufacturing, and foods and drinks production) (Mahatha *et al.*, 2015, Kim *et al.*, 2007, Kao *et al.*, 2007, Ma *et al.*, 2006, Shawgo *et al.*, 2002, Bica *et al.*, 2007, Yan and Liu, 2008, Daniel and Astruc, 2004). Several studies in literature concerning the flow of heat and mass transfer of Casson fluid, nanofluid, as well as Casson nanofluid, have been reviewed in the following section.

2 The Review

The rheological model of non-Newtonian fluid (Casson fluid) was originally introduced by Casson in research made on a flow equation for pigment oil suspension of printing ink (Casson, 1959). Bird *et al.* (1983) studied rheology and visco-plastic material of the fluid flow and concluded that Casson model constitutes a plastic fluid model that exhibits shear thinning, yield stress and high shear viscosity. In the work of Rao *et al.* (2013), it is stated that Casson fluid model reduces to the Newtonian fluid at a very high wall shear stress (when the wall shear stress is much greater than the yield stress). Saidulu and Venkata (2016) used numerical method (Keller box method) to study the effect of slip on MHD flow of a Casson fluid over an exponentially stretching sheet in the presence of thermal radiation, heat source/sink, and chemical reaction. It was found that the temperature and concentration profiles increase when the Casson parameter increases, but the reverse was the case for the velocity profile. Unsteady MHD slip flow of a Casson fluid due to a stretching sheet with suction or blowing effect was analyzed by Mahdi (2016). It was observed that increasing the slip parameter increases the fluid flow, and the thermal boundary layer becomes thinner in case of suction or blowing. Nadeem *et.al.* (2013) used the Adomian decomposition method to obtain the solution for MHD boundary layer flow of a Casson fluid over an exponentially shrinking sheet. It is noticed that when the fluid parameter approaches infinity their work reduced to the Newtonian case.

Krishna *et.al.* (2019) investigated MHD boundary layer slip Casson fluid flow over a dissipated stretched cylinder. They found that enhancing Casson parameter results in a decrease in yield stress factor which in turn reduces the fluid velocity and the boundary layer thickness. Also, Venkatesan *et.al.* (2013) carried out a comparison between Herschel Bulkely and Casson fluid model for the pulsatile flow of blood in slender conical arteries with gentle extended beyond the stenosis. It is noticed in the study that the values of the mean velocity and mean flow rate are higher while the value of plug core radius and wall shear stress are lower for the Herschel Bulkley model

The investigation of boundary layer flow involving binary mixture has gained the interest of researchers due to its applications in engineering and technology. The common examples can be found in an emulsion (the dispersion of one fluid within another fluid) and a suspension (spreading of nanoparticle within a fluid). The former is described by dispersion of oil within water, while the latter can be described by the suspension of Al_2O_3 within water. Another example is the addition of polymeric fluid to the base oil fluid to enhance the lubrication properties of mineral oil (Makinde *et.al.*, 2011). A chemical reaction is highly important in chemical processing industries to transform cheap raw materials into high-value products. This can be achieved by designing a reactor that brings reactants into close contact by providing suitable temperature and concentration fields for the duration of the process (Makinde *et.al.*, 2011). Boundary-layer flow involving binary mixture was first studied by (Bestman, 1990). The method used was the perturbation method and the effect of activation energy was examined.

Activation energy was first introduced by Svante Arrhenius in 1899. It (activation energy) was defined as energy that must be overcome for a chemical reaction to occur. In other words, activation energy is the minimum energy required to start a chemical reaction. The application of activation energy can be found in geothermal or oil reservoir engineering and mechanics of water and oil emulsions (Shafique *et al.*, 2016). Khumar *et.al.* (2018) investigated the effect of Arrhenius activation energy on chemically reactive MHD Carreau fluid flow using a numerical method. They found that the destructive binary reaction parameter favorably enhanced the species concentration. Earlier on, Shafique *et.al.* (2016) studied the boundary layer flow of Maxwell fluid in a rotating frame with binary chemical reaction and activation energy. It was observed that the solute concentration in the binary mixture is proportional to both rotating and activation energy parameters. More details on binary chemical reaction can be found in (Maleque 2013, Chetteti and Chuka, 2017, Awad *et.al.*, 2014, Abbas *et.al.*, 2016 and Mustafa *et.al.*, 2016).

The recent nanofluid models that are widely used by researchers are Buongiorno's model and Tiwari-Das's model. The former incorporates Brownian motion and thermophoretic effects (Buongiorno, 2006), while the latter takes into account the solid volume fraction (Tiwari and Das, 2007). Maiga *et al.* (2005) analyzed the laminar forced convective flow of nanofluids (Al_2O_3 - water and Al_2O_3 -Ethylene Glycol). It was concluded that nanofluid improved fluid's thermal performance as there is an increase in nanoparticle concentration. It was also reported that nanofluid, Al_2O_3 -Ethylene Glycol has a better performance than Al_2O_3 -water. Yurong *et al.* (2007) investigated heat transfer and flow behaviors of TiO_2 nanofluid flowing upward in a vertical pipe. It was observed that the nanoparticle suspension enhanced fluid thermal conductivity. It was also found that the effective thermal conductivity of nanofluid for the smaller size of nanoparticle is larger than that of the bigger size at a given concentration.

Xiao *et al.* (2009) provided a fractal model to calculate the heat flux of nanofluids. An excellent agreement between the model prediction and experimental data was obtained. The heat

transfer and laminar flow behaviors of Al_2O_3 -water nanofluid in rectangular microchannels was experimentally studied by Jung *et al.*, 2006. They discovered that the suspension of a 1.8 percent nanoparticle concentration in nanofluid improved the heat transfer coefficient by 32 percent compared with water. Oyelakin *et al.* (2016) used spectral relaxation method to study unsteady Casson nanofluid flow over a stretching sheet with thermal radiation subjected to convective and slip boundary condition. The radiation term used is a linearized Rossland radiation term, and the porous medium is not considered. They found that an increase in fluid unsteadiness has a significant influence on fluid flow, temperature and concentration profiles. More research works on nanofluid can also be found in (Akbari *et al.*, 2016, Sekrani and Poncet, 2016, Hussain *et al.*, 2016, Fetecau *et al.*, 2017, Zhao and Li, 2007).

The combination of nanofluid and porous medium enhances heat transfer characteristics (Uddin *et al.*, 2016a). Darcy's empirical flow model represents a simple linear relationship between flow rate and pressure drop in a porous medium. Any deviation from the Darcy flow concept is known as the non-Darcy flow (Animasaun, 2015b). However, if the velocity of the fluid flow is high, the relationship between fluid flow and pressure drop becomes nonlinear. This is due to the presence of inertial effect as a result of additional friction observed for high velocity flow (Uddin *et al.*, 2016b). This type of flow where the inertial effect is put into consideration is known as the Forchheimer flow model (Forchheimer, 1901). The porous medium is widely used in technical environmental applications such as filtration, thermal insulation (fabric and building materials), groundwater hydrology, oil reservoir formations, combustion, foodstuff biomechanics and various form of heat exchangers (Kole and Dey, 2010).

Runge-Kutta Fourth-Order method was used by Mishra *et al.* (2017) to study MHD stagnation-point flow past over a stretching sheet in the presence of Non-Darcy porous medium and heat source/sink. It was observed that the presence of a porous matrix decelerates the velocity profile while enhancing the temperature profile. Uddin *et al.* (2015) discussed radiative convective nanofluid flow past a stretching/shrinking sheet with slip effects using Runge-Kutta-Fehiberg Fourth-Fifth-Order. Newtonian fluid was considered. Also, the nonlinear model was assumed for the radiation and Darcy model was incorporated for the porous medium. It was found that increasing Darcy number enhances the flow due to a decrease in bulk porous medium resistance.

The fluid flow on the micro/nanoscale differs significantly from the macroscale flow. In order to incorporate the fluid flow at micro/nanoscale, the conventional no-slip boundary conditions at the fluid-solid interface must be replaced by the slip boundary conditions. Many research works on the analysis of the flow of heat and mass transfer at micro and nanoscale level have been carried out due to the wide application in micro-electro-mechanical systems (MEMS) involving slip boundary conditions (Rashidi and Mehr, 2012). This flow is applicable in heat exchangers, physiological flows, drying process and electronic cooling. According to Adesanya (2014), it is known that when Knudsen number (Kn) is zero the no-slip condition holds while if Kn is less than 0.001 the continuum flow assumption holds. However, when Kn lies between the range [0.001, 0.1] the flow is termed slip and in this regime, the standard Navier Stokes, energy as well as concentration equations hold by taking into account velocity slip, temperature slip, and mass slip.

Wen-Ming *et al.* (2012) discussed various slip models for gas microflow based on different theoretical, computational and experimental studies. Bhargava *et al.* (2012) reported that increasing the wall slip factor strongly decelerates the radial and azimuthal flow in Von-Karman swirling convection. Temperature-dependent fluid properties and slip effect on

microspherical particles were analyzed by Behzad *et al.* (2015). It was discovered that a large slip on the wall increases convection along the surface while a large temperature jump reduces the heat transfer. It was further emphasized that if temperature jump is neglected, it will lead to an overestimation of the heat transfer rate. In the above literature, linear Navier's slip boundary condition was incorporated. This condition (Navier's slip) performs well at sufficiently low shear rates. However, Navier's slip condition breaks down as the slip length increases rapidly (Gad-El-Hak, 2006). Hence, the concept of second-order slip boundary condition is considered.

Beskok and Karniadakis (1996) studied rarefaction and compressibility effects in gas microflows. The obtained results were compared with experimental data. They observed that the calculated values by using second-order velocity slip boundary conditions are close to the experimental values. Amirsom *et al.* (2019) investigated second-order slip and variable thermophysical properties effects on melting dissipative MHD bioconvection nanofluid flow. They found that the presence of second-order slip parameter weakly decreases motile micro-organism species boundary layer thickness. Second-order slip MHD flow and heat transfer of nanofluids with thermal radiation and the chemical reaction was investigated by Zhu *et al.* (2015). They reported that the thermal boundary layer thickness is found to be a decreasing function of the second-order slip parameter. Various aspect of flow, heat and mass transfer of nanofluids involving slip regimes have been investigated in (Zheng *et al.*, 2012, Ibrahim *et al.*, 2013, Mansur *et al.*, 2014, Uddin *et al.*, 2016a and Oyelakin *et al.*, 2016).

The physical properties of fluids may change significantly when exposed to temperature. For the fluids, which are important in the theory of lubrication, the heat generated by the internal friction and the corresponding rise in temperature does affect the viscosity and thermal conductivity of the fluid (Jawali and Chamkha, 2015). According to Batchelor (1987), Anyakoha and Meyers *et.al.* (2006), viscosity and thermal conductivity are most sensitive to temperature rises. The increase of temperature leads to a local increase in the transport phenomena by reducing the viscosity across the momentum boundary layer and also the heat transfer rate at the wall is also affected greatly (Animasaun, 2015b). Afify and Bazid (2014) adopted Dybbs (1987) and Chiam (1998) models for temperature dependent viscosity and thermal conductivity respectively to study the effect of variable properties on the natural convective boundary layer flow of a nanofluid past a vertical plate. It was discovered that an increase in variable viscosity parameter leads to a decrease in Nusselt and Sherwood numbers whereas opposite results were obtained for the skin-friction.

Jawali and Chamkha (2015) analyzed the combined effect of variable viscosity and thermal conductivity on free convection flow of a viscous fluid in a vertical channel. He used Attia's (2006) model for both temperature dependent viscosity and thermal conductivity. It was found that the fluid flow and heat transfer increase as the variable viscosity parameter increases, while an increase in variable thermal conductivity parameter reduces both the heat transfer and the fluid flow. Bagai and Nishad (2014) used a numerical method (i.e shooting method) to study the effect of temperature dependent viscosity on natural convective boundary layer flow over a horizontal plate embedded in a nanofluid saturated porous medium. The viscosity of the fluid is assumed to vary exponentially with temperature. The thermal conductivity was assumed constant and radiation term is neglected. It was observed that the heat and mass transfer rate increase as the viscosity parameter increases.

Casson fluid flow with variable thermo-physical properties along exponentially stretching sheet with suction and exponentially decaying internal heat generation using the homotopy analysis method (HAM) was studied by Animasaun *et al.* (2016). The radiation term used was

linear and the effect of porosity was not put into consideration. They found that an increase in the variable plastic dynamic viscosity parameter of Casson fluid leads to an increase in velocity profile and a decrease in temperature profile throughout the boundary layer. Animasaun (2015b) investigated the effect of variable viscosity and thermal conductivity on MHD free convective flow in a non-Darcian porous medium with suction and n^{th} order chemical reaction. Radiation term was not considered in this model. He discovered that the presence of the Casson fluid parameter, variable fluid viscosity, and thermal conductivity reduce temperature and concentration profiles. Amirsom *et al.* (2019) used a numerical method to show that increasing temperature-dependent viscosity parameter and Eckert number increase temperature but reduce nanoparticle volume fraction. Effect of variable properties on a couple stress fluid flow in a vertical channel was studied by Jangilis *et al.* (2019). They found that both the velocity and temperature increase with an increase in the viscosity parameter, whereas they decrease with increasing variable thermal conductivity parameter.

Most of the differential equations governing the fluid flow, heat and mass transfer are nonlinear. Therefore, the solutions may not exist in closed forms. It is then required to employ a numerical method. There are several numerical methods such as shooting method, finite difference method, finite element method and so on that are being used by researchers to provide solutions to these nonlinear differential equations. Weighted residual method (WRM) is one of the numerical methods used to obtain an approximate solution to physical engineering problems. This method (WRM) is widely used due to its accuracy and efficiency. Its solution does not require much computation time. Another advantage it has over other numerical methods is program simplicity and less computer run time. There are sub-methods under weighted residual method (WRM) which include collocation, least square, sub-domain, Galerkin and moment methods (see Finlayson, 1966).

Many researchers have used the weighted residual method to provide solutions to physical problems involving differential equations. To mention a few, Vaveri *et al.* (2012) investigated the feasibility of applying an orthogonal collocation method to solve the diffusivity equation in the radial transient flow system. Adewumi *et al.* (2016) employed Laplace weighted residual method to obtain solutions for problems with a semi-infinite domain. The results obtained are in good agreement with the exact solution. More work on the weighted residual method can be found in (Aregbesola, 2003, Odejide and Aregbesola, 2010, Leidari *et al.*, 2015, Oderinu and Aregbesola, 2012a, Oderinu and Aregbesola, 2012b, Gbasemi *et al.*, 2015).

3 Appraisal of the Literature Review

It is noted, at this juncture, from the above reviewed literature that most existing literature either neglected radiation and porous medium, or considered linear radiation effect and Darcian porous medium on nanofluid. In other words, to the best of my knowledge, no previous work takes into account the combined effect of nonlinear radiation and non Darcian porous medium. Hence, to address the lack of research in this direction, this study proposed to analyze the effect of nonlinear Rosseland radiation effect on convective non-Darcian MHD dissipative Casson (non-Newtonian) nanofluid flow past a stretching/shrinking sheet with slip and convective heating boundary conditions.

3 METHODOLOGY

1 Introduction

In this thesis, the following three main distinct models are studied.

2 Mathematical Formulation of the Models

2.1 Model 1: Investigation of the Non-Darcian MHD Dissipative Casson Nanofluid Flow over a Stretching or Shrinking Sheet with Multiple Slip Boundary Conditions

Two dimensional steady viscous incompressible laminar mixed convective boundary-layer flow of a Casson nanofluid over a stretching/shrinking sheet with slips is considered (see Figure 3.1 below). The presence of thermophoresis, Brownian motion, and viscous dissipation are observed in heat and mass transfer models. It is assumed that the surface of the sheet is coated to accommodate slip flow. It is also assumed that a magnetic field $B(\bar{x})$ of strength B_0 is applied in a transverse direction towards the flow. The magnetic Reynolds number is considered to be very small compared to applied magnetic field so as to neglect induced magnetic field effect. The electric conductivity is assumed to vary with velocity, that is $\sigma = \sigma_0 \bar{u}$. The fluid properties are assumed to be constant. The additional viscous dissipation terms in slow motion natural convective flow are considered to be small and therefore negligible. Under these assumptions together with boundary layer and Boussinesq approximation, the governing equations for Casson nanofluid flow in non-Darcian Forchheimer porous medium are (Buongiorno, 2006 and Oyelakin *et al.*, 2016):

$$\bar{u} = \lambda \bar{u}_w + \bar{u}_{slip}, T = T_w + T_{slip}, C = C_w + C_{slip}$$

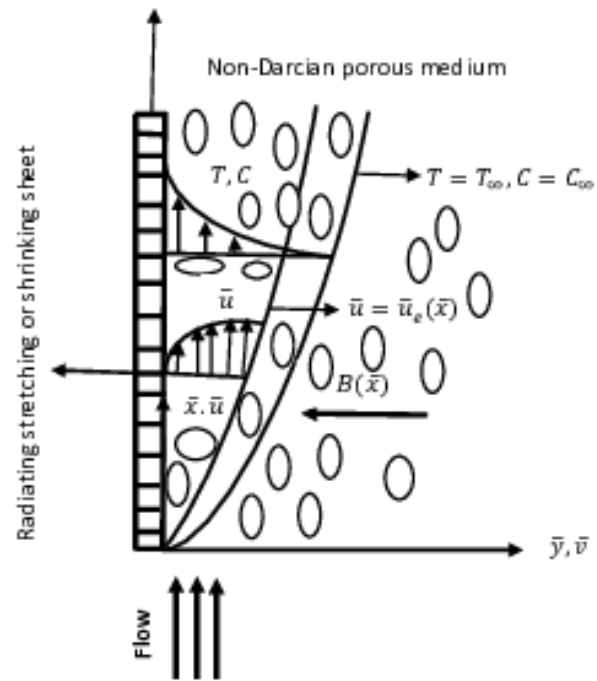


Figure 23: Flow geometry for model 1

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \quad (70)$$

$$\begin{aligned} \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = & K \bar{u}_e \frac{d \bar{u}_e}{d x} + \frac{\mu_B}{\rho_f} \left(1 + \frac{1}{\beta} \right) \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\sigma_0 B_0^2(\bar{x})}{\rho_f} (\bar{u}^2 - (K \bar{u}_e)^2) \\ & + \frac{g}{\rho_f} [(1 - C_\infty)(T - T_\infty) \rho_{f_\infty} \beta_0 - (\rho_p - \rho_{f_\infty})(C - C_\infty)] \\ & - \frac{\mu_B}{\rho_f k_p} \left(1 + \frac{1}{\beta} \right) (\bar{u} - K \bar{u}_e) - \frac{b^*}{k_p} (\bar{u}^2 - (K \bar{u}_e)^2) \end{aligned} \quad (71)$$

$$\begin{aligned} \bar{u} \frac{\partial T}{\partial x} + \bar{v} \frac{\partial T}{\partial y} = & \alpha \frac{\partial^2 T}{\partial y^2} + \tau \left[D_B \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} + \frac{D_T}{T_\infty} \left(\frac{\partial T}{\partial y} \right)^2 \right] \\ & + \frac{\mu_B}{(\rho C_p)_f} \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial \bar{u}}{\partial y} \right)^2 - \frac{1}{(\rho C_p)_f} \frac{\partial q_r}{\partial y} \end{aligned} \quad (72)$$

$$\bar{u} \frac{\partial C}{\partial x} + \bar{v} \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty} \frac{\partial^2 T}{\partial y^2} \quad (73)$$

subjected to the following appropriate boundary conditions

$$\left. \begin{aligned} \bar{u} = \lambda \bar{u}_w + \bar{u}_{slip}, T = T_w + D_1 \frac{\partial T}{\partial y}, C = C_w + E_1 \frac{\partial C}{\partial y}, \bar{v} = 0 \text{ at } \bar{y} = 0 \\ \bar{u} \rightarrow K \bar{u}_e, T \rightarrow T_\infty, C \rightarrow C_\infty \text{ as } \bar{y} \rightarrow \infty \end{aligned} \right\} \quad (74)$$

where $\bar{u}$ and $\bar{v}$ are the velocity components in $\bar{x}$ and $\bar{y}$ direction respectively, $\bar{u}_e$ is the external velocity, T and C represents fluid temperature and nanoparticle concentration respectively, T_w and C_w is the fluid temperature and nanoparticle concentration at the wall respectively, T_∞ and C_∞ is the ambient temperature and ambient nanoparticle concentration respectively, β_0 is the volumetric thermal expansion coefficient, $\tau = \frac{(\rho C_p)_p}{(\rho C_p)_f}$ is the ratio of nanoparticle heat capacity to the base fluid heat capacity, $\alpha = \frac{k}{(\rho C_p)_f}$ is the fluid thermal

diffusivity, ρ_f is the density of the base fluid, k is the thermal conductivity, $\sigma = \sigma_0 \bar{u}$ is the variable electric conductivity, σ_0 is the constant electric conductivity, $B(\bar{x}) = \frac{B_0}{\bar{x}^2}$ is the local magnetic field, B_0 is the constant magnetic field, ρ_p is the density of nanoparticle, β_0 is the volumetric thermal expansion coefficient, g is the acceleration due to gravity, k_p is the permeability of the porous medium, $\bar{u}_{slip} = N_1 \frac{\mu_B}{\rho_f} (1 + \frac{1}{\beta}) (\frac{\partial \bar{u}}{\partial y})$ is the slip velocity, $\nu = \frac{\mu_B}{\rho_f} (1 + \frac{1}{\beta})$ is the kinematic viscosity, N_1 is the velocity slip factor, D_1 is the thermal slip factor, E_1 is the solutal slip factor, b^* is the Forchheimer's inertia coefficient, D_B is the Brownian diffusion coefficient, D_T is the thermophoresis diffusion coefficient, λ is the stretching/shrinking sheet parameter, $\lambda > 0$ represents a stretching sheet, $\lambda < 0$ represents a shrinking sheet, $\lambda = 0$ stands for a static sheet. For a mixed convective flow where thermal buoyancy, concentration buoyancy and pressure gradient terms are present, the flow is a mixed convective flow when constant $K = 1$ while when $K = 0$ the flow is a natural convective flow. Rosseland radiative heat flux approximation given in Eq. 3.6 is considered in the energy equation by assuming that the fluid is gray, radiatively absorbing and emitting but non scattering (Uddin *et al.*, 2015).

$$q_r = -\frac{4\sigma_1}{3k_1} \frac{\partial T^4}{\partial y^4} \quad (75)$$

where σ_1 is the Stefan-Boltzmann constant, and k_1 is the Rossland mean absorption coefficient. Introducing the following boundary layer variables (Uddin *et al.* 2015)

$$\left. \begin{aligned} x = \frac{\bar{x}}{L}, y = \frac{\bar{y} Ra^{\frac{1}{4}}}{L}, u = \frac{\bar{u} L}{\alpha Ra^{\frac{1}{2}}}, v = \frac{\bar{v} L}{\alpha Ra^{\frac{1}{4}}}, \theta = \frac{T - T_\infty}{T_w - T_\infty}, \phi = \frac{C - C_\infty}{C_w - C_\infty}, \\ u_e = \frac{\bar{u}_e L}{\alpha Ra^{\frac{1}{2}}}, u_w = \frac{\bar{u}_w L}{\alpha Ra^{\frac{1}{2}}}, \end{aligned} \right\} \quad (76)$$

and the stream function ψ defined as

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x} \quad (77)$$

into Eqs. (3.1)–(3.5), we obtained the following dimensionless equations:

$$\begin{aligned}
\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} &= Ku_e \frac{du_e}{dx} + Pr \left(1 + \frac{1}{\beta} \right) \frac{\partial^3 \psi}{\partial y^3} \\
- \frac{\sigma_0 B_0^2}{\rho_f} \left(\left(\frac{\partial \psi}{\partial y} \right)^2 - (Ku_e)^2 \right) &+ Pr(\theta - Nr\phi) \\
- \frac{PrL^2}{k_p \sqrt{Ra}} \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial \psi}{\partial y} - Ku_e \right) - \frac{b^* L}{k_p} \left(\left(\frac{\partial \psi}{\partial y} \right)^2 - (Ku_e)^2 \right) &
\end{aligned} \tag{78}$$

$$\begin{aligned}
\frac{\partial \psi}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} &= \frac{\partial^2 \theta}{\partial y^2} + Nb \frac{\partial \theta}{\partial y} \frac{\partial \phi}{\partial y} + Nt \left(\frac{\partial \theta}{\partial y} \right)^2 + \frac{4}{3N} \frac{\partial}{\partial y} [(1 + \\
(Tr - 1)\theta)^3 \frac{\partial \theta}{\partial y}] &+ \frac{\mu \alpha Ra}{(\rho C_p)_f L^2 (T_w - T_\infty)} \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial^2 \psi}{\partial y^2} \right)^2
\end{aligned} \tag{79}$$

$$Le \frac{\partial \psi}{\partial y} \frac{\partial \phi}{\partial x} - Le \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial y} = \frac{\partial^2 \phi}{\partial y^2} + \frac{Nt}{Nb} \frac{\partial^2 \theta}{\partial y^2} \tag{80}$$

with the corresponding dimensionless boundary conditions

$$\left. \begin{aligned}
\frac{\partial \psi}{\partial y} &= \lambda u_w + N_1 \frac{\mu_B}{\rho_f} \left(1 + \frac{1}{\beta} \right) \frac{Ra^{\frac{1}{4}}}{L} \frac{\partial^2 \psi}{\partial y^2}, \frac{\partial \psi}{\partial x} = 0, \theta = 1 + \frac{Ra^{\frac{1}{4}}}{L} D_1 \frac{\partial \theta}{\partial y}, \\
\phi &= 1 + \frac{Ra^{\frac{1}{4}}}{L} E_1 \frac{\partial \phi}{\partial y} \text{ at } y = 0 \\
\frac{\partial \psi}{\partial y} &\rightarrow Ku_e, \theta \rightarrow 0, \phi \rightarrow 0 \text{ as } y \rightarrow \infty
\end{aligned} \right\} \tag{81}$$

In Eqs. (3.9)–(3.12), we have $Ra = \frac{(1 - C_\infty)g\beta(T_w - T_\infty)L^3}{\alpha\nu}$ is the Rayleigh number based on the characteristic length L , $Pr = \frac{\nu}{\alpha}$ is the Prandtl number, $Nt = \frac{\tau D_T (T_w - T_\infty)}{T_\infty \alpha}$ is the thermophoresis parameter, $Nb = \frac{\tau D_B (C_w - C_\infty)}{\alpha}$ is the Brownian motion parameter, $Le = \frac{\alpha}{D_B}$ is the Lewis number, $Nr = \frac{(\rho_p - \rho_{f_\infty})(C_w - C_\infty)}{\rho_{f_\infty} \beta_0 (T_w - T_\infty)(1 - C_\infty)}$ is the buoyancy ratio parameter and $Tr = \frac{T_w}{T_\infty}$ is

the excess wall temperature ratio parameter, $N = \frac{kk_1}{4\sigma_1 T_\infty^3}$ is the conduction radiation parameter.

The following similarity transformations (Uddin *et al.*, 2012) :

$$\eta = \frac{y}{x^{\frac{1}{4}}}, \psi = x^{\frac{3}{4}} f(\eta), \theta = \theta(\eta), \phi = \phi(\eta), D_1 = x^{\frac{1}{4}}(D_1)_o, b^* = x^{-\frac{1}{2}}(b^*)_o \quad (82)$$

$$N_1 = x^{\frac{1}{4}}(N_1)_o, u_w = x^{\frac{1}{2}}, u_e = x^{\frac{1}{2}}, E_1 = x^{\frac{1}{4}}(E_1)_o, k_p = x^{\frac{1}{2}}(k_p)_o,$$

are then introduced into Eqs. (3.9)–(3.12) to obtain

$$\begin{aligned} & \left(1 + \frac{1}{\beta}\right) f''' + \frac{1}{4Pr} (3ff'' - 2f'^2 + 2K - 4M(f'^2 - K^2) \\ & - 4Fs(f'^2 - K^2)) + \theta - Nr\phi - \frac{1}{D_a} \left(1 + \frac{1}{\beta}\right) (f' - K) = 0 \end{aligned} \quad (83)$$

$$\begin{aligned} & \theta'' + \frac{4}{3N} [(1 + (Tr - 1)\theta)^3 \theta'] + \frac{3}{4} f\theta' + Nb\theta'\phi' + Nt\theta'^2 \\ & + EcPr \left(1 + \frac{1}{\beta}\right) f''^2 = 0 \end{aligned} \quad (84)$$

$$\phi'' + \frac{3}{4} Lef\phi' + \frac{Nt}{Nb} \theta'' = 0 \quad (85)$$

with boundary conditions

$$\left. \begin{aligned} f'(0) &= \lambda + \delta_1 \left(1 + \frac{1}{\beta}\right) f''(0), f(0) = 0, \theta(0) = 1 + b\theta'(0), \\ \phi(0) &= 1 + c\phi'(0) \\ f'(\infty) &\rightarrow K, \theta(\infty) \rightarrow 0, \phi(\infty) \rightarrow 0. \end{aligned} \right\} \quad (86)$$

Similarity independent variable is denoted by η , dimensionless velocity, temperature and nanoparticle volume fraction functions by $f(\eta)$, $\theta(\eta)$ and $\phi(\eta)$ respectively, the constant thermal slip factor by $(D_1)_o$, the constant mass slip factor by $(E_1)_o$, the constant velocity slip factor by $(N_1)_o$ and the constant permeability of the porous media by $(k_p)_o$. Furthermore, in Eqs. (3.14)–(3.17), the prime symbol stands for ordinary differentiation with respect to η ,

$Ec = \frac{u_w \alpha^2 Ra}{(C_p)_f L^2 (T_w - T_\infty)}$ is the Eckert number, $Fs = \frac{(b^*)_o L}{(k_p)_o}$ is the Forchheimer parameter,

$Da = \frac{(k_p)_o \sqrt{Ra}}{L^2}$ is Darcy number, $\delta_1 = \frac{(N_1)_o \mu_b Ra^{\frac{1}{4}}}{\rho_f L}$ is the velocity slip parameter,

$b = \frac{(D_1)oRa^{\frac{1}{4}}}{L}$ is the thermal slip parameter, $c = \frac{(E_1)oRa^{\frac{1}{4}}}{L}$ is the mass slip parameter and $M = \frac{\sigma_0 B_0^2}{\rho_f}$ is the Magnetic Field parameter.

The interest of this study in engineering design quantities is heat transfer rate ($Nu_{\bar{x}}$) and mass transfer rate ($Sh_{\bar{x}}$) defined as follows

$$Nu_{\bar{x}} = \frac{\bar{x}q_w}{k(T_w - T_{\infty})} \text{ and } Sh_{\bar{x}} = \frac{\bar{x}q_m}{D_B(T_w - T_{\infty})} \quad (87)$$

where

$$q_w = -k \left[1 + \frac{16\sigma_1}{3kk_1} T^3 \right] \left(\frac{\partial T}{\partial y} \right)_{\bar{y}=0} \text{ and } q_m = -D_B \left(\frac{\partial C}{\partial y} \right)_{\bar{y}=0} \quad (88)$$

are wall heat and wall mass fluxes respectively. Using Eqs. (3.7) and (3.13) in Eq. (3.18), we have

$$Nur = Ra_{\bar{x}}^{-\frac{1}{4}} Nu_{\bar{x}} = - \left[1 + \frac{4}{3N} (1 + (Tr - 1)\theta(0))^3 \right] \theta'(0) \quad (89)$$

and

$$Shr = Ra_{\bar{x}}^{-\frac{1}{4}} Sh_{\bar{x}} = -\phi'(0) \quad (90)$$

where

$$Ra_{\bar{x}} = \frac{(1 - C_{\infty})g\beta_0(T_w - T_{\infty})\bar{x}^3}{\alpha\nu} \quad (91)$$

is the local Rayleigh number.

2.2 Model 2: Determining the Effects of Variable Thermal Conductivity and Viscosity on Casson Nanofluid Flow with Convective Heating and Velocity Slip

$$\bar{u} = \bar{u}_{slip}, -k(T) \frac{\partial T}{\partial y} = h_f(\bar{x})(T_f - T), C = C_w$$

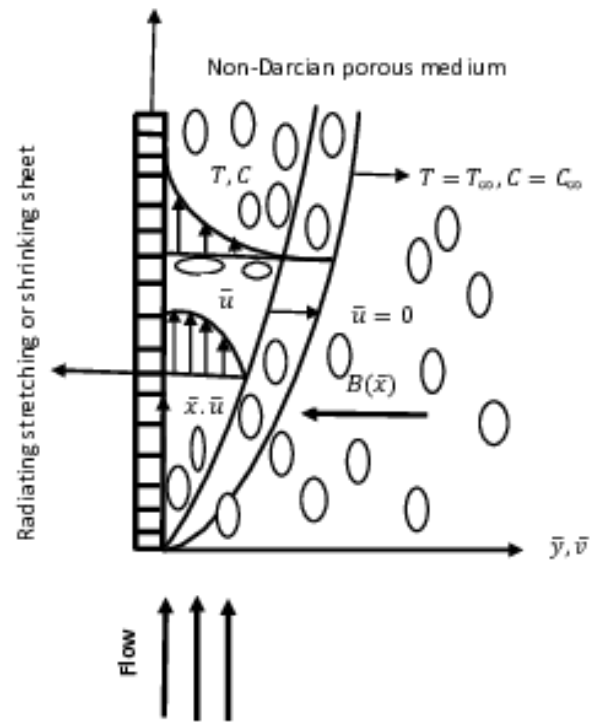


Figure 24: Flow geometry for model 2

In Figure 3.2, the two-dimensional steady laminar natural convective flow of viscous electrically conducting incompressible Casson nanofluid over a vertical flat plate is considered. All the properties of the fluid are assumed to be constant except viscosity and thermal conductivity. It is assumed that the surface of the plate is subjected to convective heating with a temperature T_f . A local magnetic field $B(\bar{x})$ is assumed to be placed in a transverse direction to the fluid flow. The induced magnetic field effect is neglected as a result of small magnetic Reynolds number. The sheet surface is assumed coated to accommodate slip flow. Nanoparticle volume fraction at the wall is C_w , while the temperature and nanoparticle volume fraction far from the wall are T_∞ and C_∞ respectively. The velocity of the fluid in $\bar{x}$ and $\bar{y}$ directions is denoted by $\bar{u}$ and $\bar{v}$ respectively. The buoyancy effect sets in as a result of temperature and concentration gradient between the plate surface and the fluid. Following the above assumptions, Boussinesq and boundary layer approximations, the equations governing the Casson nanofluid flow are written as (Animasaun *et al.*, 2016, Oyelakin *et al.*, 2016 and Uddin *et al.*, 2015)

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \quad (92)$$

$$\begin{aligned} \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} &= \frac{1}{\rho_f} \left(1 + \frac{1}{\beta} \right) \frac{\partial}{\partial \bar{y}} \left(\mu_B(T) \frac{\partial \bar{u}}{\partial \bar{y}} \right) - \frac{\sigma_0 B^2(\bar{x})}{\rho_f} \bar{u}^2 \\ &+ \frac{g}{\rho_f} [(1 - C_\infty)(T - T_\infty) \rho_{f_\infty} \beta_0 - (\rho_p - \rho_{f_\infty})(C - C_\infty)] \\ &- \frac{\mu_B(T)}{\rho_f k_p} \left(1 + \frac{1}{\beta} \right) \bar{u} - \frac{b^*}{k_p} \bar{u}^2 \end{aligned} \quad (93)$$

$$\begin{aligned} \bar{u} \frac{\partial T}{\partial \bar{x}} + \bar{v} \frac{\partial T}{\partial \bar{y}} &= \frac{1}{(\rho C_p)_f} \frac{\partial}{\partial \bar{y}} \left(k(T) \frac{\partial T}{\partial \bar{y}} \right) + \tau \left[D_B \frac{\partial C}{\partial \bar{y}} \frac{\partial T}{\partial \bar{y}} + \frac{D_T}{T_\infty} \left(\frac{\partial T}{\partial \bar{y}} \right)^2 \right] \\ &+ \left(1 + \frac{1}{\beta} \right) \frac{\mu_B(T)}{(\rho C_p)_f} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 - \frac{1}{(\rho C_p)_f} \frac{\partial q_r}{\partial \bar{y}} \end{aligned} \quad (94)$$

$$\bar{u} \frac{\partial C}{\partial \bar{x}} + \bar{v} \frac{\partial C}{\partial \bar{y}} = D_B \frac{\partial^2 C}{\partial \bar{y}^2} + \frac{D_T}{T_\infty} \frac{\partial^2 T}{\partial \bar{y}^2} \quad (95)$$

subjected to the following appropriate boundary conditions

$$\left. \begin{aligned} \bar{u} = \bar{u}_{slip}, \bar{v} = 0, -k(T) \frac{\partial T}{\partial y} = h_f(\bar{x})(T_f - T), C = C_w, \text{ at } \bar{y} = 0 \\ \bar{u} \rightarrow 0, T \rightarrow T_\infty, C \rightarrow C_\infty \text{ as } \bar{y} \rightarrow \infty \end{aligned} \right\} \quad (96)$$

The additional viscous dissipation terms due to the presence of non-Darcian porous medium and Joule heating are assumed to be neglected since they are very small in slow motion natural convective flow considered in this problem. β_0 is the volumetric thermal expansion coefficient, $\tau = \frac{(\rho c)_p}{(\rho c)_f}$ is the ratio of nanoparticle heat capacity to the base fluid heat capacity, ρ_f is the density of the base fluid, $\mu(T)_B$ is the temperature dependent dynamic viscosity, $k(T)$ is the temperature dependent thermal conductivity, $\sigma = \sigma_0 \bar{u}$ is the variable electric conductivity, σ_0 is the constant electric conductivity, $B(\bar{x}) = \frac{B_0}{\bar{x}^2}$ is the local magnetic field, B_0 is the constant magnetic field, ρ_p is the density of nanoparticle, g is the acceleration due to gravity, k_p is the permeability of the porous medium, $\bar{u}_{slip} = N_1 \frac{\mu_b(T)}{\rho_f} (1 + \frac{1}{\beta}) (\frac{\partial \bar{u}}{\partial y})$ is the slip velocity, N_1 is the slip factor, $h_f(\bar{x})$ is the heat transfer coefficient, b^* is the Forchheimer's inertia coefficient, D_B is the Brownian diffusion coefficient, D_T is the thermophoresis diffusion coefficient.

Radiative heat flux is considered by assuming that the fluid is gray, radiatively absorbing and emitting but non-scattering (Uddin *et al.*, 2015). The radiative heat flux follows Rossland approximation and is given as

$$q_r = -\frac{4\sigma_1}{3k_1} \frac{\partial T^4}{\partial y} \quad (97)$$

where σ_1 is the Stefan-Boltzmann constant, and k_1 is the Rossland mean absorption coefficient.

The assumed variable plastic dynamic viscosity and thermal conductivity used for the non-Newtonian fluid are (Jawali and Chamkha, 2015)

$$\mu_B(T) = \mu_B^* e^{-h_1(T-T_\infty)} \text{ and } k(T) = k^* e^{-h_2(T-T_\infty)} \quad (98)$$

Thermal conductivity varies linearly with temperature in the range of 0 to 400°F (Savvas *et al.*, 1994); therefore, variable thermal conductivity is approximated as

$$k(T) = k^* e^{-h_2(T-T_\infty)} \approx k^* (1 - h_2(T - T_\infty)) \quad (99)$$

where μ_B^* is the constant value of the coefficient of viscosity far from the sheet, k^* is the constant value of the coefficient of thermal conductivity far from the sheet, h_1 and h_2 are the empirical constants. The constants h_1 and h_2 ($0 < h_1, h_2 \ll 1$) may be positive values for fluids and negative values for gases (Jangilis *et al.*, 2019).

Using the following boundary layer variables

$$x = \frac{\bar{x}}{L}, y = \frac{\bar{y}Ra^{\frac{1}{4}}}{L}, u = \frac{\bar{u}L}{\alpha^*Ra^{\frac{1}{2}}}, v = \frac{\bar{v}L}{\alpha^*Ra^{\frac{1}{4}}}, \theta = \frac{T - T_{\infty}}{T_f - T_{\infty}}, \phi = \frac{C - C_{\infty}}{C_w - C_{\infty}} \quad (100)$$

and the stream function ψ defined as

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}$$

in Eqs. (3.23)-(3.30), the following dimensionless equations are obtained:

$$\begin{aligned} \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} &= Pr_{\infty} \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial^3 \psi}{\partial y^3} e^{-\gamma \theta} + \frac{\partial^2 \psi}{\partial y^2} \frac{\partial}{\partial y} e^{-\gamma \theta} \right) \\ &- \frac{\sigma_0 B_0^2}{\rho_f} \left(\frac{\partial \psi}{\partial y} \right)^2 + Pr_{\infty} (\theta - Nr \phi) - \frac{Pr_{\infty} L^2}{k_p \sqrt{Ra}} \left(1 + \frac{1}{\beta} \right) \frac{\partial \psi}{\partial y} e^{-\gamma \theta} \\ &- \frac{b^* L}{k_p} \left(\frac{\partial \psi}{\partial y} \right)^2 \end{aligned} \quad (101)$$

$$\begin{aligned} \frac{\partial \psi}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} &= (1 - \partial \theta) \frac{\partial^2 \theta}{\partial y^2} - \partial \left(\frac{\partial \theta}{\partial y} \right)^2 + Nb \frac{\partial \theta}{\partial y} \frac{\partial \phi}{\partial y} + Nt \left(\frac{\partial \theta}{\partial y} \right)^2 \\ &+ \frac{4}{3N} \frac{\partial}{\partial y} \left[(1 + (Tr - 1)\theta)^3 \frac{\partial \theta}{\partial y} \right] + \frac{\nu^* \alpha^* Ra}{(C_p)_f L^2 (T_w - T_{\infty})} \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial^2 \psi}{\partial y^2} \right)^2 e^{-\gamma \theta} \end{aligned} \quad (102)$$

$$Le \frac{\partial \psi}{\partial y} \frac{\partial \phi}{\partial x} - Le \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial y} = \frac{\partial^2 \phi}{\partial y^2} + \frac{Nt}{Nb} \frac{\partial^2 \theta}{\partial y^2} \quad (103)$$

with the corresponding dimensionless boundary conditions

$$\left. \begin{aligned} \frac{\partial \psi}{\partial y} &= N_1 \frac{\mu_B^*}{\rho_f} e^{-\gamma \theta} \left(1 + \frac{1}{\beta} \right) \frac{Ra^{\frac{1}{4}}}{L} \frac{\partial^2 \psi}{\partial y^2}, \frac{\partial \psi}{\partial x} = 0, \frac{\partial \theta}{\partial y} = \frac{-Lh_f(x)}{(1 - \partial \theta)k^* Ra^{\frac{1}{4}}} (1 - \theta), \\ \phi &= 1 \text{ at } y = 0 \\ \frac{\partial \psi}{\partial y} &\rightarrow 0, \theta \rightarrow 0, \phi \rightarrow 0 \text{ as } y \rightarrow \infty \end{aligned} \right\} \quad (104)$$

In Eqs. (3.32)-(3.35), $Ra_L = \frac{(1 - C_{\infty})g\beta_0(T_f - T_{\infty})L^3}{\alpha^* \nu^*}$ is the Rayleigh number based on the

characteristic length L , $\alpha^* = \frac{k^*}{(\rho C_p)_f}$ is the thermal diffusivity of the fluid far from the sheet,

$\nu^* = \frac{\mu_B^*}{\rho_f} \left(1 + \frac{1}{\beta} \right)$ is the kinematic viscosity of the fluid far from the sheet, $Pr_{\infty} = \frac{\nu^*}{\alpha^*}$ is the ambient

Prandtl number, $Nt = \frac{\tau D_T (T_f - T_\infty)}{T_\infty \alpha^*}$ is the thermophoresis parameter, $Nb = \frac{\tau D_B (C_w - C_\infty)}{\alpha^*}$ is the Brownian motion parameter, $Le = \frac{\alpha^*}{D_B}$ is the Lewis number, $Nr = \frac{(\rho_p - \rho_{f_\infty})(C_w - C_\infty)}{\rho_{f_\infty} \beta_0 (T_f - T_\infty)(1 - C_\infty)}$ is the buoyancy ratio parameter and $Tr = \frac{T_f}{T_\infty}$ is the excess wall temperature ratio parameter, $N = \frac{k^* k_1}{4\sigma_1 T_\infty^3}$ is the conduction radiation parameter, $\gamma = h_1(T_f - T_\infty)$ is the temperature dependent viscosity parameter and $\delta = h_2(T_f - T_\infty)$ is the temperature dependent thermal conductivity parameter.

The following similarity transformations Uddin *et al.*, 2015:

$$\eta = \frac{y}{x^{\frac{1}{4}}}, \psi = x^{\frac{3}{4}} f(\eta), \theta = \theta(\eta), \phi = \phi(\eta), D_1 = x^{\frac{1}{4}}(D_1)_o, \quad (105)$$

$$N_1 = x^{\frac{1}{4}}(N_1)_o, k_p = x^{\frac{1}{2}}(k_p)_o, h_f(x) = x^{\frac{1}{4}}(h_f)_o, b^* = x^{-\frac{1}{2}}(b^*)_o$$

are also introduced in Eqs. (3.32)-(3.35) to obtain

$$\left(1 + \frac{1}{\beta}\right)(e^{-\gamma\theta} f''' - \gamma e^{-\gamma\theta} f''\theta') + \frac{1}{4Pr_\infty}(3ff'' - 2f'^2 - 4Mf'^2 - 4Fsf'^2) \\ + \theta - Nr\phi - \frac{1}{Da}\left(1 + \frac{1}{\beta}\right)e^{-\gamma\theta} f' = 0 \quad (106)$$

$$(1 - \delta\theta)\theta'' - \delta(\theta')^2 + \frac{4}{3N}[(1 + (Tr - 1)\theta)^3\theta'] + \frac{3}{4}f\theta' + Nb\theta'\phi' + Nt\theta'^2 \\ + EcPr_\infty\left(1 + \frac{1}{\beta}\right)e^{-\gamma\theta} f''^2 = 0 \quad (107)$$

$$\phi'' + \frac{3}{4}Le f\phi' + \frac{Nt}{Nb}\theta'' = 0 \quad (108)$$

with boundary conditions

$$\left. \begin{aligned} f'(0) &= \delta_1 e^{-\gamma\theta} \left(1 + \frac{1}{\beta}\right) f''(0), f(0) = 0, \theta'(0) = \frac{-Bi}{(1 - \delta\theta(0))} (1 - \theta(0)), \\ \phi(0) &= 1 \\ f'(\infty) &\rightarrow 0, \theta(\infty) \rightarrow 0, \phi(\infty) \rightarrow 0 \end{aligned} \right\} \quad (109)$$

We denoted similarity independent variable by η , dimensionless velocity, temperature and nanoparticle volume fraction functions by $f(\eta)$, $\theta(\eta)$ and $\phi(\eta)$ respectively, the heat transfer coefficient constant factor by $(h_f)_o$, the velocity slip constant factor by $(N_1)_o$, the constant

permeability of the porous media by $(k_p)_o$ and the constant inertial Forchheimer coefficient by $(b^*)_o$. The prime symbol stands for ordinary differentiation with respect to η ,

$$Ec = \frac{\alpha^{*2} Rax^{\frac{1}{2}}}{(C_p)_f L^2 (T_f - T_\infty)} \text{ is the local Eckert number, } Fs = \frac{(b^*)_o L}{(k_p)_o} \text{ is the Forchheimer parameter,}$$

$$Da = \frac{(k_p)_o \sqrt{Ra}}{L^2} \text{ is Darcy number, } \delta_1 = \frac{(N_1)_o \mu_B^* R_a^{\frac{1}{4}}}{\rho_f L} \text{ is the velocity slip parameter, } Bi = \frac{(h_f)_o L}{k^* R_a^{\frac{1}{4}}}$$

is Biot number, $M = \frac{\sigma_0 B_0^2}{\rho_f}$ is the Magnetic Field parameter, $\gamma = a^*(T_f - T_\infty)$ is the variable viscosity parameter and $\delta = b(T_f - T_\infty)$ is the variable thermal conductivity parameter.

Variable Prandtl Number

It is remarked that Prandtl number is a function of specific heat, viscosity and thermal conductivity. Since the thermal conductivity and viscosity vary across the boundary layer, the Prandtl number varies as well. According to Pantokratoras (2004), Rahman *et.al.* (2009) and Mohammad *et.al.* (2010), the assumption of constant Prandtl number inside the boundary layer when the viscosity and thermal conductivity are temperature dependent might not give accurate result. Prandtl number is therefore represented in this present work as

$$Pr = \frac{\nu}{\alpha} = \frac{\nu^* e^{-\gamma\theta(\eta)}}{\alpha^* (1-\delta\theta(\eta))} = \frac{Pr_\infty e^{-\gamma\theta(\eta)}}{(1-\delta\theta(\eta))} \quad (110)$$

Using Eq. (3.41) in Eqs. (3.37) and (3.38), non dimensionless velocity and temperature equations with variable Prandtl number(Pr) becomes, respectively

$$\left(1 + \frac{1}{\beta}\right) (e^{-\gamma\theta} f''' - \gamma e^{-\gamma\theta} f'' \theta') + \frac{e^{-\gamma\theta}}{4Pr(1-\delta\theta)} (3ff'' - 2f'^2 - 4Mf'^2 - 4Fs f'^2) + \theta - Nr\phi - \frac{1}{Da} \left(1 + \frac{1}{\beta}\right) e^{-\gamma\theta} f' = 0, \quad (111)$$

and

$$(1-\delta\theta)\theta'' - \delta(\theta')^2 + \frac{4}{3R} [(1+(Tr-1)\theta)^3 \theta'] + \frac{3}{4} f\theta' + Nb\theta'\phi' + Nt\theta'^2 + EcPr \left(1 + \frac{1}{\beta}\right) (1-\delta\theta) f''^2 = 0 \quad (112)$$

It is noted in Eq. (3.41) that when γ and δ are set to zero, the ambient Prandtl number(Pr_∞) equals to variable Prandtl number(Pr), therefore Eqs. (3.42) and (3.43) reduce to Eqs. (3.37) and (3.38). Also, when $\eta \rightarrow \infty$ and $\theta(\eta)$ equal zero, $Pr_\infty = Pr$ irrespective of the values of γ and δ .

The quantities which are of the engineering interest in this study are Nusselt number (Nu_x) and Sherwood number (Sh_x), and these are defined as (Uddin *et al.*, 2015)

$$Nu_{\bar{x}} = \frac{\bar{x}}{T_f - T_{\infty}} \left[1 + \frac{16\sigma_1}{3k_1 k(T)} \right] \left(\frac{\partial T}{\partial y} \right)_{\bar{y}=0} \quad \text{and} \quad Sh_{\bar{x}} = \frac{\bar{x}}{C_w - C_{\infty}} \left(\frac{\partial C}{\partial y} \right)_{\bar{y}=0} \quad (113)$$

Using Eqs. (3.30), (3.31) and (3.36) in Eq. (3.44), we get

$$Nur = Nu_{\bar{x}} Ra_{\bar{x}}^{-\frac{1}{4}} = - \left[1 + \frac{4}{3N(1-\phi\theta)} (1 + (Tr-1)\theta(0))^3 \right] \theta'(0) \quad (114)$$

and

$$Shr = Sh_{\bar{x}} Ra_{\bar{x}}^{-\frac{1}{4}} = -\phi'(0) \quad (115)$$

where $Ra_{\bar{x}} = \frac{(1-C_{\infty})g\beta_0(T_w - T_{\infty})\bar{x}^3}{\alpha^* \nu^*}$ is the local Rayleigh number

2.3 Model 3: Examining the Influence of Arrhenius Activation Energy and Binary Chemical Reaction on MHD Non-Darcian Casson Nanofluid Flow with Second-order Slip condition

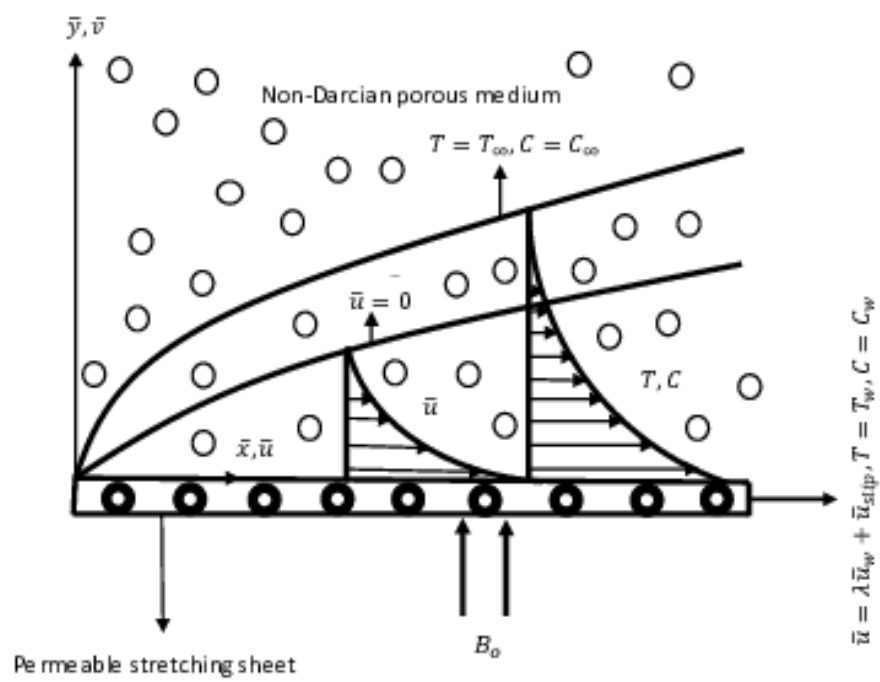


Figure 25: Flow geometry for model 3

Consider two dimensional incompressible MHD chemically reactive Casson nanofluid flow over a stretching sheet in a non-Darcian porous medium (see Fig 3.3 above). The properties of the fluid are assumed to be constant. The fluid temperature and nanoparticle volume fraction are denoted respectively by T and C while T_∞ and C_∞ stand for the ambient temperature and nanoparticle volume fraction respectively. T_w stands for the fluid temperature at the wall and C_w is for the nanoparticle volume fraction at the wall. Magnetic field of strength B_0 is placed normal to the flow direction. The magnetic Reynolds number is assumed to be small compared to the applied magnetic field in order to neglect induced magnetic field effect. Additional viscous dissipation terms due to ohmic heating and inertial force are also incorporated into this model by following (Marudappa and Jawali, 2015, Makinde *et al.*, 2015, Al-Hadhrami *et al.*, 2003). By implementing the above assumptions and boundary layer approximations, the momentum, energy, and nanoparticle concentration equations governing the problem can be written as (Oyelakin *et al.*, 2016, Mohammad *et al.*, 2015)

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = 0 \quad (116)$$

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = \frac{\mu_B}{\rho_f} \left(1 + \frac{1}{\beta}\right) \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\sigma_0 B_0^2}{\rho_f} \bar{u} - \frac{\mu_B}{\rho_f (k_p)_o} \left(1 + \frac{1}{\beta}\right) \bar{u} - \frac{b^*}{(k_p)_o} \bar{u}^2 \quad (117)$$

$$\begin{aligned} \bar{u} \frac{\partial T}{\partial x} + \bar{v} \frac{\partial T}{\partial y} = & \alpha \frac{\partial^2 T}{\partial y^2} + \tau \left[D_B \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} + \frac{D_T}{T_\infty} \left(\frac{\partial T}{\partial y} \right)^2 \right] \\ & + \left(1 + \frac{1}{\beta}\right) \frac{\mu_B}{(\rho C_p)_f} \left(\frac{\partial \bar{u}}{\partial y} \right)^2 + \frac{\sigma B_0^2}{(\rho C_p)_f} \bar{u}^2 + \left(1 + \frac{1}{\beta}\right) \frac{\mu_B}{(\rho C_p)_f (k_p)_o} \bar{u}^2 \\ & + \frac{b^*}{(k_p)_o (C_p)_f} \bar{u}^3 \end{aligned} \quad (118)$$

$$\bar{u} \frac{\partial C}{\partial x} + \bar{v} \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty} \frac{\partial^2 T}{\partial y^2} - Kr(C - C_\infty) \left(\frac{T}{T_\infty} \right)^m e^{\frac{-E_a}{\kappa T}} \quad (119)$$

where $\bar{u}$ and $\bar{v}$ are the velocity component in $\bar{x}$ and $\bar{y}$ direction respectively, $\tau = \frac{(\rho C_p)_p}{(\rho C_p)_f}$

is the ratio of nanoparticle heat capacity to the base fluid heat capacity, $\alpha = \frac{k}{(\rho C_p)_f}$ is the fluid

thermal diffusivity, ρ_f is the density of the base fluid, $\nu = \frac{\mu_B}{\rho_f} \left(1 + \frac{1}{\beta}\right)$ is the kinematic viscosity,

k is the thermal conductivity, σ_0 is the constant electric conductivity, B_0 is the constant magnetic field, ρ_p is the density of nanoparticle, g is the acceleration due to gravity, $(k_p)_o$ is the

constant permeability of the porous medium, b^* is the Forchheimer's inertia coefficient, D_B is the Brownian diffusion coefficient, D_T is the thermophoresis diffusion coefficient, λ is the stretching sheet parameter and Kr is the chemical reaction rate. The term $\left(\frac{T}{T_\infty}\right)^m e^{\frac{-E_a}{\kappa T}}$ in eq. (3.50) denotes the modified Arrhenius function in which $\kappa = 8.61 \times 10^{-5} \text{ eV} / K$ represents Boltzmann constant, m is the unitless exponent fitted rate constant ($-1 < m < 1$) and E_a is the activation energy.

The corresponding boundary conditions are

$$\left. \begin{aligned} \bar{u} &= \lambda \bar{u}_w(\bar{x}) + \bar{u}_{slip}, \bar{v} = v_w, T = T_w, C = C_w, \text{ at } \bar{y} = 0 \\ \bar{u} &\rightarrow 0, T \rightarrow T_\infty, C \rightarrow C_\infty \text{ as } \bar{y} \rightarrow \infty \end{aligned} \right\} \quad (120)$$

where $\bar{u}_w = \frac{U_r \bar{x}}{L}$ denotes velocity of the sheet, $\lambda > 0$ denotes the stretching sheet, $\bar{v}_w$ denotes suction/injection velocity and $\bar{u}_{slip}$ denotes velocity slip at the wall. Second order velocity slip used, which is valid for arbitrary small Knudsen number, is given as (Ibrahim and Gamachu, 2019)

$$\begin{aligned} \bar{u}_{slip} &= \frac{\mu_B}{\rho_f} \left(1 + \frac{1}{\beta}\right) \left[\frac{2}{3} \left(\frac{3 - \omega l^2}{\omega} - \frac{3}{2} \frac{1 - l^2}{K_n} \right) \varsigma \frac{\partial \bar{u}}{\partial \bar{y}} \right. \\ &\quad \left. - \left(l^4 + \frac{2}{K_n^2} (1 - l^2) \right) \varsigma^2 \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right] \\ \bar{u}_{slip} &= \frac{\mu_B}{\rho_f} \left(1 + \frac{1}{\beta}\right) \left[(N_1)_o \frac{\partial \bar{u}}{\partial \bar{y}} + (N_2)_o \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right] \end{aligned} \quad (121)$$

where $(N_2)_o < 0$ is the second order velocity slip constant factor and $(N_1)_o > 0$ is the first order velocity slip constant factor, K_n is Knudsen number, $l = \min(\frac{1}{K_n}, 1)$, ω is the momentum accommodation coefficient with $0 \leq \omega \leq 1$, and ς is the molecular mean free path. The following dimensionless variables (Hamad *et al.*, 2012)

$$x = \frac{\bar{x}}{L}, y = \frac{\bar{y} \sqrt{Re}}{L}, u = \frac{\bar{u}}{U_r}, v = \frac{\bar{v} \sqrt{Re}}{U_r}, \theta = \frac{T - T_\infty}{T_w - T_\infty}, \phi = \frac{C - C_\infty}{C_w - C_\infty}, \quad (122)$$

where L is the characteristic length, U_r is the reference velocity, $Re = \frac{U_r L}{\nu}$ is the Reynolds number, and the stream function ψ defined as

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}, \quad (123)$$

were introduced into Eqs. (3.47)-(3.51) to obtain the following dimensionless equations

$$\begin{aligned} \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = & \left(1 + \frac{1}{\beta}\right) \frac{\partial^3 \psi}{\partial y^3} - M \frac{\partial \psi}{\partial y} - Pp \left(1 + \frac{1}{\beta}\right) \frac{\partial \psi}{\partial y} \\ & - \frac{b^* L}{(k_p)_o} \left(\frac{\partial \psi}{\partial y}\right)^2 \end{aligned} \quad (124)$$

$$\begin{aligned} \frac{\partial \psi}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} = & \frac{1}{Pr} \frac{\partial^2 \theta}{\partial y^2} + Nb \frac{\partial \theta}{\partial y} \frac{\partial \phi}{\partial y} + Nt \left(\frac{\partial \theta}{\partial y}\right)^2 \\ & + \frac{U_r^2}{(C_p)_f (T_w - T_\infty)} \left(1 + \frac{1}{\beta}\right) \left(\frac{\partial^2 \psi}{\partial y^2}\right)^2 + \frac{\sigma B_0^2 U_r L}{\rho_f (C_p)_f (T_w - T_\infty)} \left(\frac{\partial \psi}{\partial y}\right)^2 \\ & + \frac{\nu U_r L}{(C_p)_f (k_p)_o (T_w - T_\infty)} \left(1 + \frac{1}{\beta}\right) \left(\frac{\partial \psi}{\partial y}\right)^2 + \frac{b^* U_r^2 L}{(C_p)_f (k_p)_o (T_w - T_\infty)} \left(\frac{\partial \psi}{\partial y}\right)^3 \end{aligned} \quad (125)$$

$$\begin{aligned} Sc \frac{\partial \psi}{\partial y} \frac{\partial \phi}{\partial x} - Sc \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial y} = & \frac{\partial^2 \phi}{\partial y^2} + \frac{Nt}{Nb} \frac{\partial^2 \theta}{\partial y^2} - Sc \sigma^* \phi \cdot \\ & ((Tr-1)\theta+1)^m e^{\frac{-E}{(Tr-1)\theta+1}} \end{aligned} \quad (126)$$

with the corresponding dimensionless boundary conditions

$$\left. \begin{aligned} \frac{\partial \psi}{\partial x} = \frac{\nu_w \sqrt{Re}}{U_r}, \frac{\partial \psi}{\partial y} = \lambda + \frac{\mu_B}{\rho_f} \left(1 + \frac{1}{\beta}\right) \left(\frac{(N_1)_o \sqrt{Re}}{L} \frac{\partial^2 \psi}{\partial y^2} + \frac{(N_2)_o Re}{L^2} \frac{\partial^3 \psi}{\partial y^3} \right), \\ \theta = 1, \phi = 1 \text{ at } y = 0 \\ \frac{\partial \psi}{\partial y} \rightarrow 0, \theta \rightarrow 0, \phi \rightarrow 0 \text{ as } y \rightarrow \infty \end{aligned} \right\} \quad (127)$$

In Eqs. (3.55)-(3.58), $Pr = \frac{\nu}{\alpha}$ stands for Prandtl number, $Nb = \frac{\tau D_B (C_w - C_\infty)}{\nu}$ is the Brownian motion parameter, $Nt = \frac{\tau D_T (T_w - T_\infty)}{\nu T_\infty}$ is the thermophoresis parameter, $Sc = \frac{\nu}{D_B}$ is the Schmidt number, $\sigma^* = \frac{KrL}{U_r}$ is the rate of chemical reaction parameter, $M = \frac{\sigma B_0^2 L}{\rho_f U_r}$ is the magnetic parameter, $E = \frac{E_a}{\kappa T_\infty}$ is the activation energy parameter. $Pp = \frac{\nu L}{U_r (k_p)_o}$ is the porosity parameter.

The following set of similarity transformation (Hamad *et al.*, 2012)

$$\eta = y, \psi = xf(\eta), \theta = \theta(\eta), \phi = \phi(\eta), b^* = (b^*)_o x^{-1} \quad (128)$$

were introduced into Eqs. (3.55)-(3.58) to obtain a set of ordinary nonlinear differential equations

$$\left(1 + \frac{1}{\beta}\right) f'''' + f'f' - f'^2 - Mf' - Pp\left(1 + \frac{1}{\beta}\right) f' - Fsf'^2 = 0 \quad (129)$$

$$\begin{aligned} & \frac{1}{Pr} \theta'' + f\theta' + Nb\theta'\phi' + Nt\theta''^2 + Ec\left(1 + \frac{1}{\beta}\right) f''^2 + EcMf'^2 \\ & + EcPp\left(1 + \frac{1}{\beta}\right) f'^2 + EcFsf'^3 = 0 \end{aligned} \quad (130)$$

$$\phi'' + \frac{Nt}{Nb} \theta'' + Scf\phi' - Sc\sigma^{*2} \phi((Tr-1)\theta+1)^m e^{\frac{-E}{(Tr-1)\theta+1}} = 0 \quad (131)$$

with the boundary conditions

$$\left. \begin{aligned} f(0) = S, f'(0) = \lambda + (1 + \frac{1}{\beta})(\delta_1 f''(0) + \delta_2 f'''(0)), \theta(0) = 1, \phi(0) = 1 \\ f'(\infty) \rightarrow 0, \theta(\infty) \rightarrow 0, \phi(\infty) \rightarrow 0 \end{aligned} \right\} \quad (132)$$

The similarity independent variable is denoted by η , dimensionless velocity, temperature and nanoparticle volume fraction by $f(\eta), \theta(\eta)$ and $\phi(\eta)$ respectively. The constant inertial Forchheimer coefficient is denoted by $(b^*)_o$, the prime simbol, ' , represents ordinary

differentiation with respect to η . $Ec = \frac{u_w}{(C_p)_f(T_w - T_\infty)}$ is the Eckert number and $Fs = \frac{(b^*)_o L}{(k_p)_o}$ is

the Forchheimer parameter. $\delta_1 = \frac{(N_1)_o \mu_B \sqrt{Re}}{\rho_f L} (> 0)$ is the first order velocity slip parameter,

$\delta_2 = \frac{(N_2)_o \mu_B Re}{\rho_f L^2} (< 0)$ is the second order velocity slip parameter and $S = \frac{v_w \sqrt{Re}}{U_r}$ is the suction

($S > 0$) and injection ($S < 0$) parameter

The parameters in which we are interested in this work are Nusselt number, Nu_x , and Sherwood number, Sh_x , defined as

$$Nu_x = \frac{-\bar{x}}{T_w - T_\infty} \left(\frac{\partial T}{\partial y} \right)_{y=0} \quad \text{and} \quad Sh_x = \frac{-\bar{x}}{C_w - C_\infty} \left(\frac{\partial C}{\partial y} \right)_{y=0} \quad (133)$$

Substituting Eqs. (3.53) and (3.59) into Eq. (3.64), we have

$$Nur = Re^{-\frac{1}{2}} Nu_x = -\theta'(0) \quad \text{and} \quad Shr = Re^{-\frac{1}{2}} Sh_x = -\phi'(0) \quad (134)$$

3 Method of Solution

Weighted Residual Method (WRM)

A computational approach is employed to solve the above three boundary value problems (BVP). This method is explicitly explained in the literature (Razaq and Aregbesola, 2012, Oderinu and Aregbesola, 2012b, Odejide and Aregbesola, 2010) and provides accurate results for boundary layer equations. Many linear and nonlinear differential equation problem models in science and engineering such as heat, mass and fluid flow in an infinite region are usually subjected to boundary conditions at infinity. These types of problems are mostly solved numerically using finite difference method, finite element method, shooting method and so on. However, in these aforementioned methods, solutions are obtained at preassigned particular points or the series solutions are valid within the radius of convergent. In this work, weighted residual method is employed to provide numerical solutions to the nonlinear coupled differential equations mentioned in the previous section. In this method, the solutions obtained are valid at every point in the domain. This method is simple and efficient.

Weighted residual method involves seeking an approximate solution to the differential equation of the form

$$L(\phi(x)) + f(x) = 0 \text{ in } D, \quad (135)$$

where $\phi(x)$ is the unknown dependent variable, $f(x)$ is the independent function in domain D and L is the differential operator. An approximate solution

$$\phi(x) = \sum_{i=0}^n a_i \phi_i(x) \quad (136)$$

(where a_i are constants to be determined and $\phi_i(x)$ are linear independent functions) is assumed in such a way that it satisfies the given boundary conditions. Substituting Eq. (3.67) into Eq. (3.66) results to a residual function $R(x, a_i)$ for $i = 0 \cdots n$. $R(x, a_i)$ is minimized as small as possible in the domain D by using one of the following methods.

Partition Method

The whole domain (D) is sub-divided into sub-intervals. The residual function $R(x, a_i)$ is integrated in each interval and then equate to zero.

$$\int_D R(x, a_i) = 0 \quad (137)$$

The integration of $R(x, a_i)$ in interval $[a, b]$, that is $\int_a^b R(x, a_i) dx$ is carried out using Trapezoidal or Simpson rule, while those of type $\int_b^\infty R(x, a_i) dx$ are carried out using Shifted Laguerre formula. The resulting non-linear algebraic equations are solved to obtain the values of a_i

Shifted Gauss-Laguerre Formula

Shifted Gauss-Laguerre formula is applied using the following (Oderinu and Aregbesola, 2012a):

$$\int_b^\infty R(x, a_i) = e^{-b} \sum_{k=1}^j A_k e^{x_k + b} R((x_k, a_i) + b) \quad (138)$$

where the coefficients A_k are defined as (Razaq and Aregbesola, 2012)

$$A_k = \frac{1}{L'_j(x_k)} \int_0^\infty \frac{L_j(x)e^{-x}}{x - x_k} dx = \frac{(j!)^2}{x_k (L'_j(x_k))^2}, \quad (139)$$

and x_k are the roots of j^{th} Laguerre polynomial

$$L_j = e^x \frac{d^j}{dx^j} (e^{-x} x^j). \quad (140)$$

For $j = 10$, for example, the table below shows the values of x_k and corresponding values of A_k .

Table 1: Arguments x_k and coefficients A_k

k	x_k	A_k
	0.1377 93470540492	0.3084 41115765020
	0.7294 54549503171	0.4011 19929155277
	1.8083 42901740317	0.2180 68287611810
	3.4014 33697854960	0.0620 87456098682
	5.5524 96140063418	0.0095 01516975185
	8.3301 52746764144	0.0007 53008388591
	11.843 785837899944	0.0000 28259233496
	16.279 257831377613	4.2493 13985004240 $\times 10^{-7}$
	21.996 585811980830	1.8395 64823966167 $\times 10^{-9}$
	29.920 697012273720	9.9118 27219610436 $\times 10^{-13}$

Collocation Method

In this method, after obtaining the residual by substituting the trial function into the given differential equation, the residual is minimized by setting the integral of the product of $\delta(x - x_j)$ and $R(x, a_i)$ to zero as follows

$$\int_D \delta(x - x_k) R(x, a_i) = R(x_k, a_i) = 0 \quad (141)$$

where x_j are the roots of Laguerre polynomial. Eq. (3.72) leads to a set of non-linear algebraic

equations. The resulting algebraic equations are solved to get the values of the constants a_i

Moment Method

In this method, the residual error is minimized in domain D by setting the integral of product of the weight functions $P_k(x)$ which are polynomials and residual $R(x, a_i)$ over the whole domain to zero for $k \geq 0, n$. That is

$$\int_D P_k(x) R(x, a_i) dx = 0, \quad k = 0, 1, \dots, n. \quad (142)$$

Since the boundary condition ranges from zero to infinity, the Gauss-Laguerre formula is used to integrate each of the equations in (3.73) as follows

$$\int_0^\infty F(x) \approx \sum_{k=1}^j A_k e^{x_k} F(x_k) \quad (143)$$

where $F(x)$ is the products of residual $R(x, a_i)$ and weight function $P_k(x)$. A set of non-linear algebraic equations obtained are now solved to get the values of the constants a_i .

Least-square Method

In the least-square method, the residual function $(R(x, a_i))$ is squared and integrated over the entire domain that is

$$\int_0^\infty R^2(x, a_i) = 2 \int_0^\infty R(x, a_i) \frac{\partial R}{\partial a_i} \quad (144)$$

Eq. (3.75) is then minimized by setting it to zero to have

$$\int_0^\infty R(x, a_i) \frac{\partial R}{\partial a_i} = 0 \quad (145)$$

The weight function for least-square method in Eq. (3.76) is considered as $\frac{\partial R}{\partial a_i}$ for $i = 0 \dots n$.

Eq. (3.76) results to algebraic equations and then solved to obtain the values of constants a_i

Galerkin Method

In the case of Galerkin method, the residual error is minimized in domain D by setting the integral of the product of the weight function $\phi_i(x)$ (which is the same as trial function) and residual function $R(x, a_i)$ over the entire domain D to zero for $i \geq 0, n$. That is

$$\int_D \phi_i(x) R(x, a_i) dx = 0, \quad i = 0, 1, \dots, n. \quad (146)$$

The Gauss-Laguerre formula is now employed to integrate each of the equations to have a set of nonlinear algebraic equations. These equations are solved to obtain the values of constants a_i . Galerkin method is found to be the most accurate of the above mentioned method to minimize residual error (Chakraverty *et al*, 2019). Therefore, Galerkin weighted residual method is adopted in this work.

4 Numerical Solution for Model 1

In order to obtain numerical solution for the model using weighted residual method (WRM), the following trial solutions are first assumed for $f(\eta)$, $\theta(\eta)$ and $\phi(\eta)$ as (Razaq and Aregbesola, 2012)

$$f(\eta) = K\eta + \sum_{i=0}^n a_i e^{-\frac{i\eta}{3}}, \theta(\eta) = \sum_{k=1}^n b_k e^{-\frac{k\eta}{3}} \text{ and } \phi(\eta) = \sum_{k=1}^n c_k e^{-\frac{k\eta}{3}} \quad (147)$$

Substituting Eq. (3.78) into the following boundary conditions in equation (3.17) to have

$$\left(K\eta + \sum_{i=0}^n a_i e^{-\frac{i\eta}{3}} \right)_{\eta=0} = 0 \quad (148)$$

$$\left(\frac{d}{d\eta} \left(K\eta + \sum_{i=0}^n a_i e^{-\frac{i\eta}{3}} \right) - \delta_1 \left(1 + \frac{1}{\beta} \right) \frac{d^2}{d\eta^2} \left(K\eta + \sum_{i=0}^n a_i e^{-\frac{i\eta}{3}} \right) - \lambda \right)_{\eta=0} = 0 \quad (149)$$

$$\left(\sum_{k=1}^n b_k e^{-\frac{k\eta}{3}} - b \frac{d}{d\eta} \sum_{k=1}^n b_k e^{-\frac{k\eta}{3}} - 1 \right)_{\eta=0} = 0 \quad (150)$$

$$\left(\sum_{k=1}^n c_k e^{-\frac{k\eta}{3}} - c \frac{d}{d\eta} \sum_{k=1}^n c_k e^{-\frac{k\eta}{3}} - 1 \right)_{\eta=0} = 0 \quad (151)$$

The boundary conditions at infinity in Eq. (3.17) are satisfied automatically. Substituting Eq. (3.78) into Eqs. (3.14) - (3.16) resulted to residual functions $R_{f_1}(a_i, b_k, c_k, \eta)$, $R_{\theta_1}(a_i, b_k, c_k, \eta)$, $R_{\phi_1}(a_i, b_k, c_k, \eta)$ (see Appendix A).

Using Galerkin method which involves setting the integral of the product of residual functions and the weight functions $e^{-\frac{i\eta}{3}}$ and $e^{-\frac{k\eta}{3}}$, for $i = 0, 1, 2, \dots, n-2$, $k = 0, 1, 2, \dots, n-2$ to zero, that is

$$\begin{aligned} \int_0^\infty R_{f_1} e^{-\frac{i\eta}{3}} d\eta &\approx \sum_{k=1}^j \left[A_k \left(e^\eta R_{f_1} e^{-\frac{i\eta}{3}} \right)_{\eta=x_k} \right] = 0, \\ \int_0^\infty R_{\theta_1} e^{-\frac{k\eta}{3}} d\eta &\approx \sum_{k=1}^j \left[A_k \left(e^\eta R_{\theta_1} e^{-\frac{i\eta}{3}} \right)_{\eta=x_k} \right] = 0 \text{ and} \\ \int_0^\infty R_{\phi_1} e^{-\frac{kan\eta}{3}} d\eta &\approx \sum_{k=1}^j \left[A_k \left(e^\eta R_{\phi_1} e^{-\frac{i\eta}{3}} \right)_{\eta=x_k} \right] = 0. \end{aligned} \quad (152)$$

A set of nonlinear equations $3n+1$ including Eqs. (3.79)–(3.82) are obtained. These equations are solved with the help of MATHEMATICA 11.3 symbolic package to obtain the values of constants a_i , b_k and c_k .

5 Numerical Solution for Model 2

In this section, using weighted residual method, the following trial solutions are assumed for $f(\eta)$, $\theta(\eta)$ and $\phi(\eta)$ as follows (Rasaq and Aregbesola, 2012)

$$f(\eta) = \sum_{i=0}^n a_i e^{-\frac{i\eta}{3}}, \quad \theta(\eta) = \sum_{k=1}^n b_k e^{-\frac{k\eta}{3}} \quad \text{and} \quad \phi(\eta) = \sum_{k=1}^n c_k e^{-\frac{k\eta}{3}} \quad (153)$$

Eq. (3.84) is substituted into the boundary conditions in Eq. (3.40) to obtain

$$\left(\sum_{i=0}^n a_i e^{-\frac{i\eta}{3}} \right)_{\eta=0} = 0 \quad (154)$$

$$\left(\frac{d}{d\eta} \sum_{i=0}^n a_i e^{-\frac{i\eta}{3}} - \delta_1 \left(1 + \frac{1}{\beta} \right) \exp \left(-\gamma \sum_{k=1}^n b_k e^{-\frac{k\eta}{3}} \right) \frac{d^2}{d\eta^2} \sum_{i=0}^n a_i e^{-\frac{i\eta}{3}} \right)_{\eta=0} = 0 \quad (155)$$

$$\left(\frac{d}{d\eta} \sum_{k=1}^n b_k e^{-\frac{k\eta}{3}} + Bi \left(1 - \delta \sum_{k=1}^n b_k e^{-\frac{k\eta}{3}} \right)^{-1} \left(1 - \sum_{k=1}^n b_k e^{-\frac{k\eta}{3}} \right) \right)_{\eta=0} = 0 \quad (156)$$

$$\left(\sum_{k=1}^n c_k e^{-\frac{k\eta}{3}} - 1 \right)_{\eta=0} = 0 \quad (157)$$

The boundary conditions at infinity in Eq. (3.40) are satisfied automatically. Substituting Eq. (3.84) into Eqs. (3.39), (3.42) and (3.43) resulted to residual functions $R_{f_2}(a_i, b_k, c_k, \eta)$, $R_{\theta_2}(a_i, b_k, c_k, \eta)$, $R_{\phi_2}(a_i, b_k, c_k, \eta)$ (see Appendix A).

Galerkin method is applied to minimize residual errors by first setting the integral of the product of residuals and the weight functions $e^{-\frac{i\eta}{3}}$ and $e^{-\frac{k\eta}{3}}$, for $i = 0, 1, 2, \dots, n-2$, $k = 0, 1, 2, \dots, n-2$ to zero, that is

$$\begin{aligned} \int_0^\infty R_{f_2} e^{-\frac{i\eta}{3}} d\eta &\approx \sum_{k=1}^j \left[A_k \left(e^\eta R_{f_2} e^{-\frac{i\eta}{3}} \right)_{\eta=x_k} \right] = 0, \\ \int_0^\infty R_{\theta_2} e^{-\frac{k\eta}{3}} d\eta &\approx \sum_{k=1}^j \left[A_k \left(e^\eta R_{\theta_2} e^{-\frac{i\eta}{3}} \right)_{\eta=x_k} \right] = 0 \quad \text{and} \\ \int_0^\infty R_{\phi_2} e^{-\frac{k\eta}{3}} d\eta &\approx \sum_{k=1}^j \left[A_k \left(e^\eta R_{\phi_2} e^{-\frac{i\eta}{3}} \right)_{\eta=x_k} \right] = 0 \end{aligned} \quad (158)$$

to obtain a set of algebraic equations which together with Eqs. (3.85)–(3.88) yield $3n+1$ nonlinear algebraic equations with $3n+1$ unknown coefficients $(a_i, b_k \text{ and } c_k)$. These algebraic

equations are then solved using computer assisted symbolic package MATHEMATICA 11.3 to obtain the values of constants a_i, b_k and c_k .

6 Numerical Solution for Model 3

In order to apply weighted residual method to solve the third model, in this section, trial solutions for $f(\eta)$, $\theta(\eta)$ and $\phi(\eta)$ are assumed as follows (Razaq and Aregbesola, 2012)

$$f(\eta) = \sum_{i=0}^n a_i e^{-\frac{i\eta}{3}}, \quad \theta(\eta) = \sum_{k=1}^n b_k e^{-\frac{k\eta}{3}} \quad \text{and} \quad \phi(\eta) = \sum_{k=1}^n c_k e^{-\frac{k\eta}{3}} \quad (159)$$

Eq. (3.90) is substituted into the boundary conditions in Eq. (3.63) to obtain

$$\left(\sum_{i=0}^n a_i e^{-\frac{i\eta}{3}} - S \right)_{\eta=0} = 0 \quad (160)$$

$$\left(\frac{d}{d\eta} \sum_{i=0}^n a_i e^{-\frac{i\eta}{3}} - \left(1 + \frac{1}{\beta} \right) \left(\delta_1 \frac{d^2}{d\eta^2} \sum_{i=0}^n a_i e^{-\frac{i\eta}{3}} + \delta_2 \frac{d^3}{d\eta^3} \sum_{i=0}^n a_i e^{-\frac{i\eta}{3}} \right) - \lambda \right)_{\eta=0} = 0 \quad (161)$$

$$\left(\sum_{k=1}^n b_k e^{-\frac{k\eta}{3}} - 1 \right)_{\eta=0} = 0 \quad (162)$$

$$\left(\sum_{k=1}^n c_k e^{-\frac{k\eta}{3}} - 1 \right)_{\eta=0} = 0 \quad (163)$$

The boundary conditions at infinity in Eq. (3.63) are satisfied automatically. Substituting Eq. (3.90) into Eqs. (3.60)–(3.62) resulted to residual functions $R_{f_3}(a_i, \eta)$, $R_{\theta_3}(a_i, b_k, c_k, \eta)$, $R_{\phi_3}(a_i, b_k, c_k, \eta)$ (see Appendix A).

Galerkin method is then applied to minimize residual errors by setting the integral of the product of residuals and the weight functions $e^{-\frac{i\eta}{3}}$ and $e^{-\frac{k\eta}{3}}$, for $i = 0, 1, 2, \dots, n-2$, $k = 0, 1, 2, \dots, n-2$ to zero, that is

$$\begin{aligned}
\int_0^\infty R_{f_3} e^{-\frac{i\eta}{3}} d\eta &\approx \sum_{k=1}^j \left[A_k \left(e^\eta R_{f_3} e^{-\frac{i\eta}{3}} \right)_{\eta=x_k} \right] = 0, \\
\int_0^\infty R_{\theta_3} e^{-\frac{k\eta}{3}} d\eta &\approx \sum_{k=1}^j \left[A_k \left(e^\eta R_{\theta_3} e^{-\frac{i\eta}{3}} \right)_{\eta=x_k} \right] = 0 \text{ and} \\
\int_0^\infty R_{\phi_3} e^{-\frac{k\eta}{3}} d\eta &\approx \sum_{k=1}^j \left[A_k \left(e^\eta R_{\phi_3} e^{-\frac{i\eta}{3}} \right)_{\eta=x_k} \right] = 0
\end{aligned} \tag{164}$$

In this way, Eq. (3.95) together with Eqs. (3.91)–(3.94) then produced $3n+1$ algebraic equations with $3n+1$ unknown coefficients (a_i, b_k and c_k). These equations are solved using computer assisted symbolic package MATHEMATICA 11.3 to obtain the values of the constants a_i, b_k and c_k .

4 RESULTS AND DISCUSSION

1 Introduction

In this chapter, the effect of various controlling parameters on dimensionless velocity, temperature, nanoparticle volume fraction, wall heat transfer rate (Nuselt number) and wall mass transfer rate (Sherwood number) are obtained and discussed. The special cases of these three problems which are obtained by setting some parameters to zero are solved and their results are compared with the corresponding ones found in the literature which were obtained using a different technique. This is done to validate the accuracy of the present method (GWRM).

2 Discussion of the Results Obtained for Model 1

In this section, the effect of various parameters on fluid flow, heat transfer and nanoparticle concentration were obtained and discussed. The following fixed parameters (Animasaun, 2015b, Uddin et al., 2015, Duali and Seweli, 2012) $Ec = 0.01, Le = 1, Nr = 0.1, Nb = 0.1, Nt = 0.1, Pr = 6.8, \delta_1 = 0.1, b = 0.1, c = 0.1, K = 1, Da = 10, \alpha = 0.1, N = 10, Tr$ and $n = 16$, unless otherwise stated, are used to compute the average residual errors, dimensionless velocity, temperature, nanoparticle volume fraction, wall heat transfer rate and wall mass transfer rate. The square residual errors for $f(\eta), \theta(\eta)$ and $\phi(\eta)$ are calculated respectively as follows:

$$\varepsilon_{f_1} = \int_0^\infty R_{f_1}^2(\eta) d\eta \approx \sum_{k=1}^j \left[A_k \left(e^\eta R_{f_1}^2(\eta) \right)_{\eta=x_k} \right] \tag{165}$$

$$\varepsilon_{\theta_1} = \int_0^\infty R_{\theta_1}^2(\eta) d\eta \approx \sum_{k=1}^j \left[A_k \left(e^\eta R_{\theta_1}^2(\eta) \right)_{\eta=x_k} \right] \tag{166}$$

$$\varepsilon_{\phi_1} = \int_0^\infty R_{\phi_1}^2(\eta) d\eta \approx \sum_{k=1}^j \left[A_k \left(e^{\eta} R_{\phi_1}^2(\eta) \right)_{\eta=x_k} \right] \quad (167)$$

The resulting graph of total average square residual errors ($\varepsilon_{total} = \frac{\varepsilon_{f_1} + \varepsilon_{\theta_1} + \varepsilon_{\phi_1}}{3}$) is as shown in Figure 4.1. It is seen that the total residual errors are minimized in the domain (0 to ∞). The limiting cases of heat transfer rate (Nur) in this present work is compared with published results obtained by Uddin et al. (2015) via Runge-Kutta-Fehlberg fourth-fifth-order in Table 4.1. It is found that there exists a good agreement. This, hence, shows the accuracy of the present method (WRM).

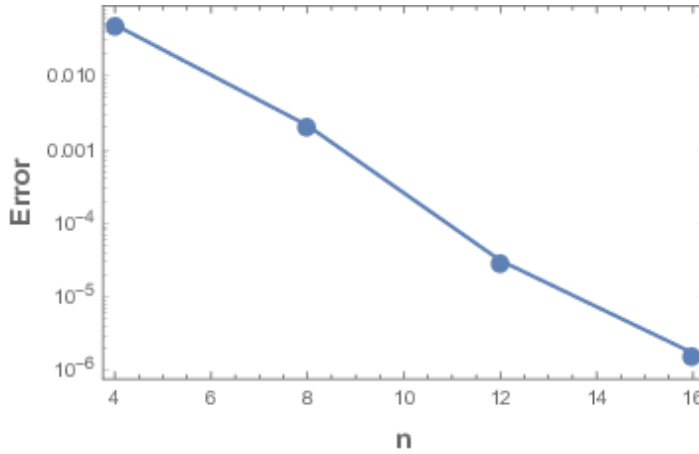


Figure 26: Average square residual errors at different order of approximation (n)

Table 2: Validation of the wall heat transfer rate ($-\theta'(0)$) for the reduced problem in which $K = Nr = Ec = M = \alpha = 0, Nt = Nt \rightarrow 0, \lambda = \delta_1 = b = c = 0, Da = N = \beta \rightarrow \infty$

Pr	Present study	Uddin et al., 2015	Relative error
.01	0.180	0.180	0.00%
.72	0.387	0.387	0.00%
	0.401	0.401	0.00%
	0.426	0.426	0.00%
	0.465	0.464	0.22%
	0.489	0.489	0.00%
	0.497	0.497	0.00%

Figures. 4.2 is plotted to study the effect of the radiation-conduction parameter (N) and stretching/shrinking parameter (λ) on the temperature profile. An increase in conduction-radiation parameter(N) decreases the temperature distribution for a shrinking, static or stretching sheet. The parameter (N), denoting the conduction-radiation parameter, is in the denominator via

the fraction $\frac{4}{3N}$ in Eq. 3.15. This implies that greater values of (N) lead to a weaker radiative heat transfer contribution which in turn leads to a decrease in nanofluid temperature. Furthermore, in Figures 4.2, temperature is found to be greatest in the case of a shrinking sheet ($\lambda < 0$) and it is least for a stretching sheet ($\lambda > 0$) (a similar result is displayed by Uddin et al.(2015)).

Figures. 4.3-4.5 depict the effect of velocity slip parameter (δ_1) and stretching/shrinking parameter (λ) on dimensionless velocity, temperature, and nanoparticle concentration profiles. It is observed from Figure 4.3 that the fluid flow increases with a shrinking, static or stretching sheet for increasing velocity slip parameter. Stretching sheet (in the absence/presence of velocity slip parameter) also enhances fluid flow, as a result of additional impulse from elastic forces received by the momentum field (which is in line with the study of Rashidi et al. 2014). The converse effect is the case for the shrinking sheet, where the retracting forces suppress the momentum boundary layer thickness (see also Gupta et al., 2014). Figure 4.4 shows the effects of the velocity slip parameter and stretching/shrinking parameter on dimensionless temperature. It is seen that the temperature increases for no-slip case ($\delta_1 = 0$) and decreases for slip case ($\delta_1 = 3$). In the stretching sheet case, the temperature reduces, whereas the shrinking sheet increases the temperature. This indicates that the shrinking sheet heats the boundary layer while the stretching sheet cools it. In figure 4.5, it is found that an increase in the velocity slip parameter reduces the nanoparticle concentration profile.

Figures 4.6-4.8 examine the influence of thermal slip parameter on dimensionless velocity, temperature and nanoparticle concentration profiles. Increasing thermal slip (b) reduces the velocity profile (see Figure 4.6) irrespective of the type of sheet considered. However, the velocity profile is enhanced for the stretching sheet case, while it is reduced in the case of a shrinking sheet. Figure 4.7 shows that an increase in thermal slip parameter(b) decreases the temperature profile. This is because as the thermal slip factor increases, there is an increase in molecular distance of the fluid from the hot surface. This then leads to less amount of heat being transferred from the hot surface to the fluid and hence leads to the reduction in temperature profile. Figure 4.8 shows that the nanoparticle concentration reduces as the thermal slip parameter increases.

The effect of Darcy number (Da) on the velocity profile (when $\delta_1 = 1.5, Tr = 3$) is presented in Figures 4.9. It is found that the velocity of the nanofluid increases with an increase in Darcy number. This is because the Darcian drag force in the momentum boundary layer (see Eq. (3.14)) decreases with an increase in Darcy number. This serves to enhance fluid flow permeability and leads to an increase in the velocity profile. It is also observed that the fluid is accelerated for the stretching sheet, and the shrinking sheet decelerates the flow (see Figures 4.9).

Figures 4.10-4.12 show the variation of Casson parameter (β) (when $\delta_1 = 1.5, Tr = 3$) on dimensionless velocity, temperature, and nanoparticle volume fraction. An increase in the Casson parameter reduces fluid flow near the sheet (see Figure 4.10). The fluid velocity begins to increase as the fluid layers move toward the free stream region. This later increment in the fluid flow is experienced as a result of the pressure gradient driving the fluid in the free stream region. Dimensionless temperature (see Figure 4.11) and nanoparticle volume fraction (see Figure 4.12) decrease as the Casson parameter increases.

Figures 4.13 discusses the effects of stretching/shrinking sheet and magnetic field parameters on dimensionless velocity. It is observed that increasing the magnetic field parameter

(M) reduces the dimensionless velocity (see Figures 4.13). This happened as a result of a magnetic drag force (Lorentz force) present in the magnetic field. The fluid flow is accelerated for the stretching sheet case, whereas it is retarded for the shrinking sheet. The influence of mass slip parameter (c) (when Lewis number, $Le=5$) on nanoparticle volume fraction is considered in Figures 4.14. Nanoparticle concentration decreases with an increase in mass slip parameter (c) (see Figure 4.14). Nanoparticle volume fraction profile decreases as a result of increasing the mass slip parameter due to the migration of nanoparticles away from the sheet surface

Figures 4.15 illustrates the effect of the Eckert number (Ec) on dimensionless temperature. As Eckert number increases, the temperature profile increases. This is because some work done by the fluid shared in the boundary layer is dissipated as heat. This generated heat leads to an increase in temperature as it is observed in Figures 4.15. Figures 4.16 displays the variation of Forchheimer parameter (Fs) (when $Tr = 3, \delta_1 = 1.5$) on dimensionless velocity. An increase in the Forchheimer parameter reduces the fluid flow. Mathematically, the Forchheimer parameter varies inversely to permeability regime term $(K_p)_0$. Increasing Forchheimer parameter (Fs) leads to a decrease in fluid permeability and this, in turn, leads to a reduction in fluid flow in the boundary layer. The stretching sheet suppresses fluid flow compared to a shrinking sheet, as it is seen in Figure 4.16.

The effect of the thermophoresis parameter (Nt) (for $Nb = 0.5$) on dimensionless temperature and nanoparticle volume fraction are shown in Figures 4.17 and 4.18. It is observed that for rising values of thermophoresis parameter, both dimensionless temperature and nanoparticle volume fraction are enhanced. Increasing thermophoresis parameter (Nt) results in the migration of nanoparticle away from the sheet wall towards the boundary layer regime and this enhances nanoparticle deposition into the nanofluid and thereby accounts for increased nanoparticle volume fraction and temperature profiles. Figures 4.19 and 4.20 depict the effect of the Brownian motion parameter (Nb) on dimensionless temperature and nanoparticle volume fraction. It is seen that the fluid temperature is appreciated for higher values of Nb . The converse behavior is observed in the case of nanoparticle concentration profile. The influence of various parameters on the wall heat transfer rate (Nur) is shown in Figure 4.21. Heat transfer rate is reduced with greater value of thermal slip (b) parameter. This is due to the fact that the presence of thermal slip reduces the temperature gradient at the wall. This is clearly observed from the boundary condition $(\theta(0) = \lambda + b\theta'(0))$. Shrinking sheet ($-\lambda$) suppresses heat transfer rate whereas the contrary behavior is observed in the case of stretching sheet (λ). It is also notice that Nur is found to be a decreasing function of thermophoresis parameter (Nt). The variation of various parameters on heat transfer rate (Nur) is also depicted in Fig 4.22. it is observed that temperature ratio (Tr) is found to elevate heat transfer rate in the presence or absence of buoyancy effect (Nr) while it (Nur) is reduced as the Brownian motion parameter increases.

In Fig 4.23 and 4.24, the influence of various parameters on the wall mass transfer rate (Shr) is presented. The mass transfer rate is appreciated with an increase in Lewis number (Le), temperature ratio (Tr), stretching (λ) and Brownian motion (Nb) parameters whereas, it is depreciated with an increase in thermal slip (c) and buoyancy ratio (Nr) parameters

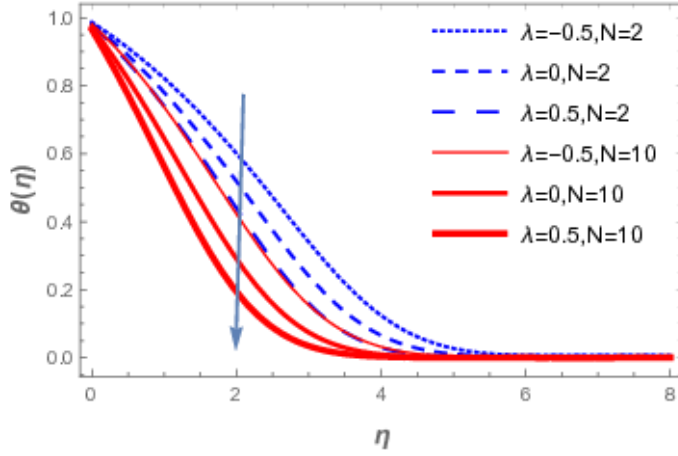


Figure 27: Temperature profile for different values of N and λ

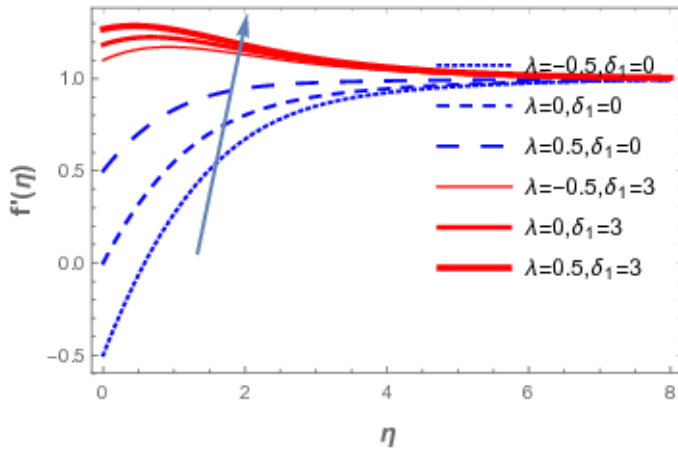


Figure 28: Velocity profile for different values of δ_1 and λ

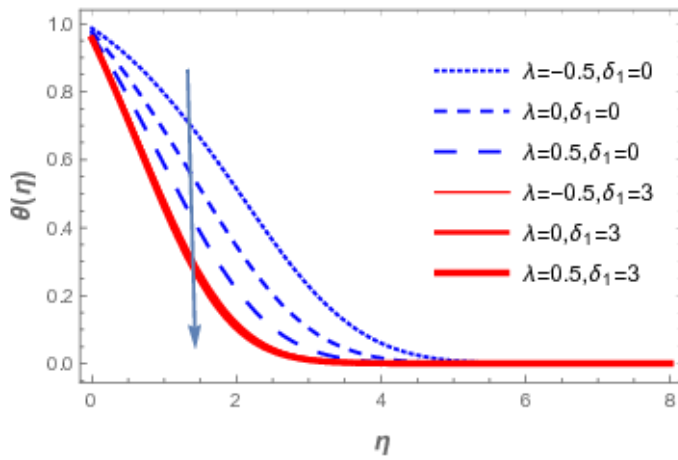


Figure 29: Temperature profile for different values of δ_1 and λ

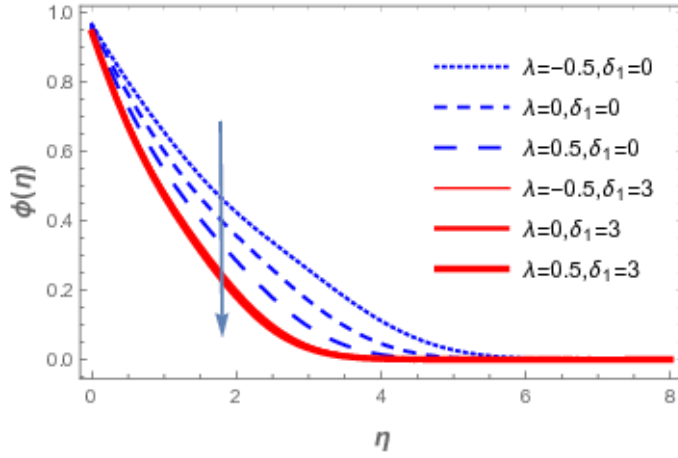


Figure 30: Nanoparticle concentration profile for different values of δ_1 and λ

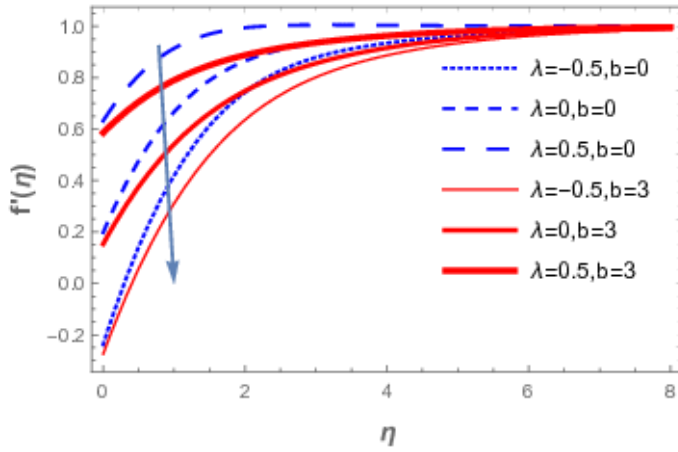


Figure 31: Velocity profile for different values of b and λ

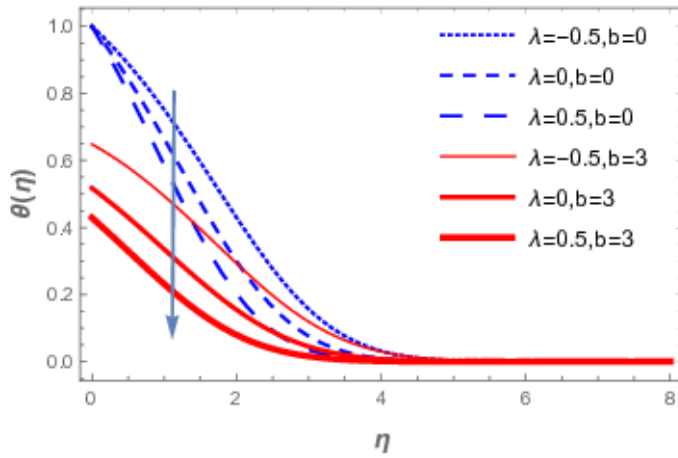


Figure 32: Temperature profile for different values of b and λ

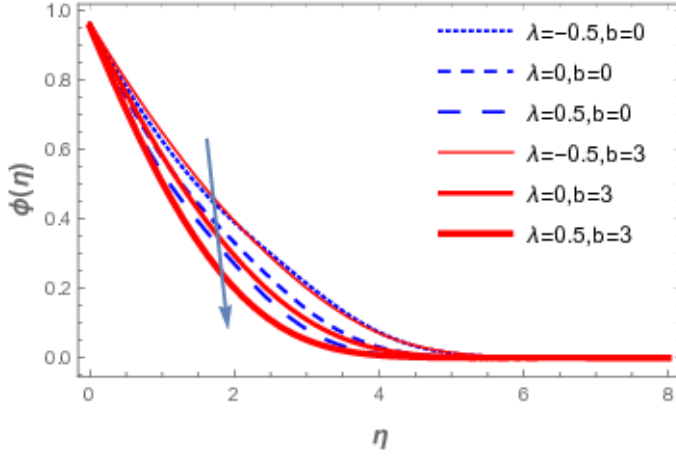


Figure 33: Nanoparticle concentration profile for different values of b and λ

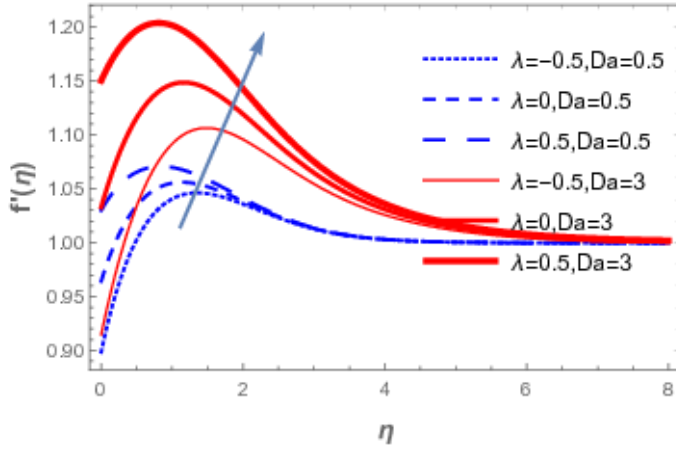


Figure 34: Velocity profile for different values of Da and λ ($\delta_1 = 1.5, Tr = 3$)

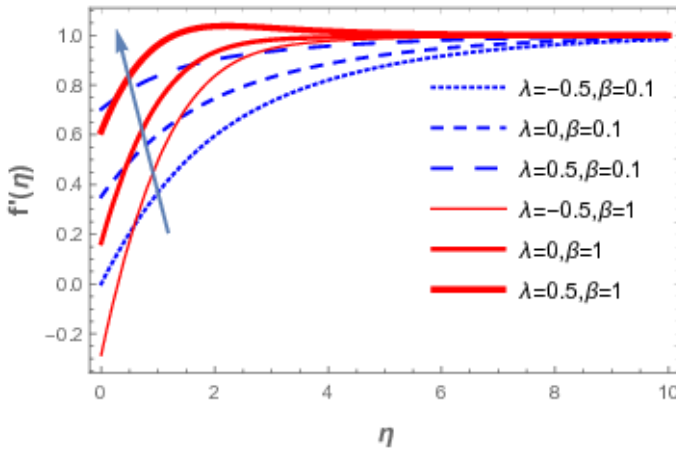


Figure 35: Velocity profile for different values of β and λ ($\delta_1 = 1.5, Tr = 3$)

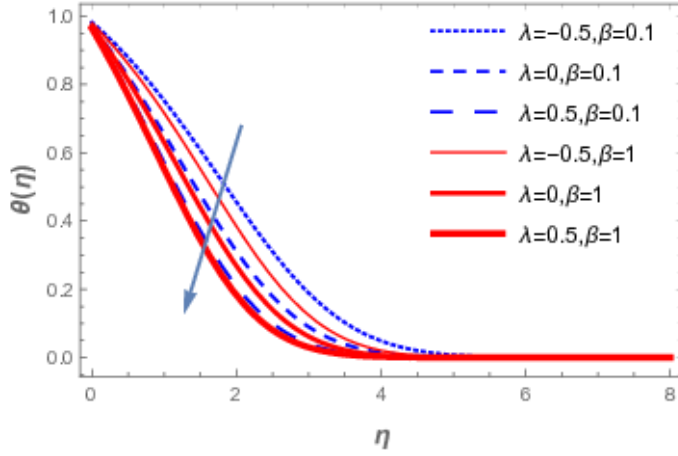


Figure 36: Temperature profile for different values of β and λ ($\delta_1 = 1.5, Tr = 3$)

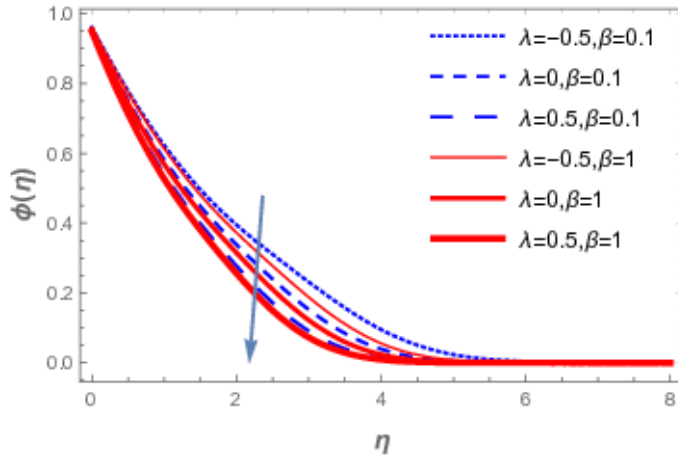


Figure 37: Nanoparticle concentration profile for different values of β and λ ($\delta_1 = 1.5, Tr = 3$)

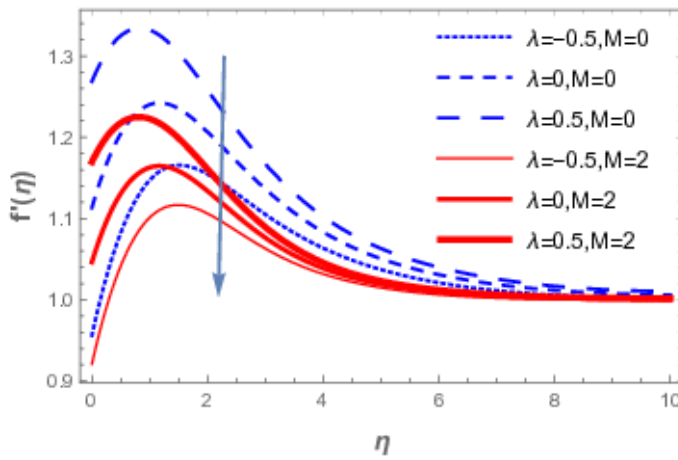


Figure 38: Velocity profile for different values of M and λ

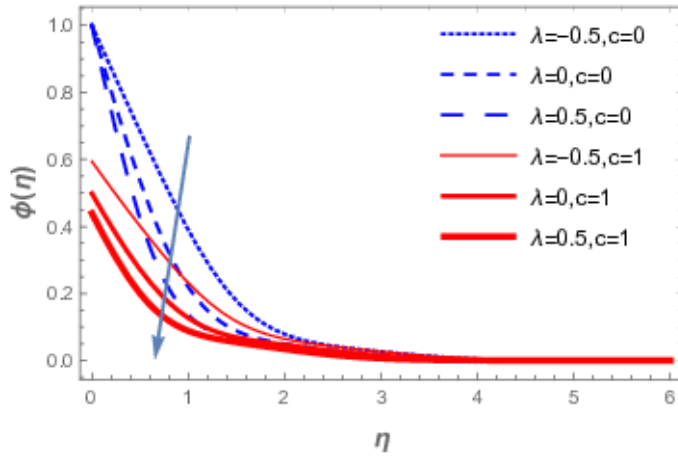


Figure 39: Nanoparticle concentration profile for different values of c and λ ($Le = 5$)

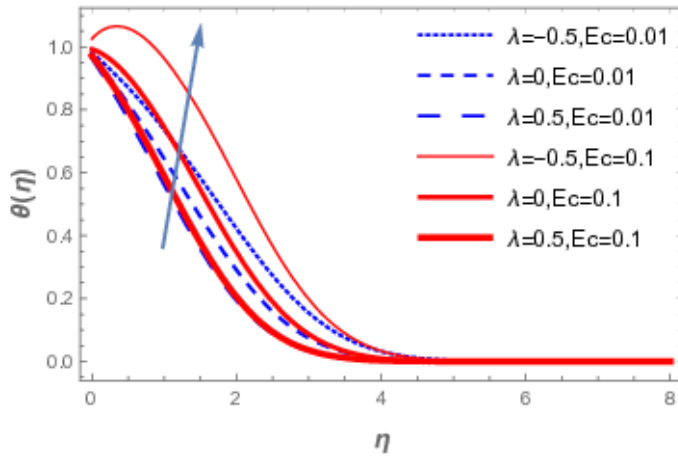


Figure 40: Temperature profile for different values of Ec and λ

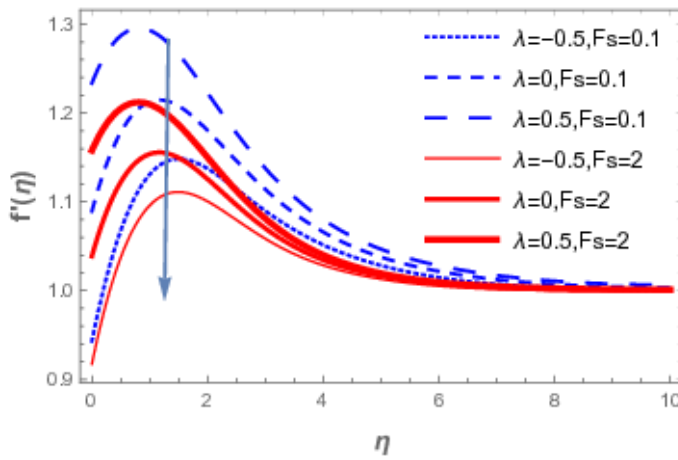


Figure 41: Velocity profile for different values of Fs and λ ($\delta_1 = 1.5, Tr = 3$)

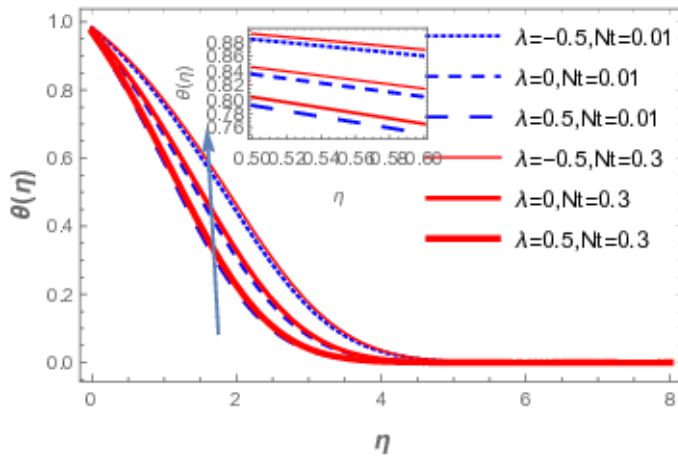


Figure 42: Temperature profile for different values of Nt and λ ($Nb = 0.5$)

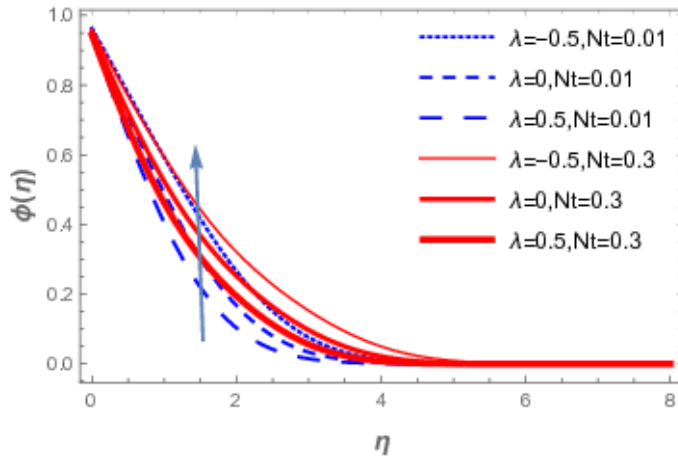


Figure 43: Nanoparticle concentration profile for different values of Nt and λ ($Nb = 0.5$)

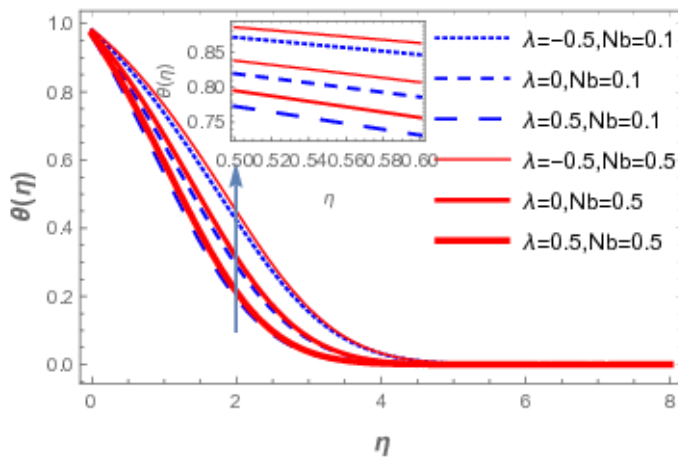


Figure 44: Temperature profile for different values of Nb and λ

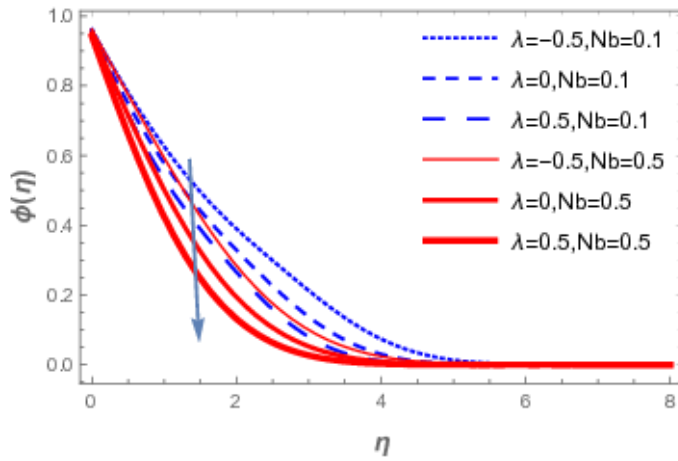


Figure 45: Nanoparticle concentration profile for different values of Nb and λ

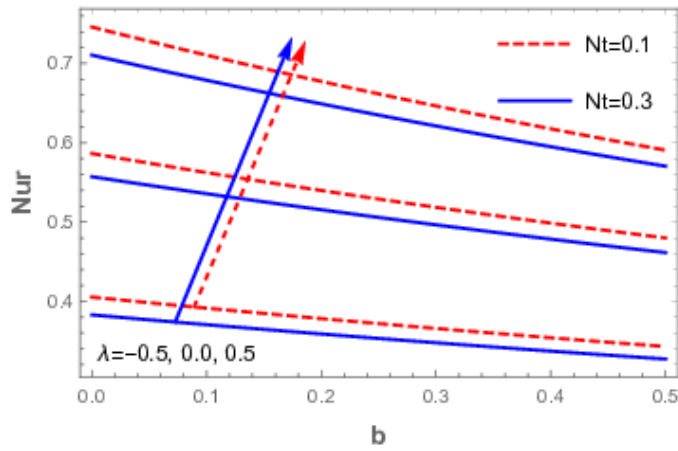


Figure 46: Nusselt number profile for different values of Nt , λ and b

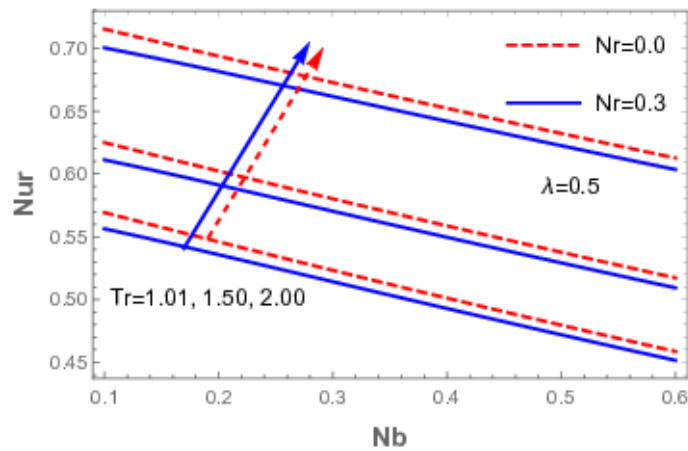


Figure 47: Nusselt number profile for different values of Nb , Tr and Nr

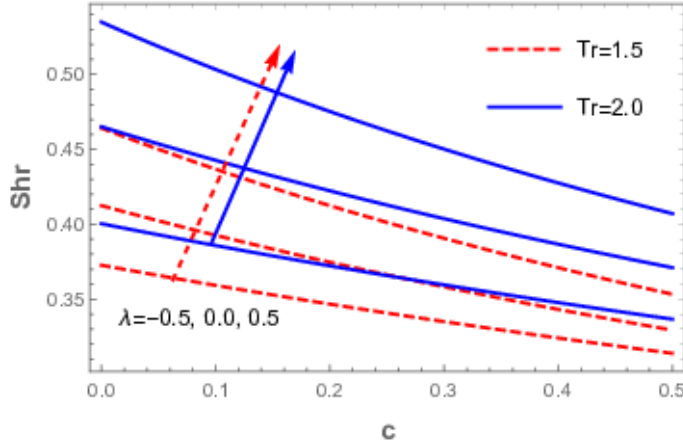


Figure 48: Sherwood number profile for different values of c , Tr and λ

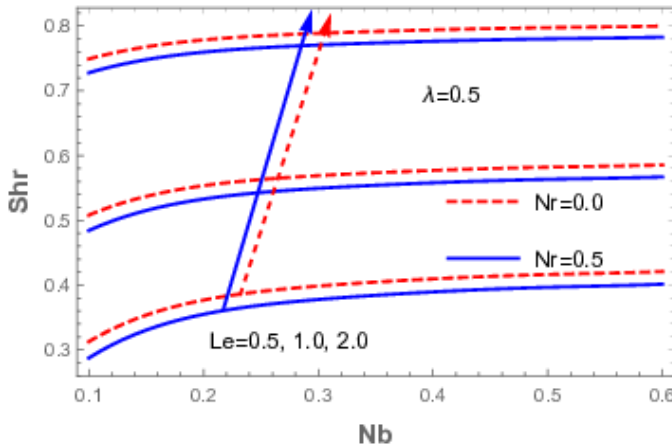


Figure 49: Sherwood number profile for different values of Nb , Le and Nr

In conclusion, MHD dissipative Casson nanofluid flow in the presence of non-Darcian porous medium over a stretching/shrinking sheet with slip boundary conditions is investigated in this section. The thermal radiation effect is incorporated by using a nonlinear Rosseland diffusion flux approximation. The momentum, energy, and nanoparticle concentration boundary-layer equations have been reduced to a nonlinear coupled ordinary differential equations. The solutions of the resulting boundary-value problem have been achieved using Galerkin weighted residual method. The results have been validated with the published results, and a good agreement is found. The effect of controlling parameters, such as Casson, velocity slip, thermal slip, mass slip, Forchheimer, and stretching/shrinking parameters as well as Darcy, Eckert and Hartmann numbers on the dimensionless velocity, temperature, nanoparticle concentration, wall heat transfer rate (Nusselt number) and wall mass transfer rate (Sherwood number) have shown below:

[label=[]]

1. An increase in the conduction-radiation parameter decreases the temperature profile;
2. Increasing velocity slip parameter enhances the velocity profile. However, it reduces both temperature and nanoparticle volume fraction profiles;
3. As the thermal slip increases, velocity, temperature, nanoparticle concentration

and rate of heat transfer profiles are found to reduce;

4. Velocity profile increases with rising values of Darcy number;
5. Increasing the Casson parameter is found to increase velocity profile and reduce both temperature and nanoparticle concentration profiles;
6. An increase in the magnetic field leads to a decrease in velocity profile;
7. Both nanoparticle volume fraction profile and Sherwood number diminish via high values of mass slip parameter;
8. Temperature profile is found to increase as Eckert number increases; and
9. Velocity profile reduces for higher appreciable values of Forchheimer parameter .
10. The results obtained also revealed that for stretching sheet case, the fluid flow is accelerated, whereas it is decelerated for shrinking sheet case
11. Temperature and nanoparticle concentration are both enhanced for shrinking sheet case, while they are suppressed for stretching sheet case
12. Finally, both Nuselt and Sherwood numbers are in favour of stretching sheet, whereas they are opposed by shrinking sheet

3 Discussion of the Results Obtained for Model 2

This section deals with analysis of the effects of various parameters on the fluid flow. The values (following Uddin *et al*, 2015, Duali and Seweli, 2012 and Animasaun, 2015b) of parameters $Le = 1$, $Nr = 0.1$, $Nb = 0.1$, $Nt = 0.1$, $Pr = 6.8$, $\delta_1 = 0.1$, $Da = 10$, $Fs = 0.1$, $N = 10$, $Tr = 2$, $\beta = 0.5$, $M = 0.5$, $Ec = 0.01$, $\delta = 0.3$, $\gamma = 1$, $Bi = 0.5$, $Ec = 0.1$, $j = 36$ and $n = 16$ are kept unchanged throughout the study unless otherwise stated. The square residual errors for $f(\eta)$, $\theta(\eta)$ and $\phi(\eta)$ are computed respectively using the followings:

$$\varepsilon_{f_2} = \int_0^\infty R_{f_2}^2(\eta) d\eta \approx \sum_{k=1}^j \left[A_k \left(e^\eta R_{f_2}^2(\eta) \right)_{\eta=x_k} \right] \quad (168)$$

$$\varepsilon_{\theta_2} = \int_0^\infty R_{\theta_2}^2(\eta) d\eta \approx \sum_{k=1}^j \left[A_k \left(e^\eta R_{\theta_2}^2(\eta) \right)_{\eta=x_k} \right] \quad (169)$$

$$\varepsilon_{\phi_2} = \int_0^\infty R_{\phi_2}^2(\eta) d\eta \approx \sum_{k=1}^j \left[A_k \left(e^\eta R_{\phi_2}^2(\eta) \right)_{\eta=x_k} \right] \quad (170)$$

Figure 4.25 shows the graph of total average square residual errors $(\varepsilon_{total} = \frac{\varepsilon_{f_2} + \varepsilon_{\theta_2} + \varepsilon_{\phi_2}}{3})$

. It was observed that the residual errors were minimized in domain $(0 \text{ to } \infty)$.

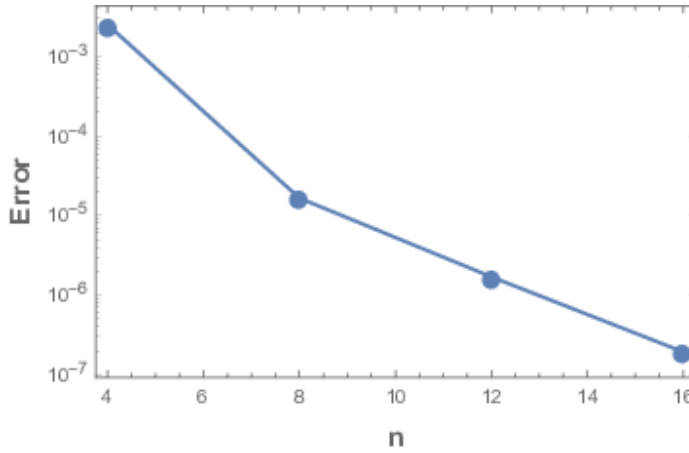


Figure 50: Average square residual errors at different order of approximation (n)

In order to ascertain the accuracy of GWRM, the computational values of the rate of heat transfer ($-\theta'(0)$) in the present work is compared with the result of Uddin *et al.* (2012) via Runge-Kutta-Fehlberg fourth-fifth-order. A good agreement is observed for the limiting situations as it is seen in Table 4.2.

Table 3: Comparison of Nusselt number values for the reduced problem with those of Uddin *et al.* (2012) when $M = Fs = \delta = \gamma = \delta_1 = Ec = 0$, $Le = Bi = 10$, $Da = N = \beta \rightarrow \infty$, $Nt = 0.1$

*Nb (lr)3- 4(lr)5-6(lr)7-8	2*Nr	Pr=1		Pr=5		Pr=10	
		Uddin <i>et al.</i> (2012)	Present work	Uddin <i>et al.</i> (2012)	Present work	Uddin <i>et al.</i> (2012)	Present work
.1	0	0.3425	0.3425	0.3839	0.3839	0.3953	0.3953
.1	0.2	0.3365	0.3365	0.3773	0.3773	0.3885	0.3885
.1	0.4	0.3301	0.3301	0.3702	0.3702	0.3813	0.3813
.3	0	0.2960	0.2959	0.3328	0.3328	0.3430	0.3430
.3	0.2	0.2917	0.2917	0.3282	0.3282	0.3382	0.3382
.3	0.4	0.2872	0.2872	0.3232	0.3232	0.3331	0.3331

Figures 4.26–4.28 show the effects of velocity slip, variable thermal conductivity and viscosity parameters (δ_1 , δ and γ) on the velocity, temperature, and nanoparticle volume fraction (nanoparticle concentration) profiles. It is found that the fluid velocity is an increasing function of the velocity slip parameter (see Figure 4.26). This is perhaps because when velocity slip occurs, the fluid particles move away from the plate and this leads to the reduction of shear force which in turn accelerates the nanofluid velocity. On the other hand, the corresponding

temperature and nanoparticle volume fraction are reducing functions of velocity slip parameter (δ_1) (see Figures 4.27 and 4.28). It is also observed that an increase in fluid variable physical properties ($\dot{\phi}$ and γ) increase the velocity profile (see Figure 4.26) and reduce both temperature and nanoparticle volume fraction profiles (see Figures 4.27 and 4.28). This may happen as a result of an increase in $\dot{\phi}$ and γ which tends to increase temperature difference ($T_f - T_\infty$) thereby weakening Casson fluid bond and decrease the strength of Casson plastic dynamic viscosity.

Figures 4.29–4.31 depict the effects of Casson and buoyancy ratio parameters (β and Nr) on dimensionless velocity, temperature, and nanoparticle volume fraction. It is observed that an increase in the Casson parameter increases the fluid flow (see Figure 4.29) and reduces both fluid temperature and nanoparticle volume fraction (see Figures 4.30 and 4.31). In the true sense, an increase in the Casson parameter (β) resists the fluid flow because it decreases the yield stress of Casson fluid and increases plastic dynamic viscosity. However, this reduction in velocity profile is overcome by the high amount of temperature being injected into the fluid due to the presence of temperature dependent viscosity and thermal conductivity parameters which in turn accelerate the fluid flow. It is also discovered in Figure 4.29 that an increase in buoyancy ratio parameter decelerates nanofluid velocity. The buoyancy ratio parameter (Nr) represents the relationship between species buoyancy force and thermal buoyancy force. When $Nr = 1$, the buoyancy forces contribute equally. When $Nr > 1$, species buoyancy dominates and vice versa for $Nr < 1$. Concentration buoyancy disappears when $Nr = 0$ and this leads to an increase in Casson nanofluid velocity. This shows that as the buoyancy ratio parameter becomes larger, the dimensionless velocity reduces. Increasing buoyancy ratio parameter (Nr) enhances both temperature and nanoparticle volume fraction profiles in Figures 4.30 and 4.31 respectively.

The effects of the conduction-radiation parameter (N) and Eckert number (Ec) on velocity and temperature profiles are shown in Figures 4.32 and 4.33. An increase in the value of (N) is found to increase the velocity profile in Figure 4.32. The temperature profile is found to reduce with an increase in the conduction-radiation parameter (see Figure 4.33). It is noted in Eq. 3.43 that the parameter (N), denoting the contribution of thermal conduction heat transfer to thermal radiation heat transfer ($N = \frac{k^* k_1}{4\sigma_1 T_\infty^3}$), is in the numerator. Therefore, an increase in the

value of (N) contributes to a decrease in the radiative mode of heat transfer which in turn reduce the fluid temperature. It is also discovered that the velocity and corresponding temperature profiles increase with an increase in Eckert number (Ec) (see Figures 4.32 and 4.33). The reason for this is due to the work done by the fluid molecules in converting kinetic energy to heat energy.

Figure 4.34 shows the effects of Darcy number (Da) and Forcheimer parameter (Fs) on the velocity profile. It is observed that the fluid velocity increases with an increase in Darcy number (Da). The Darcian drag force in Eq. 3.42 is inversely proportional to Darcy number (Da). This implies that the Darcian drag force reduces with an increase in Darcy number. Hence, this enhances fluid flow permeability and leads to an increase in Casson nanofluid velocity. It is also noticed in Figure 4.34 that the dimensionless velocity decelerates with an increase in Forchheimer parameter (Fs). This is due to the presence of an initial effect that is dragging the fluid backward. The effect of the magnetic (M) parameter on dimensionless velocity is shown in

Figure 4.35 . An increase in the value of magnetic parameter (M) leads to a reduction of dimensionless velocity. The reason for this is the presence of Lorentz force produced by a magnetic field placed in a transverse direction to the electrically conducting Casson nanofluid.

Figures 4.36 and 4.37 depict the effects of Brownian motion and thermophoresis parameters (Nb and Nt) on temperature and nanoparticle volume fraction profiles. It is found in Figure 4.36 that the temperature profile increases with an increase in Brownian motion parameter (Nb), while the opposite trend is observed for the case of nanoparticle concentration profile (see Figure 4.37). This is because thermal energy is generated as a result of an increase in nanofluid particle collision which later boosts the Casson nanofluid temperature. Also, the nanoparticles tend to move away from the plate surface and this leads to a decrease in nanoparticle volume fraction. An increase in thermophoresis parameter (Nt) increases the thermophoretic force (The force that the nanoparticles with lower temperature exert on the nanoparticles having higher temperature to move them away from the surface of the sheet) which moves nanoparticles from the region of higher temperature to the region of lower temperature, and this results to an increase in nanofluid temperature (see Figure 4.36) and corresponding nanoparticle volume fraction (see Figure 4.37).

The effect of convective heat transfer parameter Bi on dimensionless velocity and temperature are shown in Figures 4.38 and 4.39 respectively. It is found in Figure 4.38 that an increase in convective heat transfer parameter (Bi) increases dimensionless velocity as it is seen in the work of Uddin et al. (2012) This is because convective heat transfer parameter

$\left(Bi = \frac{(h_f)_o L}{k^* Ra^{\frac{1}{4}}} \right)$ in Eq. 3.40 is directly proportional to heat transfer coefficient $(h_f)_o$ and inversely

proportional to the fourth root of the Rayleigh number (Ra). An increase in convective heat transfer parameter (Bi) results to thermal convection enhancement at the wall and reduces buoyancy forces which in turn accelerates the fluid flow (see Figure 4.38) and the corresponding temperature (see Figure 4.39)

The effects of temperature ratio, Brownian motion, convective heat transfer, buoyancy ratio, thermophoresis, and Casson parameters (Tr , Nb , Bi , Nr , Nt , and β) on the Nusselt number (Nur) are shown in Figures 4.40 and 4.41. An increase in convective heat transfer, Casson and temperature ratio parameters (Bi , β and Tr) lead to an increase in Nusselt number, while an increase in Brownian motion, thermophoresis and buoyancy ratio parameters (Nb , Nt , Nr and Tr) reduces it. It is found in Figures 4.42 and 4.43 that Sherwood number increases with an increase in Lewis number, convective heat transfer, Brownian motion and Casson parameters (Le , Bi , Nb and β) respectively, while it (Sherwood number) is a decreasing function of Thermophoresis parameter (Nt) and buoyancy ratio parameter (Nr).

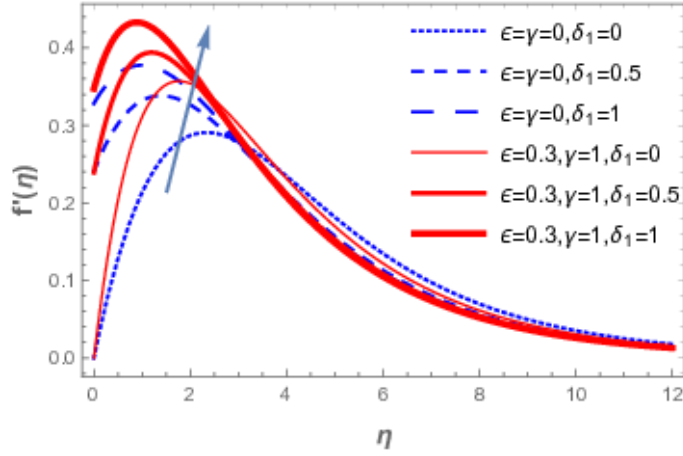


Figure 51: Velocity profile for different values of δ , γ and δ_1

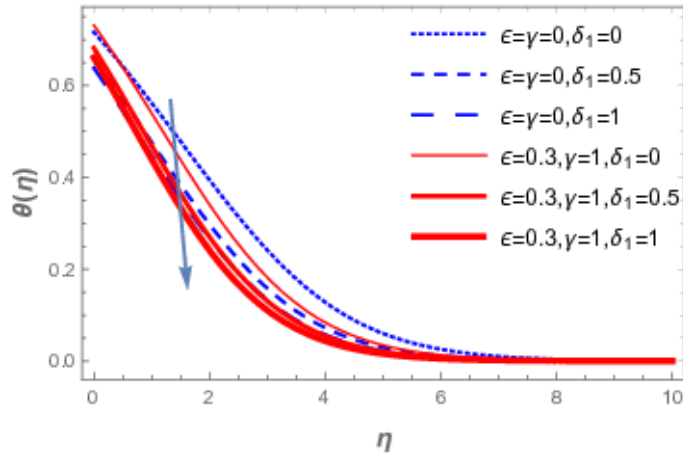


Figure 52: Temperature profile for different values of δ , γ and δ_1

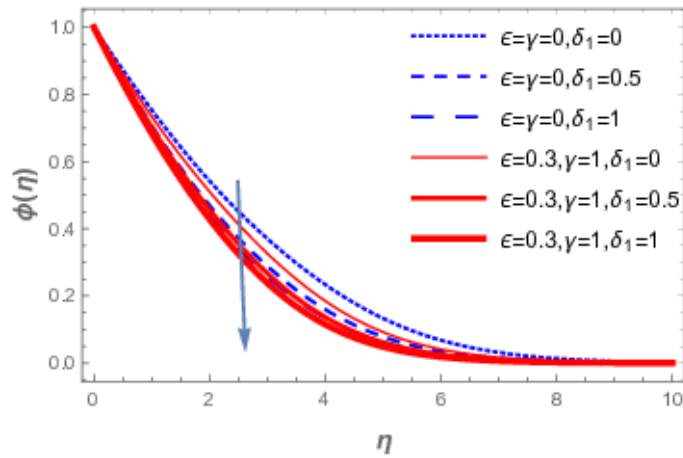


Figure 53: Nanoparticle concentration profile for different values of δ , γ and δ_1

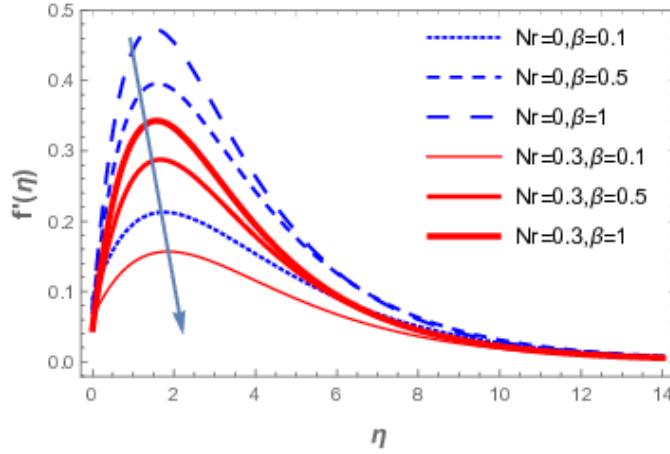


Figure 54: Velocity profile for different values of Nr and β

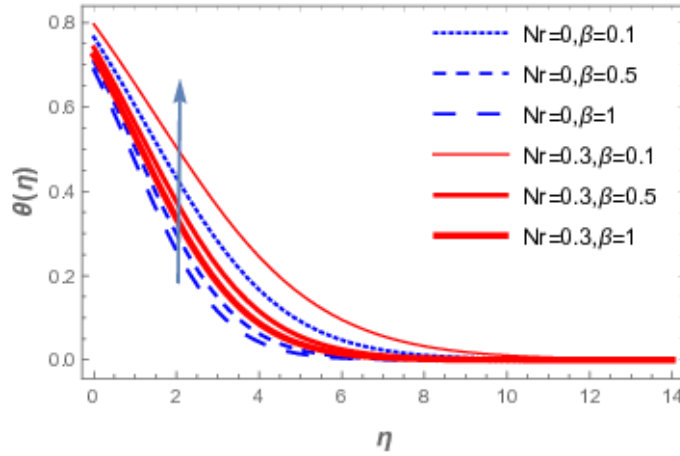


Figure 55: Temperature profile for different values of Nr and β

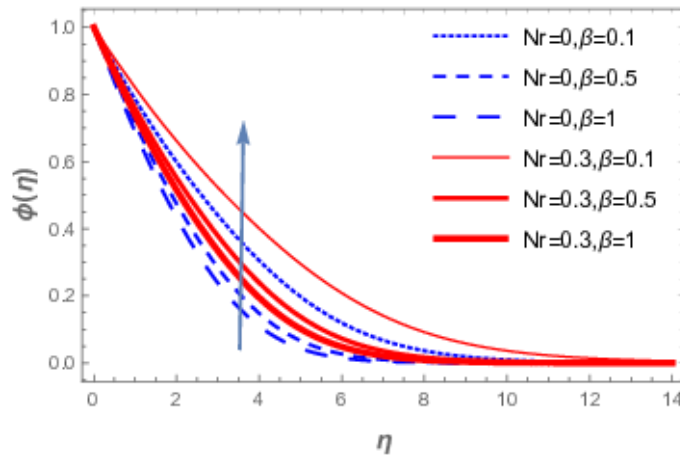


Figure 56: Nanoparticle concentration profile for different values of Nr and β

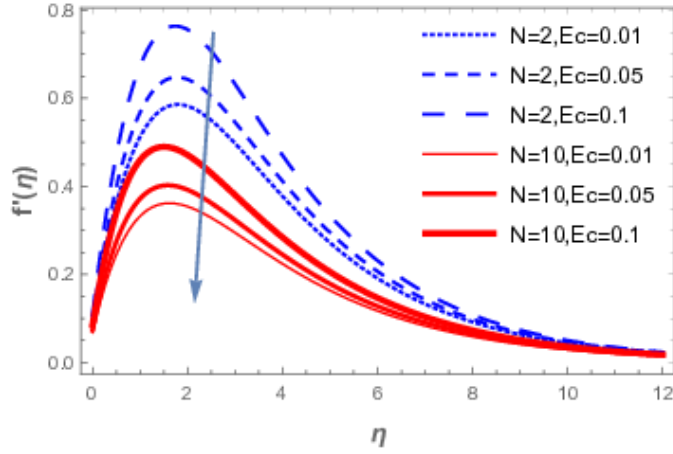


Figure 57: Velocity profile for different values of N and Ec

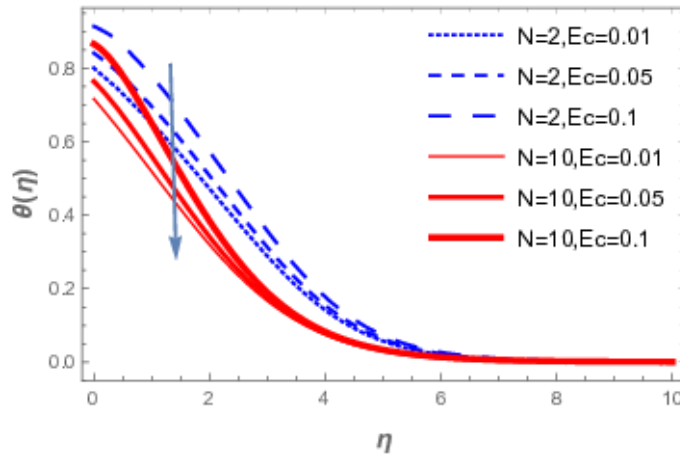


Figure 58: Temperature profile for different values of N and Ec

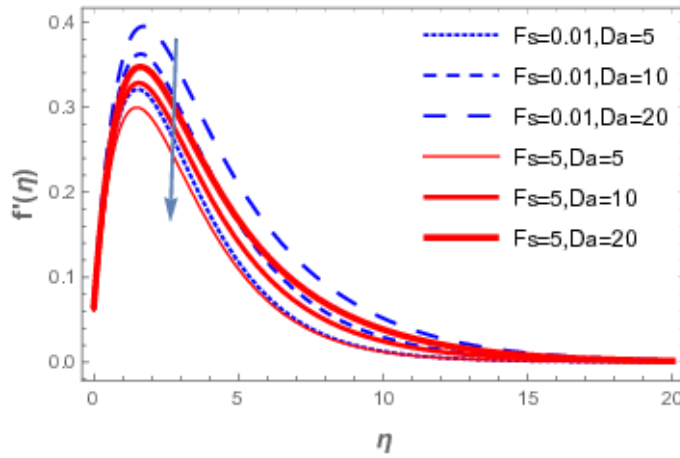


Figure 59: Velocity profile for different values of F_s and Da

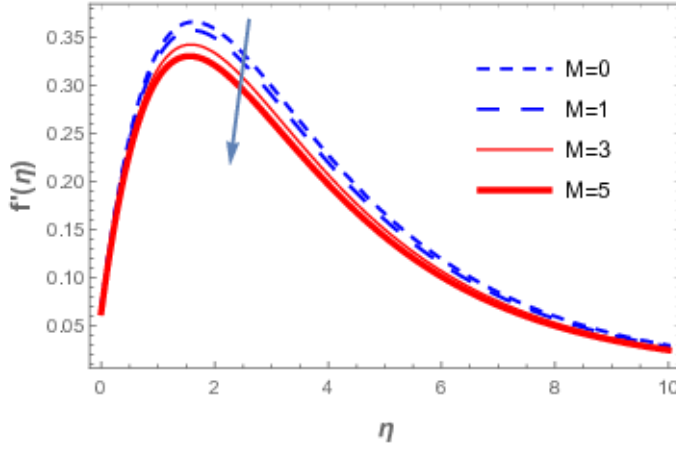


Figure 60: Velocity profile for different values of M

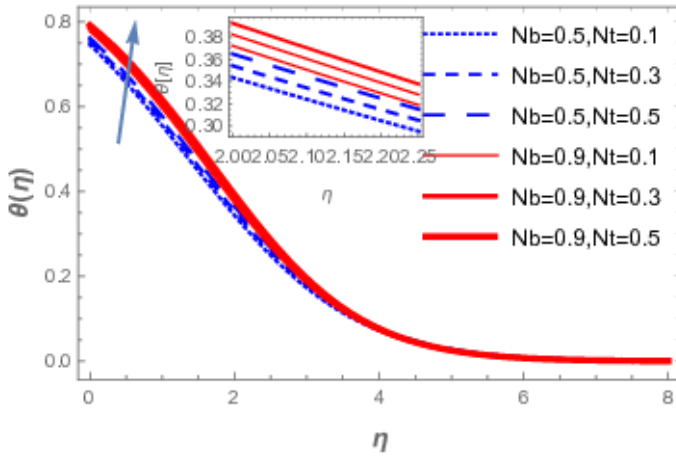


Figure 61: Temperature profile for different values of Nb and Nt

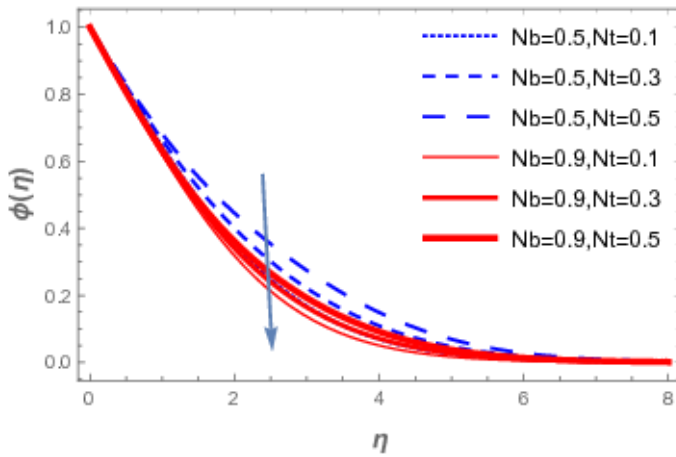


Figure 62: Nanoparticle concentration profile for different values of Nb and Nt

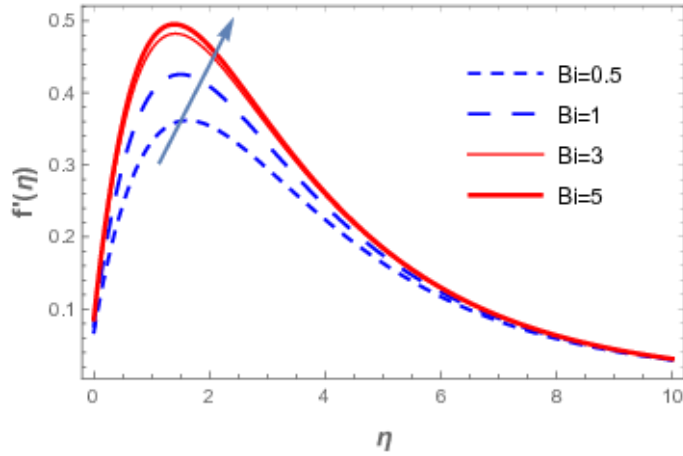


Figure 63: Velocity profile for different values of Bi

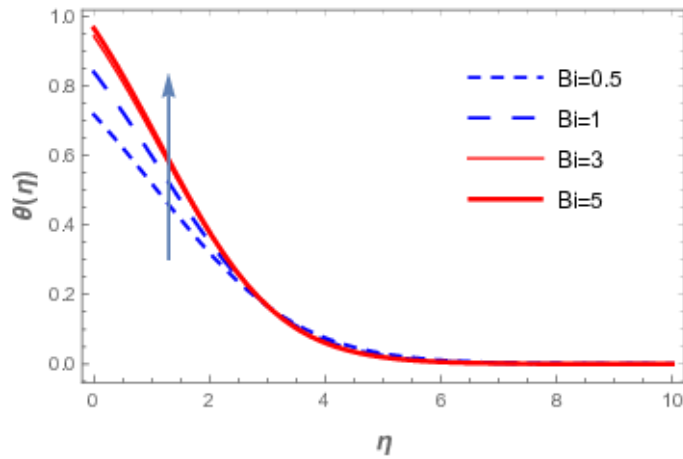


Figure 64: Temperature profile for different values of Bi

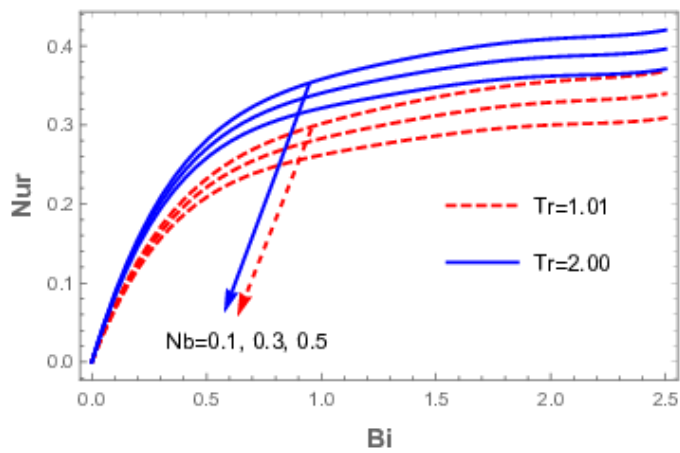


Figure 65: Nusselt number profile for different values of Bi , Tr and Nb

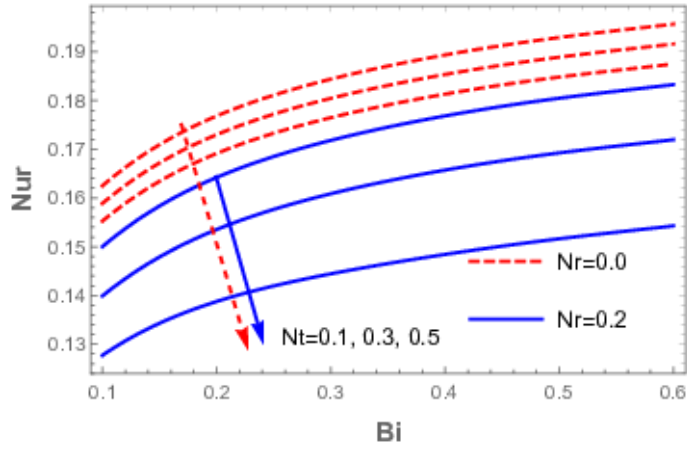


Figure 66: Nusselt number profile for different values of β , Nr and Nt

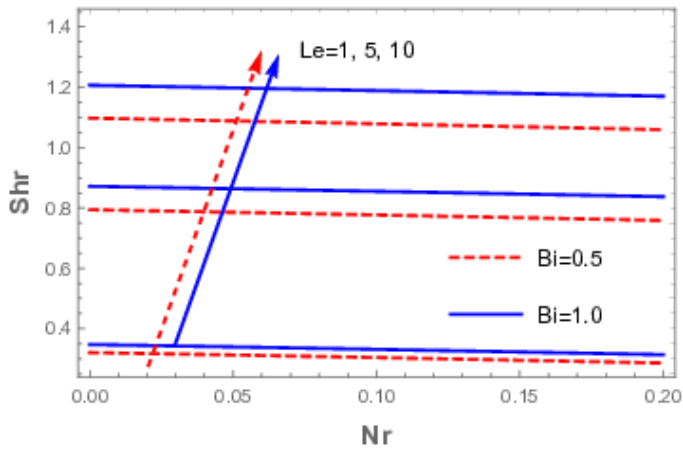


Figure 67: Sherwood number profile for different values of Bi , Nr and Le

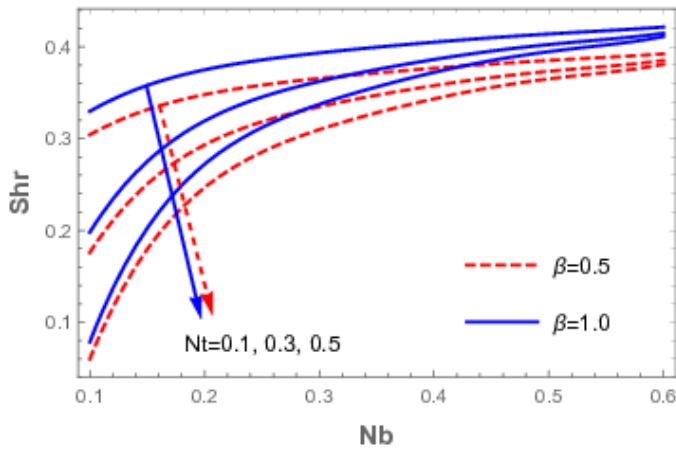


Figure 68: Sherwood number profile for different values of β , Nb and Nt

In conclusion, this section investigates the combined effect of variable thermal

conductivity and viscosity on incompressible, viscous, electrically conducting Casson nanofluid flow over a vertical plate in the presence of a transverse magnetic field, nonlinear thermal radiation, convective heating, and velocity slip. The partial differential equations governing the fluid flow, heat, and nanoparticle concentration were non-dimensionalized and later reduced to the ordinary differential equation using similarity transformation. These equations (ordinary differential equations) with corresponding boundary conditions were solved using Galerkin weighted residual method (GWRM). The effects of various controlling parameters on velocity, temperature, nanoparticle volume fraction, Nusselt number and Sherwood number profile were discussed and presented graphically. The main findings of this research problem in this section, are as follows:

[label=()]

1. Increasing velocity slip (δ_1) is found to increase the velocity profile and reduce both temperature and nanoparticle volume fraction profiles for constant or variable fluid physical properties (thermal conductivity (δ) and viscosity (γ)). It is also found that the velocity profile is increased while temperature and nanoparticle profiles are decreased as the fluid properties increase.
2. As the Casson parameter (β) increases, both the temperature and nanoparticle volume fraction are reduced while the velocity profile is increased in the presence or absence of buoyancy ratio. Buoyancy ratio parameter (Nr) is found to enhance temperature and nanoparticle volume fraction profiles while it impedes velocity profile.
3. With increasing Eckert number (Ec), both velocity and temperature profiles are increased for a fixed value of conduction-radiation (N) parameter. The velocity is found to increase with an increase in conduction-radiation parameter (N), while the temperature profile is decreased with increasing conduction-radiation (N) parameter.
4. An increase in Darcy number (Da) increases the velocity profile for a fixed value of the Forchheimer parameter (Fs). The velocity profile is found to be a decreasing function of the Forchheimer parameter (Fs).
5. The velocity profile is reduced as the magnetic parameter increases. Increasing the convective heating parameter (Bi) leads to an increase in both the velocity and temperature profiles.
6. An increase in the thermophoresis parameter (Nt) enhances both temperature and nanoparticle volume fraction profiles for a fixed value of the Brownian motion parameter (Nb). An increase in the Brownian motion parameter increases the temperature profile and decreases the nanoparticle volume fraction profile.
7. Nusselt number profile is increased with increasing convective heating (Bi), Casson (β) and temperature ratio (Tr) parameters while it is reduced with increasing Brownian motion (Nb), thermophoresis (Nt) and buoyancy ratio (Nr) parameters
8. Sherwood number profile is found to be an increasing function of Lewis number (Le), convective heating (Bi), Brownian motion (Nb), and Casson (β) parameters while it is a decreasing function of thermophoresis (Nt) and buoyancy ratio (Nr) parameters.

4 Discussion of the Results Obtained for Model 3

In the present section, the variations of the controlling fluid parameters were carried out

and corresponding solutions are presented using graphs and a table. The following values of the parameters

$Sc = 5, Pr = 6.8, Tr = 2, Ec = 0.1, m = 0.5, \lambda = 1, \alpha = 0.1, \delta_2 = -0.5, S = 0, Nt = 0.01, Nb = 0.01, Pp = 0.1, \delta_1 = 0.1, \lambda = 0.1$

and $n = 16$ are kept unchanged throughout the study unless otherwise stated. The square residual errors for $f(\eta), \theta(\eta)$ and $\phi(\eta)$ are computed respectively using the following formulae

$$\varepsilon_{f_3} = \int_0^\infty R_{f_3}^2(\eta) d\eta \approx \sum_{k=1}^j \left[A_k \left(e^{\eta} R_{f_3}^2(\eta) \right)_{\eta=x_k} \right] \quad (171)$$

$$\varepsilon_{\theta_3} = \int_0^\infty R_{\theta_3}^2(\eta) d\eta \approx \sum_{k=1}^j \left[A_k \left(e^{\eta} R_{\theta_3}^2(\eta) \right)_{\eta=x_k} \right] \quad (172)$$

$$\varepsilon_{\phi_3} = \int_0^\infty R_{\phi_3}^2(\eta) d\eta \approx \sum_{k=1}^j \left[A_k \left(e^{\eta} R_{\phi_3}^2(\eta) \right)_{\eta=x_k} \right] \quad (173)$$

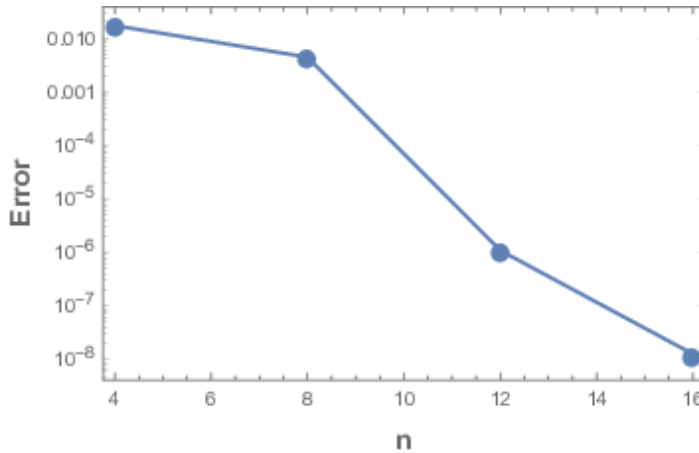


Figure 69: Average square residual errors at different order of approximation (n)

Figure 4.44 shows the graph of the total average square residual error ($\varepsilon_{total} = \frac{\varepsilon_{f_3} + \varepsilon_{\theta_3} + \varepsilon_{\phi_3}}{3}$). It was observed that the total average square residual error reduces as the value of n is increased. The values of the Nusselt numbers for the reduced problem were compared with the published results of Abolbashari *et al.* (2015) via homotopy analysis method to ascertain the accuracy of the used method. A good agreement was found at it is seen in Table 4.3.

Table 4: Comparison of reduced wall heat transfer rate ($-\theta'(0)$) in the absence of nanoparticle for different values of Pr when $M = Pp = Fs = S = \delta_1 = \delta_2 = 0, \beta \rightarrow \infty$

Pr	Present study	Abolbashari <i>et al.</i>	Relative error
----	---------------	---------------------------	----------------

		2015	
.2	0.1691	0.1691	0.00%
.7	0.4539	0.4539	0.00%
.0	0.9114	0.9114	0.00%
.0	1.8954	1.8954	0.00%

Figures 4.45–4.47 depict the effects of porosity (Pp) and first-order velocity slip (δ_1) parameters on dimensionless velocity, temperature, and nanoparticle volume fraction. An increase in porosity parameter (Pp) reduces the velocity profile (see Figure 4.45) and increases both the temperature (see Figure 4.46) and nanoparticle volume fraction (see Figure 4.47) profiles. Porosity parameter (Pp) is inversely proportional to fluid permeability (k_p). Increasing porosity parameter, manifests in increased resistance to the Casson nanofluid flow which leads to flow retardation. The fluid temperature increases since the flow is slightly stagnated and allow more of the heat to be conducted to the fluid. Furthermore, an increase in the first-order slip parameter reduces the velocity profile (see Figure 4.45) and increases both temperature and nanoparticle volume fraction profiles (see Figures 4.46 and 4.47). These phenomena are so because not all the pulling force from the stretching sheet can be transmitted to the Casson nanofluid. This leads to a decrease in velocity profile and a corresponding increase in temperature and nanoparticle volume fraction profiles.

Figures 4.48-4.50 illustrate the effects of second-order slip and wall transpiration (suction/injection) parameters (δ_2 and S) on velocity, temperature, and nanoparticle volume fraction profiles. The velocity profile is decreased (see Figure 4.48), while both the temperature and nanoparticle volume fraction profiles are increased (see Figures 4.49 and 4.50) as the second-order slip parameter increases for both wall injection ($S < 0$) and suction ($S > 0$). Also, the velocity profile rises with injection while it suppresses with suction. The opposite is the case for both temperature and nanoparticle volume fraction profiles.

Figures 4.51-4.53 analyze the effects of Casson and Magnetic field parameters (β and M) on velocity, temperature, and nanoparticle volume fraction profiles. An increased Casson parameter (β) leads to a decrease in velocity profile and an increase in temperature and nanoparticle volume fraction profiles in the presence or absence of the magnetic field. This can be attributed to the fact that increasing Casson parameter (β) resists the fluid flow as it decreases the yield stress of Casson fluid and increases plastic dynamic viscosity. This results to the retardation of the velocity profile (see Figure 4.51) and in turn, enhances both temperature and nanoparticle volume fraction profiles (see Figures 4.52 and 4.53). An increase in the value of magnetic parameter (M) leads to a reduction of dimensionless velocity (see Figure 4.51). This is due to the presence of Lorentz force produced by a magnetic field placed in a transverse direction to the electrically conducting Casson nanofluid. The additional work done in dragging the fluid against the Lorentzian drag force heats the fluid and later leads to a rise in temperature profile (see Figure 4.52) and corresponding nanoparticle concentration profile (see Figure 4.53).

Figures 4.54-4.56 illustrate the effects of stretching and Forchheimer parameters (λ and F_s) on velocity, temperature, and nanoparticle volume fraction profiles. The velocity profile is found to increase (see Figure 4.54) while both the temperature and nanoparticle volume fraction profiles are decreased (see Figures 4.55 and 4.56) with increasing stretching parameter (λ). This is because an increased value of (λ) enhances momentum diffusion while it impedes heat and

species diffusion. An increase in Forchheimer parameter (F_s) reduces the velocity profile while it increases both temperature and nanoparticle volume fraction profiles (see Figures 4.55-4.56). This is a result of an increase in fluid inertial force which in turn suppresses the velocity profile and increases both temperature and nanoparticle volume fraction profiles. The effect of Eckert number (Ec) on temperature profile is analyzed in Figure 4.57. Increasing the value of the Eckert number (Ec) is found to elevate temperature profile. This is because the thermal boundary layer is increased with greater viscous heating effect due to the conversion of kinetic energy into heat energy in the boundary layer.

Figure 4.58 illustrates the impacts of chemical reaction (σ^*) and activation energy (E) on the nanoparticle volume fraction profile. It is observed in the last part of Eq. (3.62) that $\sigma^{*2} \exp\left(\frac{-E}{(Tr-1)\theta+1}\right)$ is increased as σ^* increases. This leads to a decrease in the nanoparticle volume fraction profile. Nanoparticle volume fraction profile is increased with increasing activation energy parameter (E) as it is observed in Figure 4.58. In this expression $\sigma^{*2} \exp\left(\frac{-E}{(Tr-1)\theta+1}\right)$ (which is in the last part of Eq. (3.62)), chemical reaction rate (σ^*) decreases with increase in activation energy which in turn results to an increase in nanoparticle volume fraction profile. The effects of thermophoresis and Brownian motion parameters (Nt and Nb) on temperature and nanoparticle volume fraction profiles are depicted in Figures 4.59 and 4.60. Temperature and nanoparticle volume fraction profiles are increased (see Figures 4.59 and 4.60) with increasing thermophoresis parameter (Nt). However, an increase in the Brownian motion parameter (Nb) enhances the temperature profile (see Figure 4.59) while it impedes nanoparticle volume fraction profile (see Figure 4.60).

Figures 4.61 and 4.62 present the effects of these various parameters Nt , Nb , Ec , Pp , M and S on Nusselt number. It is observed that Nusselt number is a decreasing functions of Brownian motion, thermophoresis, Eckert number, Magnetic field, porosity and injection parameters (Nb , Nt , Ec , M , Pp , $S < 0$), whereas it is an increasing function of only suction parameter ($S > 0$). Also, Figures 4.63 and 4.64 depict the effects of parameters σ^* , E , Nt , Kr , Pp , M and S on Sherwood number. It is noticed that Sherwood number is increased as each of chemical reaction rate parameter (σ^*), thermophoresis parameter (Nt) and suction parameter ($S > 0$) increases while it is decreased with increasing activation energy parameter (E), porosity parameter (Pp), Magnetic field parameter (M) and injection parameter ($S < 0$).

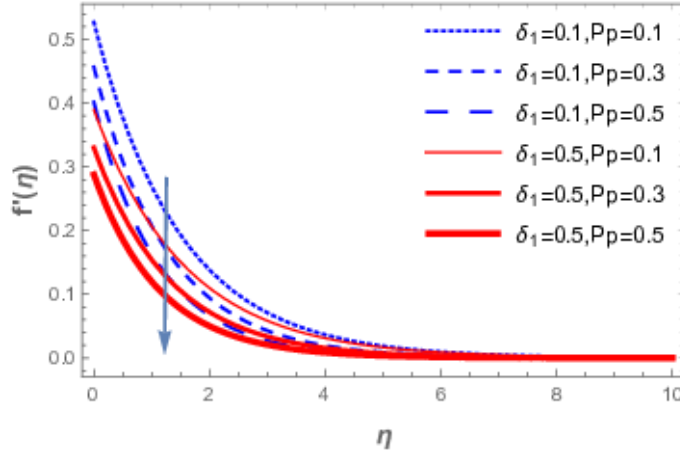


Figure 70: Velocity profile for different values of Pp and δ_1

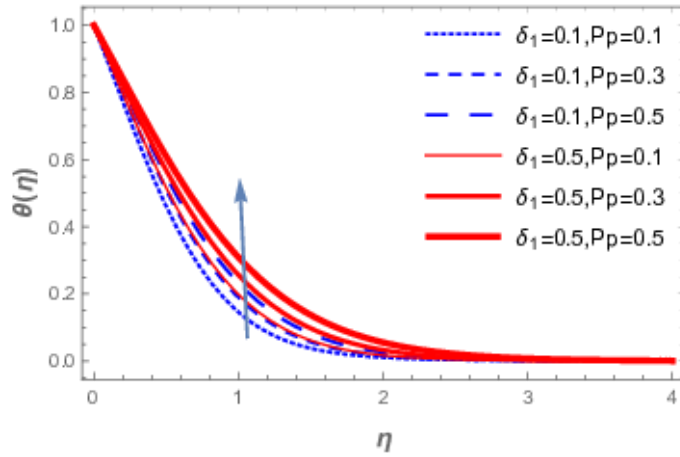


Figure 71: Temperature profile for different values of Pp and δ_1

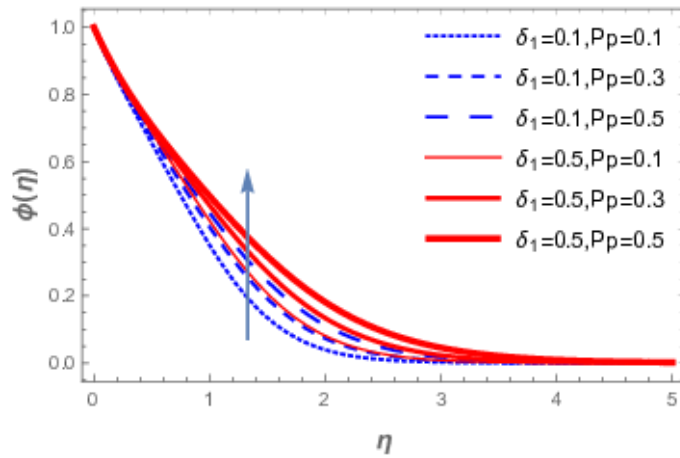


Figure 72: Nanoparticle concentration profile for different values of Pp and δ_1

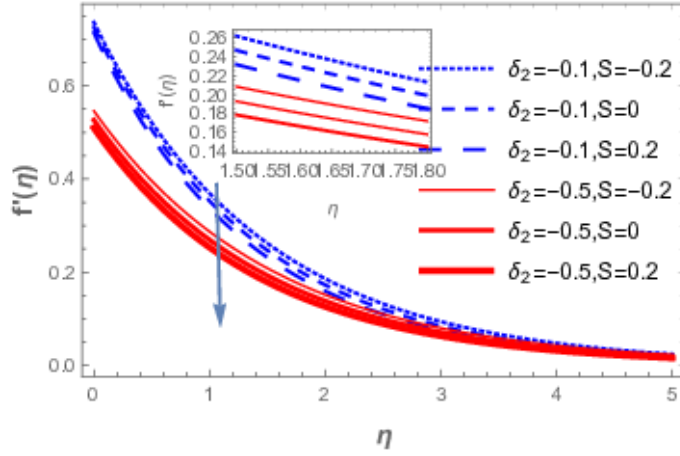


Figure 73: Velocity profile for different values of δ_2 and S

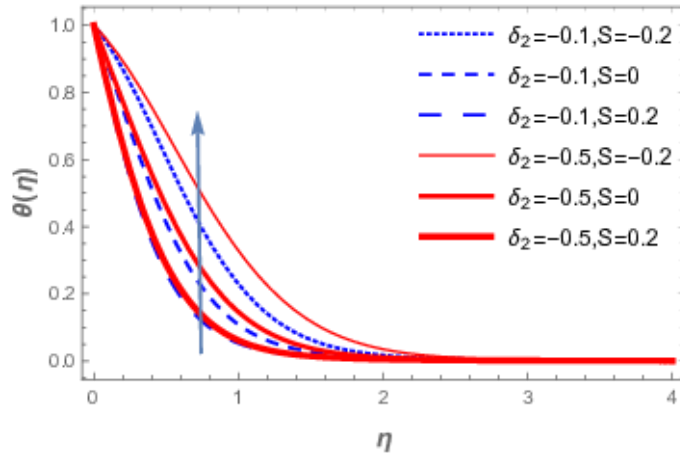


Figure 74: Temperature profile for different values of δ_2 and S

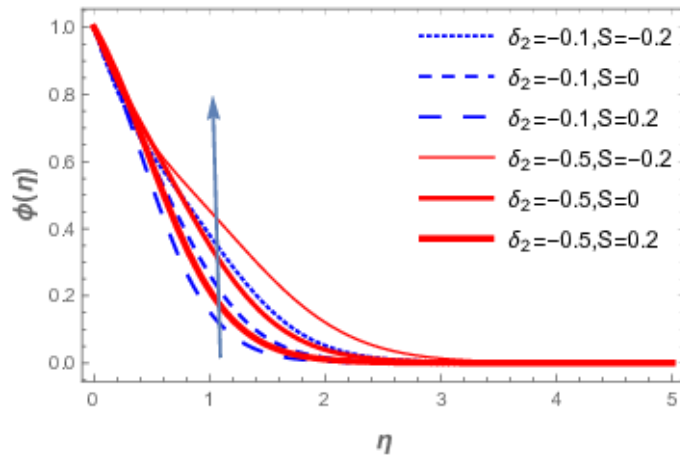


Figure 75: Nanoparticle concentration profile for different values of δ_2 and S

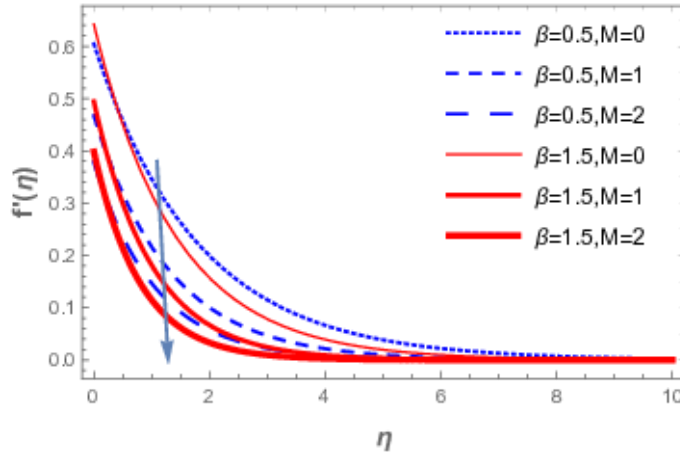


Figure 76: Velocity profile for different values of β and M

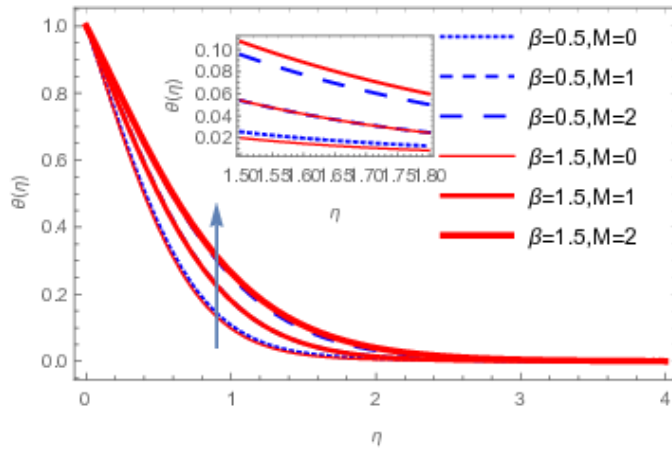


Figure 77: Temperature profile for different values of β and M

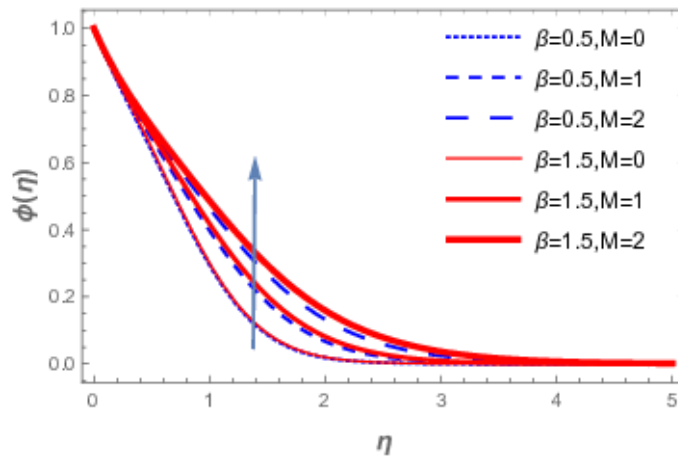


Figure 78: Nanoparticle concentration profile for different values of β and M

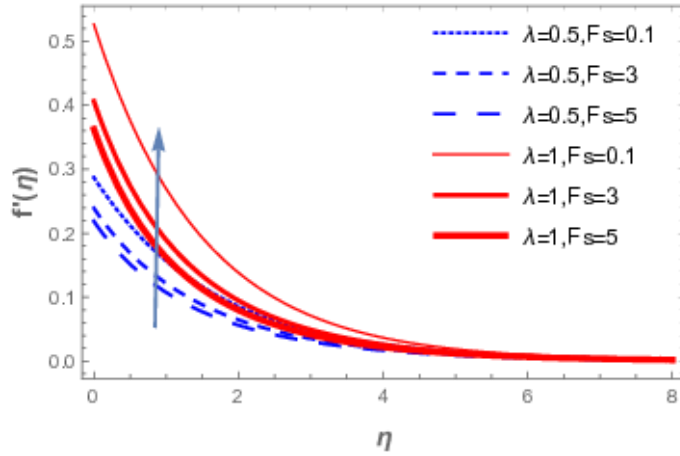


Figure 79: Velocity profile for different values of λ and F_s

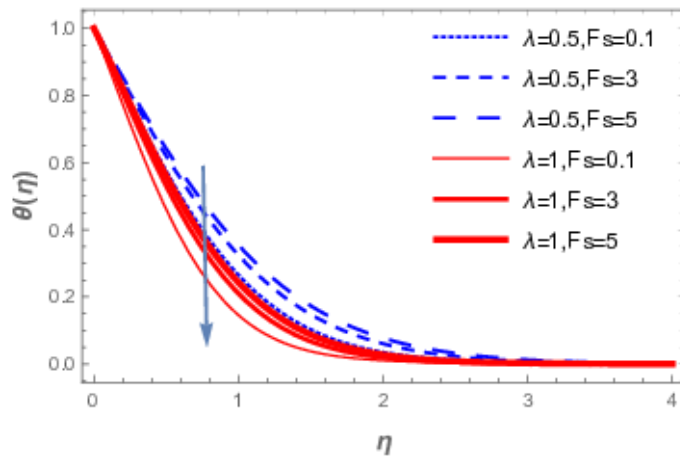


Figure 80: Temperature profile for different values of λ and F_s

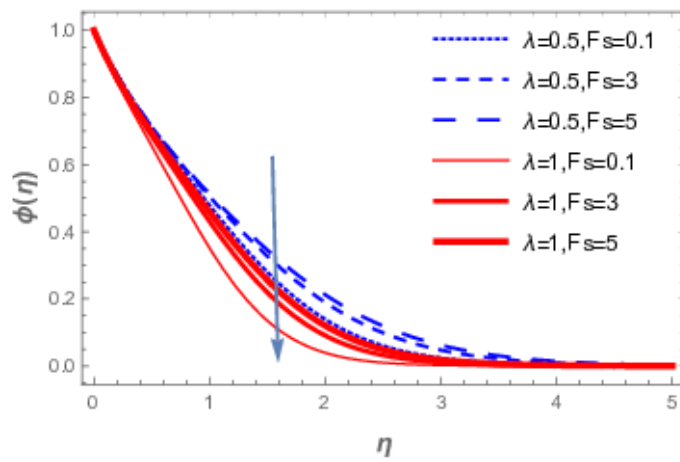


Figure 81: Nanoparticle concentration profile for different values of λ and F_s

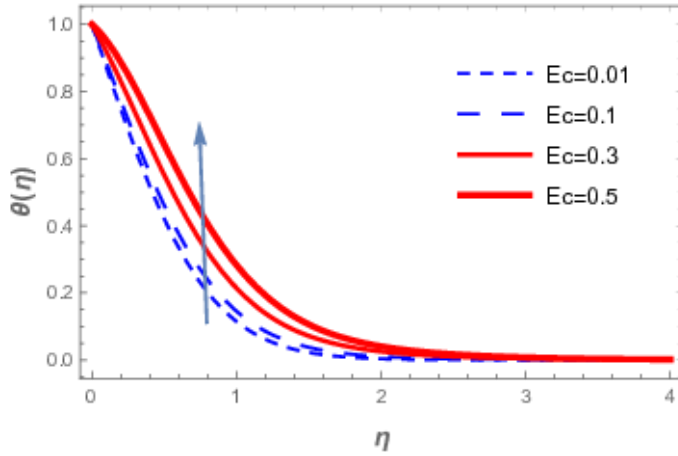


Figure 82: Temperature profile for different values of Ec

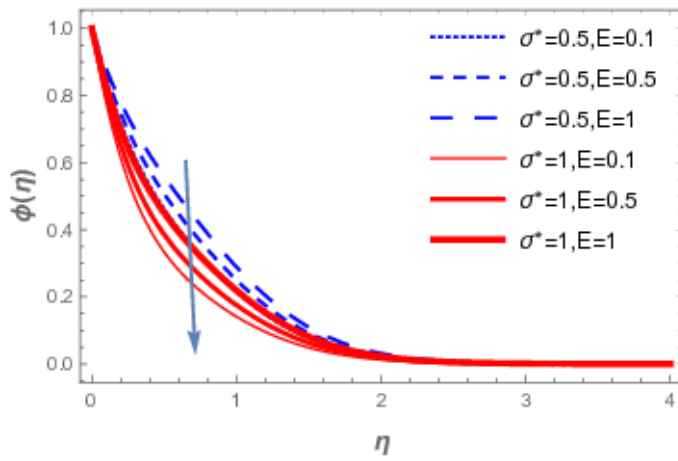


Figure 83: Nanoparticle concentration profile for different values of σ^* and E

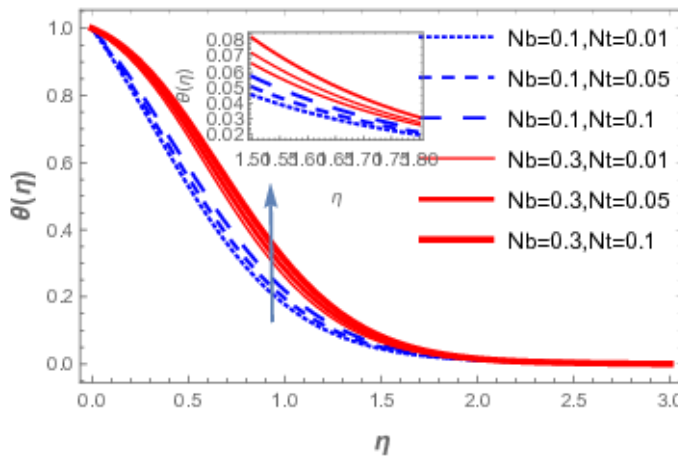


Figure 84: Temperature profile for different values of Nt and Nb

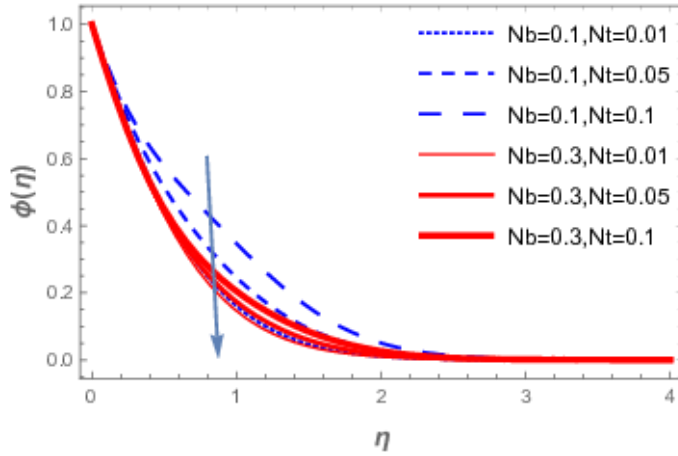


Figure 85: Nanoparticle concentration profile for different values of Nt and Nb

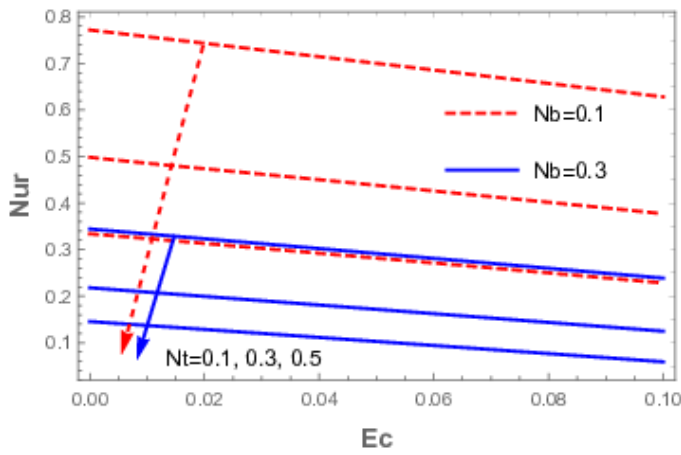


Figure 86: Nusselt number profile for different values of Nt , Nb and Ec

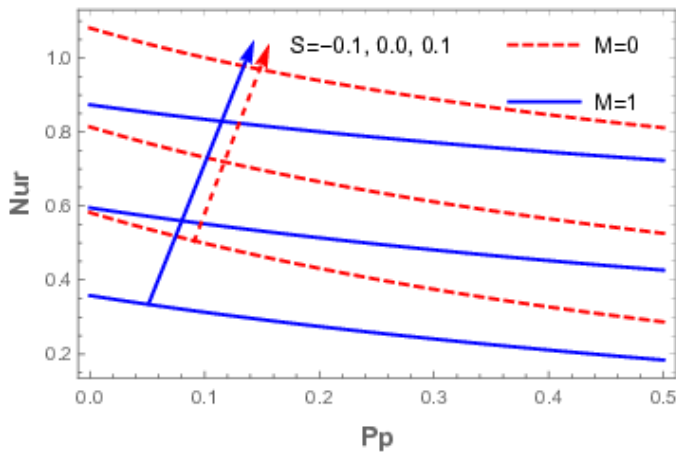


Figure 87: Nusselt number profile for different values of Pp , S and M

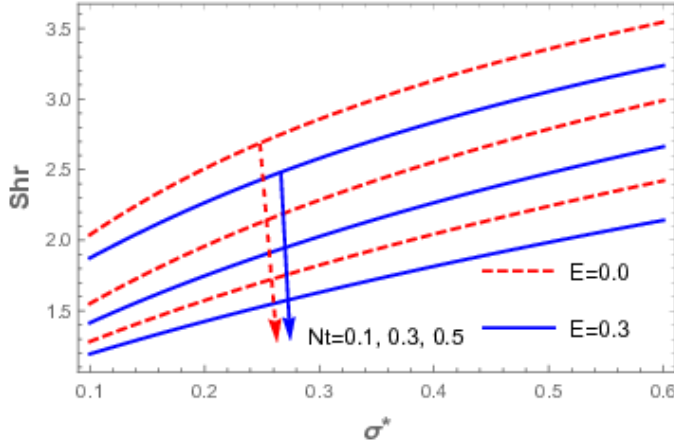


Figure 88: Sherwood number profile for different values of Nt , E and σ^*

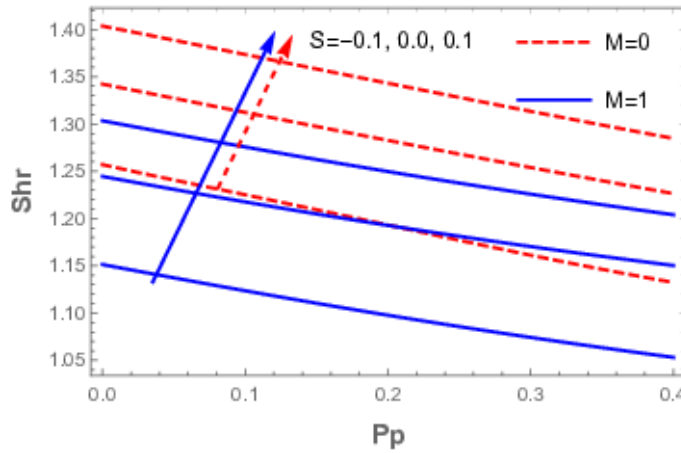


Figure 89: Sherwood number profile for different values of Pp , S and M

In conclusion, this section deals with the effects of a binary chemical reaction and activation energy on MHD dissipative Casson nanofluid flow over a stretching sheet in a non-Darcian porous medium with a second-order slip boundary condition. The resulting partial differential equations governing the flow, heat and nanoparticle volume fraction were non-dimensionalized and reduced to ordinary differential equations. The said equations (ordinary differential equations) together with corresponding boundary conditions were solved using Galerkin weighted residual method. The obtained results were presented and discussed graphically. The major findings are as follows: [label=()]

1. An increase in first-order velocity slip parameter (δ_1) decreases the velocity profile while it increases temperature and nanoparticle volume fraction profiles for a fixed value of the porosity parameter (Pp). The temperature and nanoparticle volume fraction are increased while the velocity profile is reduced as the porosity parameter (Pp) increases.

2. Increasing the second order velocity slip parameter (δ_2) reduces the velocity profile and enhances both temperature and nanoparticle volume fraction in the presence of wall suction ($S > 0$) or injection ($S < 0$) parameter. The velocity profile is increased in the case of the wall injection parameter while it is reduced for suction. Both the temperature and

nanoparticle volume fraction profiles are increased in the presence of the suction parameter but reduced for injection case.

3. The velocity profile is reduced while temperature and nanoparticle volume fraction profiles increased with increasing Casson parameter (β) in the presence or absence of a magnetic field (M). An increase in the Magnetic field is found to increase both temperature and nanoparticle volume fraction profiles while it reduces the velocity profile.

4. With increasing Forchheimer parameter (Fs), the velocity profile is reduced while both temperature and nanoparticle profiles are increased. Increasing the stretching parameter (λ) enhances the velocity profile and impede temperature and nanoparticle volume fraction profiles.

5. An increase in the Eckert number parameter (Ec) increases the temperature profile.

6. The nanoparticle volume fraction is increased with increasing activation energy parameter (E). It is also found that increasing the chemical reaction rate parameter (σ^*) reduces the nanoparticle volume fraction.

7. An increase in thermophoresis parameter (Nt) increases both temperature and nanoparticle volume fraction. Increasing the Brownian motion parameter (Nb) leads to an increase in temperature profile while it impedes the nanoparticle volume fraction profile.

8. The Nusselt number profile is reduced with increasing thermophoresis parameter (Nt), Eckert number (Ec), Magnetic field parameter (M), porosity parameter (Pp) and injection parameter ($S < 0$). Increasing the suction parameter ($S > 0$) enhances the Nusselt number profile.

9. Increasing chemical reaction rate parameter (σ^*), thermophoresis parameter (Nt) and suction parameter ($S > 0$) increase Sherwood number profile while increasing activation energy parameter (E), porosity parameter (Pp), Magnetic field parameter (M) and injection parameter ($S < 0$) reduce Sherwood number.

5 SUMMARY, CONCLUSION AND RECOMENDATIONS

1 Introduction

This chapter provides a detailed summary of the work carried out in this thesis. The major findings of this research, contribution to knowledge and recommendations are also presented.

2 Summary of the Work Done

This work focused on the combination of one of the non-Newtonian fluids (Casson fluid) with nanoparticles (metallic or nonmetallic substance) to yield what is known as Casson nanofluid. Conventional heat transfer fluids (Newtonian or non-Newtonian fluids) are low thermal conductivity fluids. Therefore, their applications in cooling tools result in increased manufacturing and operating costs. For these reasons, engineers have started developing novel technologies based on nanofluid to enhance the thermal performance of relevant fluids. Specifically, in this thesis, three interesting problems viz (i) Non-Darcian MHD dissipative Casson nanofluid flow along a stretching or shrinking sheet in the presence of multiple Navier slip boundary conditions (ii) MHD Casson nanofluid flow with variable properties and (iii) MHD non-Darcian Casson nanofluid flow in the presence of non-Navier (second-order) slip boundary condition are studied. The nonlinear partial differential equations governing the problems were first reduced to nonlinear ordinary

differential equations with relevant boundary conditions and then solved using a numerical technique (GWRM). The effects of pertinent controlling parameters in each of the problems were presented graphically and analyzed.

3 Conclusion

This thesis investigated the analysis of non-Darcian MHD dissipative Casson nanofluid flow past a stretching or shrinking sheet with slip and convective heating boundary conditions. The obtained governing partial differential equations for continuity, momentum, energy, and nanoparticle concentration equations are nondimensionalized and then reduced to ordinary nonlinear differential equations using a similarity transformation. The resulting nonlinear ordinary differential equations are solved numerically using Galerking weighted residual method (GWRM). The numerical results for the velocity, temperature and nanoparticle volume fraction profiles, wall heat transfer rate (Nusselt number) and wall mass transfer rate (Sherwood number) are obtained and presented graphically to show the effects of pertinent parameters such as Casson parameter (β), Eckert number (Ec), convective heating parameter (Bi), Darcy number (Da), Forchheimer parameter (Fs), conduction-radiation parameter (N), first order velocity slip parameter (δ_1), second order velocity slip parameter (δ_2), thermal slip parameter (b), mass slip parameter (c), Brownian motion parameter (Nb), thermophoresis parameter (Nt), temperature dependent viscosity parameter (γ), temperature dependent thermal conductivity parameter (ϕ) parameter, chemical reaction rate parameter (σ^*), activation energy parameter (E) and suction/injection parameter (S). The major conclusions are shown as follows: [label=()]

1. An increase in Forchheimer (non-Darcian porous medium) parameter reduces the velocity, while it enhances both temperature and nanoparticle volume fraction.
2. Increasing temperature-dependent viscosity and thermal conductivity increase velocity and suppress both the temperature and nanoparticle volume fraction.
3. Increasing chemical reaction parameter reduces concentration profile while increasing activation energy parameter leads to an increase in concentration profile.
4. An increase in the thermal radiation parameter reduces the temperature.

4 Recommendations

It is observed in this study that the various pertinent fluid controlling parameters considered in this work have significant effects on the heat and mass transfer of Casson nanofluid for applications in industrial coolant, brake fluid, cancer therapy, MHD generator, nanodrug delivery and so on. It is then recommended that those parameters should be considered when carrying out models involving heat and mass transfer of Casson nanofluid flow.

5 Contribution to Knowledge

Some of the importance of this thesis work are highlighted below. [label=()]

1. It is revealed in this work that an increase in the magnetic field reduces the velocity profile and enhances temperature and nanoparticle volume fraction profiles. This implies that this model will be useful in some engineering applications (such as solar film collector, heat exchanger technology, geothermal energy storage, engine coolant) dealing with the cooling and heating process.
2. It is well known that when an external magnetic field is imposed on a ferrofluid (magnetic nanofluid), it results in a nonuniform magnetic body force which leads to a form of heat transfer called thermomagnetic convection. This is applicable in enhancing conventional

convective heat transfer. This present work is useful in this area when considering magnetic nanoparticles

3. Another promising approach for cancer treatment is the magnetic drug targeting (MDT). Magnetite (Fe_3O_4), maghemite ($\gamma-Fe_2O_3$), and other ferrites are commonly used for MDT because they can be guided by an external magnetic field (Dorniani *et al.*, 2012). MDT delivery includes the following steps: attachment of a cytotoxic drug to bio-compatible magnetic nanoparticles; injection of ferrofluid into patient's body; application of high-gradient magnetic field for conduction of drugs to the target tissue; release of drug in the target tissue (Sun *et al.*, 2008).

Casson fluid model being considered in this research is widely used in medical therapy to model blood flow. This research work is applicable in this area (MDT) especially when the Casson fluid model is being combined with magnetic nanoparticle in the presence of an external magnetic field. The blood flow can be represented by Casson fluid while the magnetic nanoparticles will be considered as a drug carrier and an external magnetic field can be used to control the drug to reach its target.

4. It is found in this work that an increase in porosity parameter (P_p), implies a lower permeability in the porous medium. This is equivalent to an increase in the presence of solid fibers which consequently reduces the fluid flow and increases both temperature and nanoparticle concentration. The fluid flow in the presence of a porous medium may be exploited to regulate the fluid temperature and nanoparticle concentration for manufacturing nanomaterials for different technological applications including thermal insulation, combustion, biomechanics, and heat exchangers

6 Sugestion for future studies

This research work investigated the steady boundary layer flow of Casson nanofluid subjected to various pertinent parameters. Further study that can be considered is to investigate the unsteady boundary layer Casson nanofluid flow. It will also be interesting to study Casson nanofluid flow in a cylindrical channel. Other weighted residual methods such as the collocation method, moment method, partitioning method and a widely used semi-analytical method (Homotopy analysis method) can be considered for future research.

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0683-6.

Appendix A

$$\begin{aligned}
R_{f_1}(a_i, b_k, c_k, \eta) = & \frac{1}{\text{Da}} \left((1 + \frac{1}{\beta}) \left(K + \frac{1}{3} e^{-\frac{\eta}{3}} a_1 + \frac{2}{3} e^{-\frac{2\eta}{3}} a_2 + e^{-\eta} a_3 + \frac{4}{3} e^{-\frac{4\eta}{3}} a_4 + \frac{5}{3} e^{-\frac{5\eta}{3}} a_5 \right. \right. \\
& + 2e^{-2\eta} a_6 + \frac{7}{3} e^{-\frac{7\eta}{3}} a_7 + \frac{8}{3} e^{-\frac{8\eta}{3}} a_8 + 3e^{-3\eta} a_9 + \frac{10}{3} e^{-\frac{10\eta}{3}} a_{10} + \frac{11}{3} e^{-\frac{11\eta}{3}} a_{11} + 4e^{-4\eta} a_{12} \\
& + \frac{13}{3} e^{-\frac{13\eta}{3}} a_{13} + \frac{14}{3} e^{-\frac{14\eta}{3}} a_{14} + 5e^{-5\eta} a_{15} + \frac{16}{3} e^{-\frac{16\eta}{3}} a_{16} \Big) + (1 + \frac{1}{\beta}) \left(-\frac{1}{27} e^{-\frac{\eta}{3}} a_1 - \frac{8}{27} e^{-\frac{2\eta}{3}} a_2 \right. \\
& - e^{-\eta} a_3 - \frac{64}{27} e^{-\frac{4\eta}{3}} a_4 - \frac{125}{27} e^{-\frac{5\eta}{3}} a_5 - 8e^{-2\eta} a_6 - \frac{343}{27} e^{-\frac{7\eta}{3}} a_7 - \frac{512}{27} e^{-\frac{8\eta}{3}} a_8 - 27e^{-3\eta} a_9 \\
& - \frac{1000}{27} e^{-\frac{10\eta}{3}} a_{10} - \frac{1331}{27} e^{-\frac{11\eta}{3}} a_{11} - 64e^{-4\eta} a_{12} - \frac{2197}{27} e^{-\frac{13\eta}{3}} a_{13} - \frac{2744}{27} e^{-\frac{14\eta}{3}} a_{14} - 125e^{-5\eta} a_{15} \\
& - \frac{4096}{27} e^{-\frac{16\eta}{3}} a_{16} \Big) + \frac{1}{4\text{Pr}} \left(-2 \left(-\frac{1}{3} e^{-\frac{\eta}{3}} a_1 - \frac{2}{3} e^{-\frac{2\eta}{3}} a_2 - e^{-\eta} a_3 - \frac{4}{3} e^{-\frac{4\eta}{3}} a_4 - \frac{5}{3} e^{-\frac{5\eta}{3}} a_5 - 2e^{-2\eta} a_6 \right. \right. \\
& - \frac{7}{3} e^{-\frac{7\eta}{3}} a_7 - \frac{8}{3} e^{-\frac{8\eta}{3}} a_8 - 3e^{-3\eta} a_9 - \frac{10}{3} e^{-\frac{10\eta}{3}} a_{10} - \frac{11}{3} e^{-\frac{11\eta}{3}} a_{11} - 4e^{-4\eta} a_{12} - \frac{13}{3} e^{-\frac{13\eta}{3}} a_{13} \\
& - \frac{14}{3} e^{-\frac{14\eta}{3}} a_{14} - 5e^{-5\eta} a_{15} - \frac{16}{3} e^{-\frac{16\eta}{3}} a_{16} \Big)^2 + 2K + 3(K\eta + a_0 + e^{-\frac{\eta}{3}} a_1 + e^{-\frac{2\eta}{3}} a_2 + e^{-\eta} a_3 \\
& + e^{-\frac{4\eta}{3}} a_4 + e^{-\frac{5\eta}{3}} a_5 + e^{-2\eta} a_6 + e^{-\frac{7\eta}{3}} a_7 + e^{-\frac{8\eta}{3}} a_8 + e^{-3\eta} a_9 + e^{-\frac{10\eta}{3}} a_{10} + e^{-\frac{11\eta}{3}} a_{11} + e^{-4\eta} a_{12} \\
& + e^{-\frac{13\eta}{3}} a_{13} + e^{-\frac{14\eta}{3}} a_{14} + e^{-5\eta} a_{15} + e^{-\frac{16\eta}{3}} a_{16} \Big) \left(\frac{1}{9} e^{-\frac{\eta}{3}} a_1 + \frac{4}{9} e^{-\frac{2\eta}{3}} a_2 + e^{-\eta} a_3 + \frac{16}{9} e^{-\frac{4\eta}{3}} a_4 \right. \\
& + \frac{25}{9} e^{-\frac{5\eta}{3}} a_5 + 4e^{-2\eta} a_6 + \frac{49}{9} e^{-\frac{7\eta}{3}} a_7 + \frac{64}{9} e^{-\frac{8\eta}{3}} a_8 + 9e^{-3\eta} a_9 + \frac{100}{9} e^{-\frac{10\eta}{3}} a_{10} + \frac{121}{9} e^{-\frac{11\eta}{3}} a_{11} \\
& + 16e^{-4\eta} a_{12} + \frac{169}{9} e^{-\frac{13\eta}{3}} a_{13} + \frac{196}{9} e^{-\frac{14\eta}{3}} a_{14} + 25e^{-5\eta} a_{15} + \frac{256}{9} e^{-\frac{16\eta}{3}} a_{16} \Big) + 4M(K^2 \\
& - \left(-\frac{1}{3} e^{-\frac{\eta}{3}} a_1 - \frac{2}{3} e^{-\frac{2\eta}{3}} a_2 - e^{-\eta} a_3 - \frac{4}{3} e^{-\frac{4\eta}{3}} a_4 - \frac{5}{3} e^{-\frac{5\eta}{3}} a_5 - 2e^{-2\eta} a_6 - \frac{7}{3} e^{-\frac{7\eta}{3}} a_7 - \frac{8}{3} e^{-\frac{8\eta}{3}} a_8 \right. \\
& - 3e^{-3\eta} a_9 - \frac{10}{3} e^{-\frac{10\eta}{3}} a_{10} - \frac{11}{3} e^{-\frac{11\eta}{3}} a_{11} - 4e^{-4\eta} a_{12} - \frac{13}{3} e^{-\frac{13\eta}{3}} a_{13} - \frac{14}{3} e^{-\frac{14\eta}{3}} a_{14} - 5e^{-5\eta} a_{15} \\
& - \frac{16}{3} e^{-\frac{16\eta}{3}} a_{16} \Big)^2 \Big) + 4Fs(K^2 - \left(-\frac{1}{3} e^{-\frac{\eta}{3}} a_1 - \frac{2}{3} e^{-\frac{2\eta}{3}} a_2 - e^{-\eta} a_3 - \frac{4}{3} e^{-\frac{4\eta}{3}} a_4 - \frac{5}{3} e^{-\frac{5\eta}{3}} a_5 - 2e^{-2\eta} a_6 \right. \\
& - \frac{7}{3} e^{-\frac{7\eta}{3}} a_7 - \frac{8}{3} e^{-\frac{8\eta}{3}} a_8 - 3e^{-3\eta} a_9 - \frac{10}{3} e^{-\frac{10\eta}{3}} a_{10} - \frac{11}{3} e^{-\frac{11\eta}{3}} a_{11} - 4e^{-4\eta} a_{12} - \frac{13}{3} e^{-\frac{13\eta}{3}} a_{13} \\
& - \frac{14}{3} e^{-\frac{14\eta}{3}} a_{14} - 5e^{-5\eta} a_{15} - \frac{16}{3} e^{-\frac{16\eta}{3}} a_{16} \Big)^2 \Big) + e^{-\frac{\eta}{3}} b_1 + e^{-\frac{2\eta}{3}} b_2 + e^{-\eta} b_3 + e^{-\frac{4\eta}{3}} b_4 + e^{-\frac{5\eta}{3}} b_5 + e^{-2\eta} b_6 \\
& + e^{-\frac{7\eta}{3}} b_7 + e^{-\frac{8\eta}{3}} b_8 + e^{-3\eta} b_9 + e^{-\frac{10\eta}{3}} b_{10} + e^{-\frac{11\eta}{3}} b_{11} + e^{-4\eta} b_{12} + e^{-\frac{13\eta}{3}} b_{13} + e^{-\frac{14\eta}{3}} b_{14}
\end{aligned}$$

$$\begin{aligned}
& +e^{-5\eta}b_{15}+e^{-\frac{16\eta}{3}}b_{16}-\text{Nr}(e^{\frac{\eta}{3}}c_1+e^{-\frac{2\eta}{3}}c_2+e^{-\eta}c_3+e^{-\frac{4\eta}{3}}c_4+e^{-\frac{5\eta}{3}}c_5+e^{-2\eta}c_6 \\
& +e^{-\frac{7\eta}{3}}c_7+e^{-\frac{8\eta}{3}}c_8+e^{-3\eta}c_9+e^{-\frac{10\eta}{3}}c_{10}+e^{-\frac{11\eta}{3}}c_{11}+e^{-4\eta}c_{12}+e^{-\frac{13\eta}{3}}c_{13}+e^{-\frac{14\eta}{3}}c_{14} \\
& +e^{-5\eta}c_{15}+e^{-\frac{16\eta}{3}}c_{16}),
\end{aligned}$$

$$\begin{aligned}
R_{\theta_1}(a_i, b_k, c_k, \eta) = & \text{EcPr}\left(1+\frac{1}{\beta}\right)\left(\frac{1}{9}e^{-\frac{\eta}{3}}a_1+\frac{4}{9}e^{-\frac{2\eta}{3}}a_2+e^{-\eta}a_3+\frac{16}{9}e^{-\frac{4\eta}{3}}a_4+\frac{25}{9}e^{-\frac{5\eta}{3}}a_5\right. \\
& +4e^{-2\eta}a_6+\frac{49}{9}e^{-\frac{7\eta}{3}}a_7+\frac{64}{9}e^{-\frac{8\eta}{3}}a_8+9e^{-3\eta}a_9+\frac{100}{9}e^{-\frac{10\eta}{3}}a_{10}+\frac{121}{9}e^{-\frac{11\eta}{3}}a_{11}+16e^{-4\eta}a_{12} \\
& +\frac{169}{9}e^{-\frac{13\eta}{3}}a_{13}+\frac{196}{9}e^{-\frac{14\eta}{3}}a_{14}+25e^{-5\eta}a_{15}+\frac{256}{9}e^{-\frac{16\eta}{3}}a_{16})^2+\text{Nt}(-\frac{1}{3}e^{-\frac{\eta}{3}}b_1-\frac{2}{3}e^{-\frac{2\eta}{3}}b_2 \\
& -e^{-\eta}b_3-\frac{4}{3}e^{-\frac{4\eta}{3}}b_4-\frac{5}{3}e^{-\frac{5\eta}{3}}b_5-2e^{-2\eta}b_6-\frac{7}{3}e^{-\frac{7\eta}{3}}b_7-\frac{8}{3}e^{-\frac{8\eta}{3}}b_8-3e^{-3\eta}b_9-\frac{10}{3}e^{-\frac{10\eta}{3}}b_{10} \\
& -\frac{11}{3}e^{-\frac{11\eta}{3}}b_{11}-4e^{-4\eta}b_{12}-\frac{13}{3}e^{-\frac{13\eta}{3}}b_{13}-\frac{14}{3}e^{-\frac{14\eta}{3}}b_{14}-5e^{-5\eta}b_{15}-\frac{16}{3}e^{-\frac{16\eta}{3}}b_{16})^2+\frac{1}{9}e^{-\frac{\eta}{3}}b_1 \\
& +\frac{4}{9}e^{-\frac{2\eta}{3}}b_2+e^{-\eta}b_3+\frac{16}{9}e^{-\frac{4\eta}{3}}b_4+\frac{25}{9}e^{-\frac{5\eta}{3}}b_5+4e^{-2\eta}b_6+\frac{49}{9}e^{-\frac{7\eta}{3}}b_7+\frac{64}{9}e^{-\frac{8\eta}{3}}b_8+9e^{-3\eta}b_9 \\
& +\frac{100}{9}e^{-\frac{10\eta}{3}}b_{10}+\frac{121}{9}e^{-\frac{11\eta}{3}}b_{11}+16e^{-4\eta}b_{12}+\frac{169}{9}e^{-\frac{13\eta}{3}}b_{13}+\frac{196}{9}e^{-\frac{14\eta}{3}}b_{14}+25e^{-5\eta}b_{15} \\
& +\frac{256}{9}e^{-\frac{16\eta}{3}}b_{16}+\frac{3}{4}(K\eta+a_0+e^{-\frac{\eta}{3}}a_1+e^{-\frac{2\eta}{3}}a_2+e^{-\eta}a_3+e^{-\frac{4\eta}{3}}a_4+e^{-\frac{5\eta}{3}}a_5+e^{-2\eta}a_6 \\
& +e^{-\frac{7\eta}{3}}a_7+e^{-\frac{8\eta}{3}}a_8+e^{-3\eta}a_9+e^{-\frac{10\eta}{3}}a_{10}+e^{-\frac{11\eta}{3}}a_{11}+e^{-4\eta}a_{12}+e^{-\frac{13\eta}{3}}a_{13}+e^{-\frac{14\eta}{3}}a_{14} \\
& +e^{-5\eta}a_{15}+e^{-\frac{16\eta}{3}}a_{16})(-\frac{1}{3}e^{-\frac{\eta}{3}}b_1-\frac{2}{3}e^{-\frac{2\eta}{3}}b_2-e^{-\eta}b_3-\frac{4}{3}e^{-\frac{4\eta}{3}}b_4-\frac{5}{3}e^{-\frac{5\eta}{3}}b_5-2e^{-2\eta}b_6 \\
& -\frac{7}{3}e^{-\frac{7\eta}{3}}b_7-\frac{8}{3}e^{-\frac{8\eta}{3}}b_8-3e^{-3\eta}b_9-\frac{10}{3}e^{-\frac{10\eta}{3}}b_{10}-\frac{11}{3}e^{-\frac{11\eta}{3}}b_{11}-4e^{-4\eta}b_{12}-\frac{13}{3}e^{-\frac{13\eta}{3}}b_{13} \\
& -\frac{14}{3}e^{-\frac{14\eta}{3}}b_{14}-5e^{-5\eta}b_{15}-\frac{16}{3}e^{-\frac{16\eta}{3}}b_{16})+\frac{4}{3N}((\frac{1}{9}e^{-\frac{\eta}{3}}b_1+\frac{4}{9}e^{-\frac{2\eta}{3}}b_2+e^{-\eta}b_3+\frac{16}{9}e^{-\frac{4\eta}{3}}b_4 \\
& +\frac{25}{9}e^{-\frac{5\eta}{3}}b_5+4e^{-2\eta}b_6+\frac{49}{9}e^{-\frac{7\eta}{3}}b_7+\frac{64}{9}e^{-\frac{8\eta}{3}}b_8+9e^{-3\eta}b_9+\frac{100}{9}e^{-\frac{10\eta}{3}}b_{10}+\frac{121}{9}e^{-\frac{11\eta}{3}}b_{11} \\
& +16e^{-4\eta}b_{12}+\frac{169}{9}e^{-\frac{13\eta}{3}}b_{13}+\frac{196}{9}e^{-\frac{14\eta}{3}}b_{14}+25e^{-5\eta}b_{15}+\frac{256}{9}e^{-\frac{16\eta}{3}}b_{16})((\text{Tr}-1)(e^{-\frac{\eta}{3}}b_1
\end{aligned}$$

$$\begin{aligned}
& +e^{-\frac{2\eta}{3}}b_2 + e^{-\eta}b_3 + e^{-\frac{4\eta}{3}}b_4 + e^{-\frac{5\eta}{3}}b_5 + e^{-2\eta}b_6 + e^{-\frac{7\eta}{3}}b_7 + e^{-\frac{8\eta}{3}}b_8 + e^{-3\eta}b_9 + e^{-\frac{10\eta}{3}}b_{10} \\
& +e^{-\frac{11\eta}{3}}b_{11} + e^{-4\eta}b_{12} + e^{-\frac{13\eta}{3}}b_{13} + e^{-\frac{14\eta}{3}}b_{14} + e^{-5\eta}b_{15} + e^{-\frac{16\eta}{3}}b_{16}) + 1)^3 + 3(\text{Tr} - 1)(\\
& -\frac{1}{3}e^{-\frac{\eta}{3}}b_1 - \frac{2}{3}e^{-\frac{2\eta}{3}}b_2 - e^{-\eta}b_3 - \frac{4}{3}e^{-\frac{4\eta}{3}}b_4 - \frac{5}{3}e^{-\frac{5\eta}{3}}b_5 - 2e^{-2\eta}b_6 - \frac{7}{3}e^{-\frac{7\eta}{3}}b_7 - \frac{8}{3}e^{-\frac{8\eta}{3}}b_8 \\
& -3e^{-3\eta}b_9 - \frac{10}{3}e^{-\frac{10\eta}{3}}b_{10} - \frac{11}{3}e^{-\frac{11\eta}{3}}b_{11} - 4e^{-4\eta}b_{12} - \frac{13}{3}e^{-\frac{13\eta}{3}}b_{13} - \frac{14}{3}e^{-\frac{14\eta}{3}}b_{14} - 5e^{-5\eta}b_{15} \\
& -\frac{16}{3}e^{-\frac{16\eta}{3}}b_{16})^2((\text{Tr} - 1)(e^{-\frac{\eta}{3}}b_1 + e^{-\frac{2\eta}{3}}b_2 + e^{-\eta}b_3 + e^{-\frac{4\eta}{3}}b_4 + e^{-\frac{5\eta}{3}}b_5 + e^{-2\eta}b_6 + e^{-\frac{7\eta}{3}}b_7 \\
& +e^{-\frac{8\eta}{3}}b_8 + e^{-3\eta}b_9 + e^{-\frac{10\eta}{3}}b_{10} + e^{-\frac{11\eta}{3}}b_{11} + e^{-4\eta}b_{12} + e^{-\frac{13\eta}{3}}b_{13} + e^{-\frac{14\eta}{3}}b_{14} + e^{-5\eta}b_{15} \\
& +e^{-\frac{16\eta}{3}}b_{16}) + 1)^2) + \text{Nb}(-\frac{1}{3}e^{-\frac{\eta}{3}}b_1 - \frac{2}{3}e^{-\frac{2\eta}{3}}b_2 - e^{-\eta}b_3 - \frac{4}{3}e^{-\frac{4\eta}{3}}b_4 - \frac{5}{3}e^{-\frac{5\eta}{3}}b_5 - 2e^{-2\eta}b_6 \\
& -\frac{7}{3}e^{-\frac{7\eta}{3}}b_7 - \frac{8}{3}e^{-\frac{8\eta}{3}}b_8 - 3e^{-3\eta}b_9 - \frac{10}{3}e^{-\frac{10\eta}{3}}b_{10} - \frac{11}{3}e^{-\frac{11\eta}{3}}b_{11} - 4e^{-4\eta}b_{12} - \frac{13}{3}e^{-\frac{13\eta}{3}}b_{13} \\
& -\frac{14}{3}e^{-\frac{14\eta}{3}}b_{14} - 5e^{-5\eta}b_{15} - \frac{16}{3}e^{-\frac{16\eta}{3}}b_{16})(-\frac{1}{3}e^{-\frac{\eta}{3}}c_1 - \frac{2}{3}e^{-\frac{2\eta}{3}}c_2 - e^{-\eta}c_3 - \frac{4}{3}e^{-\frac{4\eta}{3}}c_4 \\
& -\frac{5}{3}e^{-\frac{5\eta}{3}}c_5 - 2e^{-2\eta}c_6 - \frac{7}{3}e^{-\frac{7\eta}{3}}c_7 - \frac{8}{3}e^{-\frac{8\eta}{3}}c_8 - 3e^{-3\eta}c_9 - \frac{10}{3}e^{-\frac{10\eta}{3}}c_{10} - \frac{11}{3}e^{-\frac{11\eta}{3}}c_{11} - 4e^{-4\eta}c_{12} \\
& -\frac{13}{3}e^{-\frac{13\eta}{3}}c_{13} - \frac{14}{3}e^{-\frac{14\eta}{3}}c_{14} - 5e^{-5\eta}c_{15} - \frac{16}{3}e^{-\frac{16\eta}{3}}c_{16}),
\end{aligned}$$

$$\begin{aligned}
R_{\phi_1}(a_i, b_k, c_k, \eta) = & \frac{\text{Nt}}{\text{Nb}}(\frac{1}{9}e^{-\frac{\eta}{3}}b_1 + \frac{4}{9}e^{-\frac{2\eta}{3}}b_2 + e^{-\eta}b_3 + \frac{16}{9}e^{-\frac{4\eta}{3}}b_4 + \frac{25}{9}e^{-\frac{5\eta}{3}}b_5 + 4e^{-2\eta}b_6 \\
& +\frac{49}{9}e^{-\frac{7\eta}{3}}b_7 + \frac{64}{9}e^{-\frac{8\eta}{3}}b_8 + 9e^{-3\eta}b_9 + \frac{100}{9}e^{-\frac{10\eta}{3}}b_{10} + \frac{121}{9}e^{-\frac{11\eta}{3}}b_{11} + 16e^{-4\eta}b_{12} + \frac{169}{9}e^{-\frac{13\eta}{3}}b_{13} \\
& +\frac{196}{9}e^{-\frac{14\eta}{3}}b_{14} + 25e^{-5\eta}b_{15} + \frac{256}{9}e^{-\frac{16\eta}{3}}b_{16}) + \frac{1}{9}e^{-\frac{\eta}{3}}c_1 + \frac{4}{9}e^{-\frac{2\eta}{3}}c_2 + e^{-\eta}c_3 + \frac{16}{9}e^{-\frac{4\eta}{3}}c_4 \\
& +\frac{25}{9}e^{-\frac{5\eta}{3}}c_5 + 4e^{-2\eta}c_6 + \frac{49}{9}e^{-\frac{7\eta}{3}}c_7 + \frac{64}{9}e^{-\frac{8\eta}{3}}c_8 + 9e^{-3\eta}c_9 + \frac{100}{9}e^{-\frac{10\eta}{3}}c_{10} + \frac{121}{9}e^{-\frac{11\eta}{3}}c_{11} \\
& +16e^{-4\eta}c_{12} + \frac{169}{9}e^{-\frac{13\eta}{3}}c_{13} + \frac{196}{9}e^{-\frac{14\eta}{3}}c_{14} + 25e^{-5\eta}c_{15} + \frac{256}{9}e^{-\frac{16\eta}{3}}c_{16} + \frac{3}{4}\text{Le}(K\eta + a_0 + e^{-\frac{\eta}{3}}a_1
\end{aligned}$$

$$\begin{aligned}
& +e^{\frac{2\eta}{3}}a_2 + e^{-\eta}a_3 + e^{\frac{-4\eta}{3}}a_4 + e^{\frac{-5\eta}{3}}a_5 + e^{-2\eta}a_6 + e^{\frac{-7\eta}{3}}a_7 + e^{\frac{-8\eta}{3}}a_8 + e^{-3\eta}a_9 + e^{\frac{-10\eta}{3}}a_{10} \\
& +e^{\frac{-11\eta}{3}}a_{11} + e^{-4\eta}a_{12} + e^{\frac{-13\eta}{3}}a_{13} + e^{\frac{-14\eta}{3}}a_{14} + e^{-5\eta}a_{15} + e^{\frac{-16\eta}{3}}a_{16}) \left(-\frac{1}{3}e^{\frac{-\eta}{3}}c_1 - \frac{2}{3}e^{\frac{-2\eta}{3}}c_2 \right. \\
& -e^{-\eta}c_3 - \frac{4}{3}e^{\frac{-4\eta}{3}}c_4 - \frac{5}{3}e^{\frac{-5\eta}{3}}c_5 - 2e^{-2\eta}c_6 - \frac{7}{3}e^{\frac{-7\eta}{3}}c_7 - \frac{8}{3}e^{\frac{-8\eta}{3}}c_8 - 3e^{-3\eta}c_9 - \frac{10}{3}e^{\frac{-10\eta}{3}}c_{10} \\
& \left. -\frac{11}{3}e^{\frac{-11\eta}{3}}c_{11} - 4e^{-4\eta}c_{12} - \frac{13}{3}e^{\frac{-13\eta}{3}}c_{13} - \frac{14}{3}e^{\frac{-14\eta}{3}}c_{14} - 5e^{-5\eta}c_{15} - \frac{16}{3}e^{\frac{-16\eta}{3}}c_{16}\right)
\end{aligned}$$

$$\begin{aligned}
R_{f_2}(a_i, b_k, c_k, \eta) = & -\frac{1}{\text{Da}} \left(\exp(-\gamma(e^{\frac{\eta}{3}}b_1 + e^{\frac{2\eta}{3}}b_2 + e^{-\eta}b_3 + e^{\frac{-4\eta}{3}}b_4 + e^{\frac{-5\eta}{3}}b_5 + e^{-2\eta}b_6 \right. \\
& + e^{\frac{-7\eta}{3}}b_7 + e^{\frac{-8\eta}{3}}b_8 + e^{-3\eta}b_9 + e^{\frac{-10\eta}{3}}b_{10} + e^{\frac{-11\eta}{3}}b_{11} + e^{-4\eta}b_{12} + e^{\frac{-13\eta}{3}}b_{13} + e^{\frac{-14\eta}{3}}b_{14} \\
& + e^{-5\eta}b_{15} + e^{\frac{-16\eta}{3}}b_{16})) \left(1 + \frac{1}{\beta}\right) \left(-\frac{1}{3}e^{\frac{-\eta}{3}}a_1 - \frac{2}{3}e^{\frac{-2\eta}{3}}a_2 - e^{-\eta}a_3 - \frac{4}{3}e^{\frac{-4\eta}{3}}a_4 - \frac{5}{3}e^{\frac{-5\eta}{3}}a_5 \right. \\
& - 2e^{-2\eta}a_6 - \frac{7}{3}e^{\frac{-7\eta}{3}}a_7 - \frac{8}{3}e^{\frac{-8\eta}{3}}a_8 - 3e^{-3\eta}a_9 - \frac{10}{3}e^{\frac{-10\eta}{3}}a_{10} - \frac{11}{3}e^{\frac{-11\eta}{3}}a_{11} - 4e^{-4\eta}a_{12} \\
& - \frac{13}{3}e^{\frac{-13\eta}{3}}a_{13} - \frac{14}{3}e^{\frac{-14\eta}{3}}a_{14} - 5e^{-5\eta}a_{15} - \frac{16}{3}e^{\frac{-16\eta}{3}}a_{16})) + e^{\frac{\eta}{3}}b_1 + e^{\frac{2\eta}{3}}b_2 + e^{-\eta}b_3 \\
& + e^{\frac{-4\eta}{3}}b_4 + e^{\frac{-5\eta}{3}}b_5 + e^{-2\eta}b_6 + e^{\frac{7\eta}{3}}b_7 + e^{\frac{-8\eta}{3}}b_8 + e^{-3\eta}b_9 + e^{\frac{-10\eta}{3}}b_{10} + e^{\frac{-11\eta}{3}}b_{11} + e^{-4\eta}b_{12} \\
& + e^{\frac{-13\eta}{3}}b_{13} + e^{\frac{-14\eta}{3}}b_{14} + e^{-5\eta}b_{15} + e^{\frac{-16\eta}{3}}b_{16} + \left(1 + \frac{1}{\beta}\right) \left(\exp(-\gamma(e^{\frac{\eta}{3}}b_1 + e^{\frac{2\eta}{3}}b_2 + e^{-\eta}b_3 \right. \\
& + e^{\frac{-4\eta}{3}}b_4 + e^{\frac{-5\eta}{3}}b_5 + e^{-2\eta}b_6 + e^{\frac{7\eta}{3}}b_7 + e^{\frac{-8\eta}{3}}b_8 + e^{-3\eta}b_9 + e^{\frac{-10\eta}{3}}b_{10} + e^{\frac{-11\eta}{3}}b_{11} + e^{-4\eta}b_{12} \\
& + e^{\frac{-13\eta}{3}}b_{13} + e^{\frac{-14\eta}{3}}b_{14} + e^{-5\eta}b_{15} + e^{\frac{-16\eta}{3}}b_{16})) \left(-\frac{1}{27}e^{\frac{-\eta}{3}}a_1 - \frac{8}{27}e^{\frac{-2\eta}{3}}a_2 - e^{-\eta}a_3 - \frac{64}{27}e^{\frac{-4\eta}{3}}a_4 \right. \\
& - \frac{125}{27}e^{\frac{-5\eta}{3}}a_5 - 8e^{-2\eta}a_6 - \frac{343}{27}e^{\frac{-7\eta}{3}}a_7 - \frac{512}{27}e^{\frac{-8\eta}{3}}a_8 - 27e^{-3\eta}a_9 - \frac{1000}{27}e^{\frac{-10\eta}{3}}a_{10} \\
& - \frac{1331}{27}e^{\frac{-11\eta}{3}}a_{11} - 64e^{-4\eta}a_{12} - \frac{2197}{27}e^{\frac{-13\eta}{3}}a_{13} - \frac{2744}{27}e^{\frac{-14\eta}{3}}a_{14} - 125e^{-5\eta}a_{15} - \frac{4096}{27}e^{\frac{-16\eta}{3}}a_{16}) \\
& - \exp(-\gamma(e^{\frac{\eta}{3}}b_1 + e^{\frac{2\eta}{3}}b_2 + e^{-\eta}b_3 + e^{\frac{-4\eta}{3}}b_4 + e^{\frac{-5\eta}{3}}b_5 + e^{-2\eta}b_6 + e^{\frac{-7\eta}{3}}b_7 + e^{\frac{-8\eta}{3}}b_8 + e^{-3\eta}b_9 \\
& + e^{\frac{-10\eta}{3}}b_{10} + e^{\frac{-11\eta}{3}}b_{11} + e^{-4\eta}b_{12} + e^{\frac{-13\eta}{3}}b_{13} + e^{\frac{-14\eta}{3}}b_{14} + e^{-5\eta}b_{15} + e^{\frac{-16\eta}{3}}b_{16})) \gamma \left(\frac{1}{9}e^{\frac{-\eta}{3}}a_1 \right.
\end{aligned}$$

$$\begin{aligned}
& +\frac{4}{9}e^{-\frac{2\eta}{3}}a_2+e^{-\eta}a_3+\frac{16}{9}e^{-\frac{4\eta}{3}}a_4+\frac{25}{9}e^{-\frac{5\eta}{3}}a_5+4e^{-2\eta}a_6+\frac{49}{9}e^{-\frac{7\eta}{3}}a_7+\frac{64}{9}e^{-\frac{8\eta}{3}}a_8+9e^{-3\eta}a_9 \\
& +\frac{100}{9}e^{-\frac{10\eta}{3}}a_{10}+\frac{121}{9}e^{-\frac{11\eta}{3}}a_{11}+16e^{-4\eta}a_{12}+\frac{169}{9}e^{-\frac{13\eta}{3}}a_{13}+\frac{196}{9}e^{-\frac{14\eta}{3}}a_{14}+25e^{-5\eta}a_{15} \\
& +\frac{256}{9}e^{-\frac{16\eta}{3}}a_{16})\left(-\frac{1}{3}e^{-\frac{\eta}{3}}b_1-\frac{2}{3}e^{-\frac{2\eta}{3}}b_2-e^{-\eta}b_3-\frac{4}{3}e^{-\frac{4\eta}{3}}b_4-\frac{5}{3}e^{-\frac{5\eta}{3}}b_5-2e^{-2\eta}b_6-\frac{7}{3}e^{-\frac{7\eta}{3}}b_7 \right. \\
& -\frac{8}{3}e^{-\frac{8\eta}{3}}b_8-3e^{-3\eta}b_9-\frac{10}{3}e^{-\frac{10\eta}{3}}b_{10}-\frac{11}{3}e^{-\frac{11\eta}{3}}b_{11}-4e^{-4\eta}b_{12}-\frac{13}{3}e^{-\frac{13\eta}{3}}b_{13}-\frac{14}{3}e^{-\frac{14\eta}{3}}b_{14} \\
& \left.-5e^{-5\eta}b_{15}-\frac{16}{3}e^{-\frac{16\eta}{3}}b_{16})\right)-\text{Nr}(e^{-\frac{\eta}{3}}c_1+e^{-\frac{2\eta}{3}}c_2+e^{-\eta}c_3+e^{-\frac{4\eta}{3}}c_4+e^{-\frac{5\eta}{3}}c_5+e^{-2\eta}c_6 \\
& +e^{-\frac{7\eta}{3}}c_7+e^{-\frac{8\eta}{3}}c_8+e^{-3\eta}c_9+e^{-\frac{10\eta}{3}}c_{10}+e^{-\frac{11\eta}{3}}c_{11}+e^{-4\eta}c_{12}+e^{-\frac{13\eta}{3}}c_{13}+e^{-\frac{14\eta}{3}}c_{14}+e^{-5\eta}c_{15} \\
& +e^{-\frac{16\eta}{3}}c_{16})+(4\text{Pr}(1-\delta(e^{-\frac{\eta}{3}}b_1+e^{-\frac{2\eta}{3}}b_2+e^{-\eta}b_3+e^{-\frac{4\eta}{3}}b_4+e^{-\frac{5\eta}{3}}b_5+e^{-2\eta}b_6+e^{-\frac{7\eta}{3}}b_7 \\
& +e^{-\frac{8\eta}{3}}b_8+e^{-3\eta}b_9+e^{-\frac{10\eta}{3}}b_{10}+e^{-\frac{11\eta}{3}}b_{11}+e^{-4\eta}b_{12}+e^{-\frac{13\eta}{3}}b_{13}+e^{-\frac{14\eta}{3}}b_{14}+e^{-5\eta}b_{15}+e^{-\frac{16\eta}{3}}b_{16})))^{-1} \\
& \left(\exp(-\gamma(e^{-\frac{\eta}{3}}b_1+e^{-\frac{2\eta}{3}}b_2+e^{-\eta}b_3+e^{-\frac{4\eta}{3}}b_4+e^{-\frac{5\eta}{3}}b_5+e^{-2\eta}b_6+e^{-\frac{7\eta}{3}}b_7+e^{-\frac{8\eta}{3}}b_8 \right. \\
& \left.+e^{-3\eta}b_9+e^{-\frac{10\eta}{3}}b_{10}+e^{-\frac{11\eta}{3}}b_{11}+e^{-4\eta}b_{12}+e^{-\frac{13\eta}{3}}b_{13}+e^{-\frac{14\eta}{3}}b_{14}+e^{-5\eta}b_{15}+e^{-\frac{16\eta}{3}}b_{16}))\right) \cdot \\
& (-4\text{Fs}(-\frac{1}{3}e^{-\frac{\eta}{3}}a_1-\frac{2}{3}e^{-\frac{2\eta}{3}}a_2-e^{-\eta}a_3-\frac{4}{3}e^{-\frac{4\eta}{3}}a_4-\frac{5}{3}e^{-\frac{5\eta}{3}}a_5-2e^{-2\eta}a_6-\frac{7}{3}e^{-\frac{7\eta}{3}}a_7-\frac{8}{3}e^{-\frac{8\eta}{3}}a_8 \\
& -3e^{-3\eta}a_9-\frac{10}{3}e^{-\frac{10\eta}{3}}a_{10}-\frac{11}{3}e^{-\frac{11\eta}{3}}a_{11}-4e^{-4\eta}a_{12}-\frac{13}{3}e^{-\frac{13\eta}{3}}a_{13}-\frac{14}{3}e^{-\frac{14\eta}{3}}a_{14}-5e^{-5\eta}a_{15} \\
& -\frac{16}{3}e^{-\frac{16\eta}{3}}a_{16})^2-4M(-\frac{1}{3}e^{-\frac{\eta}{3}}a_1-\frac{2}{3}e^{-\frac{2\eta}{3}}a_2-e^{-\eta}a_3-\frac{4}{3}e^{-\frac{4\eta}{3}}a_4-\frac{5}{3}e^{-\frac{5\eta}{3}}a_5-2e^{-2\eta}a_6 \\
& -\frac{7}{3}e^{-\frac{7\eta}{3}}a_7-\frac{8}{3}e^{-\frac{8\eta}{3}}a_8-3e^{-3\eta}a_9-\frac{10}{3}e^{-\frac{10\eta}{3}}a_{10}-\frac{11}{3}e^{-\frac{11\eta}{3}}a_{11}-4e^{-4\eta}a_{12}-\frac{13}{3}e^{-\frac{13\eta}{3}}a_{13} \\
& -\frac{14}{3}e^{-\frac{14\eta}{3}}a_{14}-5e^{-5\eta}a_{15}-\frac{16}{3}e^{-\frac{16\eta}{3}}a_{16})^2-2(-\frac{1}{3}e^{-\frac{\eta}{3}}a_1-\frac{2}{3}e^{-\frac{2\eta}{3}}a_2-e^{-\eta}a_3-\frac{4}{3}e^{-\frac{4\eta}{3}}a_4 \\
& -\frac{5}{3}e^{-\frac{5\eta}{3}}a_5-2e^{-2\eta}a_6-\frac{7}{3}e^{-\frac{7\eta}{3}}a_7-\frac{8}{3}e^{-\frac{8\eta}{3}}a_8-3e^{-3\eta}a_9-\frac{10}{3}e^{-\frac{10\eta}{3}}a_{10}-\frac{11}{3}e^{-\frac{11\eta}{3}}a_{11} \\
& -4e^{-4\eta}a_{12}-\frac{13}{3}e^{-\frac{13\eta}{3}}a_{13}-\frac{14}{3}e^{-\frac{14\eta}{3}}a_{14}-5e^{-5\eta}a_{15}-\frac{16}{3}e^{-\frac{16\eta}{3}}a_{16})^2+3(a_0+e^{-\frac{\eta}{3}}a_1 \\
& +e^{-\frac{2\eta}{3}}a_2+e^{-\eta}a_3+e^{-\frac{4\eta}{3}}a_4+e^{-\frac{5\eta}{3}}a_5+e^{-2\eta}a_6+e^{-\frac{7\eta}{3}}a_7+e^{-\frac{8\eta}{3}}a_8+e^{-3\eta}a_9+e^{-\frac{10\eta}{3}}a_{10} \\
& +e^{-\frac{11\eta}{3}}a_{11}+e^{-4\eta}a_{12}+e^{-\frac{13\eta}{3}}a_{13}+e^{-\frac{14\eta}{3}}a_{14}+e^{-5\eta}a_{15}+e^{-\frac{16\eta}{3}}a_{16})\left(\frac{1}{9}e^{-\frac{\eta}{3}}a_1+\frac{4}{9}e^{-\frac{2\eta}{3}}a_2 \right.
\end{aligned}$$

$$\begin{aligned}
& +e^{-\eta}a_3+\frac{16}{9}e^{-\frac{4\eta}{3}}a_4+\frac{25}{9}e^{-\frac{5\eta}{3}}a_5+4e^{-2\eta}a_6+\frac{49}{9}e^{-\frac{7\eta}{3}}a_7+\frac{64}{9}e^{-\frac{8\eta}{3}}a_8+9e^{-3\eta}a_9 \\
& +\frac{100}{9}e^{-\frac{10\eta}{3}}a_{10}+\frac{121}{9}e^{-\frac{11\eta}{3}}a_{11}+16e^{-4\eta}a_{12}+\frac{169}{9}e^{-\frac{13\eta}{3}}a_{13}+\frac{196}{9}e^{-\frac{14\eta}{3}}a_{14}+25e^{-5\eta}a_{15} \\
& +\frac{256}{9}e^{-\frac{16\eta}{3}}a_{16})))
\end{aligned}$$

$$\begin{aligned}
R_{\theta_2}(a_i, b_k, c_k, \eta) = & \text{Ec Pr} \left(1 + \frac{1}{\beta} \right) (1 - \partial (e^{-\frac{\eta}{3}} b_1 + e^{-\frac{2\eta}{3}} b_2 + e^{-\eta} b_3 + e^{-\frac{4\eta}{3}} b_4 + e^{-\frac{5\eta}{3}} b_5 + e^{-2\eta} b_6 \\
& + e^{-\frac{7\eta}{3}} b_7 + e^{-\frac{8\eta}{3}} b_8 + e^{-3\eta} b_9 + e^{-\frac{10\eta}{3}} b_{10} + e^{-\frac{11\eta}{3}} b_{11} + e^{-4\eta} b_{12} + e^{-\frac{13\eta}{3}} b_{13} + e^{-\frac{14\eta}{3}} b_{14} + e^{-5\eta} b_{15} \\
& + e^{-\frac{16\eta}{3}} b_{16})) (\frac{1}{9} e^{-\frac{\eta}{3}} a_1 + \frac{4}{9} e^{-\frac{2\eta}{3}} a_2 + e^{-\eta} a_3 + \frac{16}{9} e^{-\frac{4\eta}{3}} a_4 + \frac{25}{9} e^{-\frac{5\eta}{3}} a_5 + 4e^{-2\eta} a_6 + \frac{49}{9} e^{-\frac{7\eta}{3}} a_7 \\
& + \frac{64}{9} e^{-\frac{8\eta}{3}} a_8 + 9e^{-3\eta} a_9 + \frac{100}{9} e^{-\frac{10\eta}{3}} a_{10} + \frac{121}{9} e^{-\frac{11\eta}{3}} a_{11} + 16e^{-4\eta} a_{12} + \frac{169}{9} e^{-\frac{13\eta}{3}} a_{13} \\
& + \frac{196}{9} e^{-\frac{14\eta}{3}} a_{14} + 25e^{-5\eta} a_{15} + \frac{256}{9} e^{-\frac{16\eta}{3}} a_{16})^2 + \text{Nt} (-\frac{1}{3} e^{-\frac{\eta}{3}} b_1 - \frac{2}{3} e^{-\frac{2\eta}{3}} b_2 - e^{-\eta} b_3 - \frac{4}{3} e^{-\frac{4\eta}{3}} b_4 \\
& - \frac{5}{3} e^{-\frac{5\eta}{3}} b_5 - 2e^{-2\eta} b_6 - \frac{7}{3} e^{-\frac{7\eta}{3}} b_7 - \frac{8}{3} e^{-\frac{8\eta}{3}} b_8 - 3e^{-3\eta} b_9 - \frac{10}{3} e^{-\frac{10\eta}{3}} b_{10} - \frac{11}{3} e^{-\frac{11\eta}{3}} b_{11} - 4e^{-4\eta} b_{12} \\
& - \frac{13}{3} e^{-\frac{13\eta}{3}} b_{13} - \frac{14}{3} e^{-\frac{14\eta}{3}} b_{14} - 5e^{-5\eta} b_{15} - \frac{16}{3} e^{-\frac{16\eta}{3}} b_{16})^2 - \partial (-\frac{1}{3} e^{-\frac{\eta}{3}} b_1 - \frac{2}{3} e^{-\frac{2\eta}{3}} b_2 - e^{-\eta} b_3 \\
& - \frac{4}{3} e^{-\frac{4\eta}{3}} b_4 - \frac{5}{3} e^{-\frac{5\eta}{3}} b_5 - 2e^{-2\eta} b_6 - \frac{7}{3} e^{-\frac{7\eta}{3}} b_7 - \frac{8}{3} e^{-\frac{8\eta}{3}} b_8 - 3e^{-3\eta} b_9 - \frac{10}{3} e^{-\frac{10\eta}{3}} b_{10} - \frac{11}{3} e^{-\frac{11\eta}{3}} b_{11} \\
& - 4e^{-4\eta} b_{12} - \frac{13}{3} e^{-\frac{13\eta}{3}} b_{13} - \frac{14}{3} e^{-\frac{14\eta}{3}} b_{14} - 5e^{-5\eta} b_{15} - \frac{16}{3} e^{-\frac{16\eta}{3}} b_{16})^2 + \frac{3}{4} (a_0 + e^{-\frac{\eta}{3}} a_1 + e^{-\frac{2\eta}{3}} a_2 \\
& + e^{-\eta} a_3 + e^{-\frac{4\eta}{3}} a_4 + e^{-\frac{5\eta}{3}} a_5 + e^{-2\eta} a_6 + e^{-\frac{7\eta}{3}} a_7 + e^{-\frac{8\eta}{3}} a_8 + e^{-3\eta} a_9 + e^{-\frac{10\eta}{3}} a_{10} + e^{-\frac{11\eta}{3}} a_{11} \\
& + e^{-4\eta} a_{12} + e^{-\frac{13\eta}{3}} a_{13} + e^{-\frac{14\eta}{3}} a_{14} + e^{-5\eta} a_{15} + e^{-\frac{16\eta}{3}} a_{16}) (-\frac{1}{3} e^{-\frac{\eta}{3}} b_1 - \frac{2}{3} e^{-\frac{2\eta}{3}} b_2 - e^{-\eta} b_3 \\
& - \frac{4}{3} e^{-\frac{4\eta}{3}} b_4 - \frac{5}{3} e^{-\frac{5\eta}{3}} b_5 - 2e^{-2\eta} b_6 - \frac{7}{3} e^{-\frac{7\eta}{3}} b_7 - \frac{8}{3} e^{-\frac{8\eta}{3}} b_8 - 3e^{-3\eta} b_9 - \frac{10}{3} e^{-\frac{10\eta}{3}} b_{10} - \frac{11}{3} e^{-\frac{11\eta}{3}} b_{11} \\
& - 4e^{-4\eta} b_{12} - \frac{13}{3} e^{-\frac{13\eta}{3}} b_{13} - \frac{14}{3} e^{-\frac{14\eta}{3}} b_{14} - 5e^{-5\eta} b_{15} - \frac{16}{3} e^{-\frac{16\eta}{3}} b_{16}) + (\frac{1}{9} e^{-\frac{\eta}{3}} b_1 + \frac{4}{9} e^{-\frac{2\eta}{3}} b_2 \\
& + e^{-\eta} b_3 + \frac{16}{9} e^{-\frac{4\eta}{3}} b_4 + \frac{25}{9} e^{-\frac{5\eta}{3}} b_5 + 4e^{-2\eta} b_6 + \frac{49}{9} e^{-\frac{7\eta}{3}} b_7 + \frac{64}{9} e^{-\frac{8\eta}{3}} b_8 + 9e^{-3\eta} b_9 + \frac{100}{9} e^{-\frac{10\eta}{3}} b_{10} \\
& + \frac{121}{9} e^{-\frac{11\eta}{3}} b_{11} + 16e^{-4\eta} b_{12} + \frac{169}{9} e^{-\frac{13\eta}{3}} b_{13} + \frac{196}{9} e^{-\frac{14\eta}{3}} b_{14} + 25e^{-5\eta} b_{15} + \frac{256}{9} e^{-\frac{16\eta}{3}} b_{16}) (1 \\
& - \partial (e^{-\frac{\eta}{3}} b_1 + e^{-\frac{2\eta}{3}} b_2 + e^{-\eta} b_3 + e^{-\frac{4\eta}{3}} b_4 + e^{-\frac{5\eta}{3}} b_5 + e^{-2\eta} b_6 + e^{-\frac{7\eta}{3}} b_7 + e^{-\frac{8\eta}{3}} b_8 + e^{-3\eta} b_9 \\
& + e^{-\frac{10\eta}{3}} b_{10} + e^{-\frac{11\eta}{3}} b_{11} + e^{-4\eta} b_{12} + e^{-\frac{13\eta}{3}} b_{13} + e^{-\frac{14\eta}{3}} b_{14} + e^{-5\eta} b_{15} + e^{-\frac{16\eta}{3}} b_{16})) \\
& + \frac{4}{3N} (((\frac{1}{9} e^{-\frac{\eta}{3}} b_1 + \frac{4}{9} e^{-\frac{2\eta}{3}} b_2 + e^{-\eta} b_3 + \frac{16}{9} e^{-\frac{4\eta}{3}} b_4 + \frac{25}{9} e^{-\frac{5\eta}{3}} b_5 + 4e^{-2\eta} b_6 + \frac{49}{9} e^{-\frac{7\eta}{3}} b_7
\end{aligned}$$

$$\begin{aligned}
& + \frac{64}{9} e^{-\frac{8\eta}{3}} b_8 + 9e^{-3\eta} b_9 + \frac{100}{9} e^{-\frac{10\eta}{3}} b_{10} + \frac{121}{9} e^{-\frac{11\eta}{3}} b_{11} + 16e^{-4\eta} b_{12} + \frac{169}{9} e^{-\frac{13\eta}{3}} b_{13} \\
& + \frac{196}{9} e^{-\frac{14\eta}{3}} b_{14} + 25e^{-5\eta} b_{15} + \frac{256}{9} e^{-\frac{16\eta}{3}} b_{16}) ((\text{Tr}-1)(e^{-\frac{\eta}{3}} b_1 + e^{-\frac{2\eta}{3}} b_2 + e^{-\eta} b_3 + e^{-\frac{4\eta}{3}} b_4 \\
& + e^{-\frac{5\eta}{3}} b_5 + e^{-2\eta} b_6 + e^{-\frac{7\eta}{3}} b_7 + e^{-\frac{8\eta}{3}} b_8 + e^{-3\eta} b_9 + e^{-\frac{10\eta}{3}} b_{10} + e^{-\frac{11\eta}{3}} b_{11} + e^{-4\eta} b_{12} + e^{-\frac{13\eta}{3}} b_{13} \\
& + e^{-\frac{14\eta}{3}} b_{14} + e^{-5\eta} b_{15} + e^{-\frac{16\eta}{3}} b_{16}) + 1)^3 + 3(\text{Tr}-1)(-\frac{1}{3} e^{-\frac{\eta}{3}} b_1 - \frac{2}{3} e^{-\frac{2\eta}{3}} b_2 - e^{-\eta} b_3 - \frac{4}{3} e^{-\frac{4\eta}{3}} b_4 \\
& - \frac{5}{3} e^{-\frac{5\eta}{3}} b_5 - 2e^{-2\eta} b_6 - \frac{7}{3} e^{-\frac{7\eta}{3}} b_7 - \frac{8}{3} e^{-\frac{8\eta}{3}} b_8 - 3e^{-3\eta} b_9 - \frac{10}{3} e^{-\frac{10\eta}{3}} b_{10} - \frac{11}{3} e^{-\frac{11\eta}{3}} b_{11} - 4e^{-4\eta} b_{12} \\
& - \frac{13}{3} e^{-\frac{13\eta}{3}} b_{13} - \frac{14}{3} e^{-\frac{14\eta}{3}} b_{14} - 5e^{-5\eta} b_{15} - \frac{16}{3} e^{-\frac{16\eta}{3}} b_{16})^2 ((\text{Tr}-1)(e^{-\frac{\eta}{3}} b_1 + e^{-\frac{2\eta}{3}} b_2 + e^{-\eta} b_3 \\
& + e^{-\frac{4\eta}{3}} b_4 + e^{-\frac{5\eta}{3}} b_5 + e^{-2\eta} b_6 + e^{-\frac{7\eta}{3}} b_7 + e^{-\frac{8\eta}{3}} b_8 + e^{-3\eta} b_9 + e^{-\frac{10\eta}{3}} b_{10} + e^{-\frac{11\eta}{3}} b_{11} + e^{-4\eta} b_{12} \\
& + e^{-\frac{13\eta}{3}} b_{13} + e^{-\frac{14\eta}{3}} b_{14} + e^{-5\eta} b_{15} + e^{-\frac{16\eta}{3}} b_{16}) + 1)^2)) + \text{Nb}(-\frac{1}{3} e^{-\frac{\eta}{3}} b_1 - \frac{2}{3} e^{-\frac{2\eta}{3}} b_2 - e^{-\eta} b_3 \\
& - \frac{4}{3} e^{-\frac{4\eta}{3}} b_4 - \frac{5}{3} e^{-\frac{5\eta}{3}} b_5 - 2e^{-2\eta} b_6 - \frac{7}{3} e^{-\frac{7\eta}{3}} b_7 - \frac{8}{3} e^{-\frac{8\eta}{3}} b_8 - 3e^{-3\eta} b_9 - \frac{10}{3} e^{-\frac{10\eta}{3}} b_{10} - \frac{11}{3} e^{-\frac{11\eta}{3}} b_{11} \\
& - 4e^{-4\eta} b_{12} - \frac{13}{3} e^{-\frac{13\eta}{3}} b_{13} - \frac{14}{3} e^{-\frac{14\eta}{3}} b_{14} - 5e^{-5\eta} b_{15} - \frac{16}{3} e^{-\frac{16\eta}{3}} b_{16}) (-\frac{1}{3} e^{-\frac{\eta}{3}} c_1 - \frac{2}{3} e^{-\frac{2\eta}{3}} c_2 \\
& - e^{-\eta} c_3 - \frac{4}{3} e^{-\frac{4\eta}{3}} c_4 - \frac{5}{3} e^{-\frac{5\eta}{3}} c_5 - 2e^{-2\eta} c_6 - \frac{7}{3} e^{-\frac{7\eta}{3}} c_7 - \frac{8}{3} e^{-\frac{8\eta}{3}} c_8 - 3e^{-3\eta} c_9 - \frac{10}{3} e^{-\frac{10\eta}{3}} c_{10} \\
& - \frac{11}{3} e^{-\frac{11\eta}{3}} c_{11} - 4e^{-4\eta} c_{12} - \frac{13}{3} e^{-\frac{13\eta}{3}} c_{13} - \frac{14}{3} e^{-\frac{14\eta}{3}} c_{14} - 5e^{-5\eta} c_{15} - \frac{16}{3} e^{-\frac{16\eta}{3}} c_{16}),
\end{aligned}$$

$$\begin{aligned}
R_{\phi_2}(a_i, b_k, c_k, \eta) &= \frac{\text{Nt}}{\text{Nb}} ((\frac{1}{9} e^{-\frac{\eta}{3}} b_1 + \frac{4}{9} e^{-\frac{2\eta}{3}} b_2 + e^{-\eta} b_3 + \frac{16}{9} e^{-\frac{4\eta}{3}} b_4 + \frac{25}{9} e^{-\frac{5\eta}{3}} b_5 + 4e^{-2\eta} b_6 \\
& + \frac{49}{9} e^{-\frac{7\eta}{3}} b_7 + \frac{64}{9} e^{-\frac{8\eta}{3}} b_8 + 9e^{-3\eta} b_9 + \frac{100}{9} e^{-\frac{10\eta}{3}} b_{10} + \frac{121}{9} e^{-\frac{11\eta}{3}} b_{11} + 16e^{-4\eta} b_{12} + \frac{169}{9} e^{-\frac{13\eta}{3}} b_{13} \\
& + \frac{196}{9} e^{-\frac{14\eta}{3}} b_{14} + 25e^{-5\eta} b_{15} + \frac{256}{9} e^{-\frac{16\eta}{3}} b_{16})) + \frac{1}{9} e^{-\frac{\eta}{3}} c_1 + \frac{4}{9} e^{-\frac{2\eta}{3}} c_2 + e^{-\eta} c_3 + \frac{16}{9} e^{-\frac{4\eta}{3}} c_4 \\
& + \frac{25}{9} e^{-\frac{5\eta}{3}} c_5 + 4e^{-2\eta} c_6 + \frac{49}{9} e^{-\frac{7\eta}{3}} c_7 + \frac{64}{9} e^{-\frac{8\eta}{3}} c_8 + 9e^{-3\eta} c_9 + \frac{100}{9} e^{-\frac{10\eta}{3}} c_{10} + \frac{121}{9} e^{-\frac{11\eta}{3}} c_{11} \\
& + 16e^{-4\eta} c_{12} + \frac{169}{9} e^{-\frac{13\eta}{3}} c_{13} + \frac{196}{9} e^{-\frac{14\eta}{3}} c_{14} + 25e^{-5\eta} c_{15} + \frac{256}{9} e^{-\frac{16\eta}{3}} c_{16} + \frac{3}{4} \text{Le}(a_0 + e^{-\frac{\eta}{3}} a_1
\end{aligned}$$

$$\begin{aligned}
& +e^{\frac{-2\eta}{3}}a_2 + e^{-\eta}a_3 + e^{\frac{-4\eta}{3}}a_4 + e^{\frac{-5\eta}{3}}a_5 + e^{-2\eta}a_6 + e^{\frac{-7\eta}{3}}a_7 + e^{\frac{-8\eta}{3}}a_8 + e^{-3\eta}a_9 + e^{\frac{-10\eta}{3}}a_{10} \\
& +e^{\frac{-11\eta}{3}}a_{11} + e^{-4\eta}a_{12} + e^{\frac{-13\eta}{3}}a_{13} + e^{\frac{-14\eta}{3}}a_{14} + e^{-5\eta}a_{15} + e^{\frac{-16\eta}{3}}a_{16}) \left(-\frac{1}{3}e^{\frac{-\eta}{3}}c_1 - \frac{2}{3}e^{\frac{-2\eta}{3}}c_2 \right. \\
& -e^{-\eta}c_3 - \frac{4}{3}e^{\frac{-4\eta}{3}}c_4 - \frac{5}{3}e^{\frac{-5\eta}{3}}c_5 - 2e^{-2\eta}c_6 - \frac{7}{3}e^{\frac{-7\eta}{3}}c_7 - \frac{8}{3}e^{\frac{-8\eta}{3}}c_8 - 3e^{-3\eta}c_9 - \frac{10}{3}e^{\frac{-10\eta}{3}}c_{10} \\
& \left. - \frac{11}{3}e^{\frac{-11\eta}{3}}c_{11} - 4e^{-4\eta}c_{12} - \frac{13}{3}e^{\frac{-13\eta}{3}}c_{13} - \frac{14}{3}e^{\frac{-14\eta}{3}}c_{14} - 5e^{-5\eta}c_{15} - \frac{16}{3}e^{\frac{-16\eta}{3}}c_{16}\right),
\end{aligned}$$

$$\begin{aligned}
R_{f_3}(a_i, \eta) = & -Fs\left(-\frac{1}{3}e^{\frac{-\eta}{3}}a_1 - \frac{2}{3}e^{\frac{-2\eta}{3}}a_2 - e^{-\eta}a_3 - \frac{4}{3}e^{\frac{-4\eta}{3}}a_4 - \frac{5}{3}e^{\frac{-5\eta}{3}}a_5 - 2e^{-2\eta}a_6 - \frac{7}{3}e^{\frac{-7\eta}{3}}a_7 \right. \\
& - \frac{8}{3}e^{\frac{-8\eta}{3}}a_8 - 3e^{-3\eta}a_9 - \frac{10}{3}e^{\frac{-10\eta}{3}}a_{10} - \frac{11}{3}e^{\frac{-11\eta}{3}}a_{11} - 4e^{-4\eta}a_{12} - \frac{13}{3}e^{\frac{-13\eta}{3}}a_{13} - \frac{14}{3}e^{\frac{-14\eta}{3}}a_{14} \\
& \left. - 5e^{-5\eta}a_{15} - \frac{16}{3}e^{\frac{-16\eta}{3}}a_{16}\right)^2 - \left(-\frac{1}{3}e^{\frac{-\eta}{3}}a_1 - \frac{2}{3}e^{\frac{-2\eta}{3}}a_2 - e^{-\eta}a_3 - \frac{4}{3}e^{\frac{-4\eta}{3}}a_4 - \frac{5}{3}e^{\frac{-5\eta}{3}}a_5 - 2e^{-2\eta}a_6 \right. \\
& - \frac{7}{3}e^{\frac{-7\eta}{3}}a_7 - \frac{8}{3}e^{\frac{-8\eta}{3}}a_8 - 3e^{-3\eta}a_9 - \frac{10}{3}e^{\frac{-10\eta}{3}}a_{10} - \frac{11}{3}e^{\frac{-11\eta}{3}}a_{11} - 4e^{-4\eta}a_{12} - \frac{13}{3}e^{\frac{-13\eta}{3}}a_{13} \\
& \left. - \frac{14}{3}e^{\frac{-14\eta}{3}}a_{14} - 5e^{-5\eta}a_{15} - \frac{16}{3}e^{\frac{-16\eta}{3}}a_{16}\right)^2 - M\left(-\frac{1}{3}e^{\frac{-\eta}{3}}a_1 - \frac{2}{3}e^{\frac{-2\eta}{3}}a_2 - e^{-\eta}a_3 - \frac{4}{3}e^{\frac{-4\eta}{3}}a_4 \right. \\
& - \frac{5}{3}e^{\frac{-5\eta}{3}}a_5 - 2e^{-2\eta}a_6 - \frac{7}{3}e^{\frac{-7\eta}{3}}a_7 - \frac{8}{3}e^{\frac{-8\eta}{3}}a_8 - 3e^{-3\eta}a_9 - \frac{10}{3}e^{\frac{-10\eta}{3}}a_{10} - \frac{11}{3}e^{\frac{-11\eta}{3}}a_{11} - 4e^{-4\eta}a_{12} \\
& - \frac{13}{3}e^{\frac{-13\eta}{3}}a_{13} - \frac{14}{3}e^{\frac{-14\eta}{3}}a_{14} - 5e^{-5\eta}a_{15} - \frac{16}{3}e^{\frac{-16\eta}{3}}a_{16}) - \text{Pp}\left(1 + \frac{1}{\beta}\right)\left(-\frac{1}{3}e^{\frac{-\eta}{3}}a_1 - \frac{2}{3}e^{\frac{-2\eta}{3}}a_2 \right. \\
& - e^{-\eta}a_3 - \frac{4}{3}e^{\frac{-4\eta}{3}}a_4 - \frac{5}{3}e^{\frac{-5\eta}{3}}a_5 - 2e^{-2\eta}a_6 - \frac{7}{3}e^{\frac{-7\eta}{3}}a_7 - \frac{8}{3}e^{\frac{-8\eta}{3}}a_8 - 3e^{-3\eta}a_9 - \frac{10}{3}e^{\frac{-10\eta}{3}}a_{10} \\
& - \frac{11}{3}e^{\frac{-11\eta}{3}}a_{11} - 4e^{-4\eta}a_{12} - \frac{13}{3}e^{\frac{-13\eta}{3}}a_{13} - \frac{14}{3}e^{\frac{-14\eta}{3}}a_{14} - 5e^{-5\eta}a_{15} - \frac{16}{3}e^{\frac{-16\eta}{3}}a_{16}) + (a_0 \\
& + e^{\frac{-\eta}{3}}a_1 + e^{\frac{-2\eta}{3}}a_2 + e^{-\eta}a_3 + e^{\frac{-4\eta}{3}}a_4 + e^{\frac{-5\eta}{3}}a_5 + e^{-2\eta}a_6 + e^{\frac{-7\eta}{3}}a_7 + e^{\frac{-8\eta}{3}}a_8 + e^{-3\eta}a_9 + e^{\frac{-10\eta}{3}}a_{10} \\
& \left. + e^{\frac{-11\eta}{3}}a_{11} + e^{-4\eta}a_{12} + e^{\frac{-13\eta}{3}}a_{13} + e^{\frac{-14\eta}{3}}a_{14} + e^{-5\eta}a_{15} + e^{\frac{-16\eta}{3}}a_{16}\right)\left(\frac{1}{9}e^{\frac{-\eta}{3}}a_1 + \frac{4}{9}e^{\frac{-2\eta}{3}}a_2 + e^{-\eta}a_3 \right.
\end{aligned}$$

$$\begin{aligned}
& +\frac{16}{9}e^{-\frac{4\eta}{3}}a_4+\frac{25}{9}e^{-\frac{5\eta}{3}}a_5+4e^{-2\eta}a_6+\frac{49}{9}e^{-\frac{7\eta}{3}}a_7+\frac{64}{9}e^{-\frac{8\eta}{3}}a_8+9e^{-3\eta}a_9+\frac{100}{9}e^{-\frac{10\eta}{3}}a_{10} \\
& +\frac{121}{9}e^{-\frac{11\eta}{3}}a_{11}+16e^{-4\eta}a_{12}+\frac{169}{9}e^{-\frac{13\eta}{3}}a_{13}+\frac{196}{9}e^{-\frac{14\eta}{3}}a_{14}+25e^{-5\eta}a_{15}+\frac{256}{9}e^{-\frac{16\eta}{3}}a_{16}) \\
& +(1+\frac{1}{\beta})(-\frac{1}{27}e^{-\frac{\eta}{3}}a_1-\frac{8}{27}e^{-\frac{2\eta}{3}}a_2-e^{-\eta}a_3-\frac{64}{27}e^{-\frac{4\eta}{3}}a_4-\frac{125}{27}e^{-\frac{5\eta}{3}}a_5-8e^{-2\eta}a_6 \\
& -\frac{343}{27}e^{-\frac{7\eta}{3}}a_7-\frac{512}{27}e^{-\frac{8\eta}{3}}a_8-27e^{-3\eta}a_9-\frac{1000}{27}e^{-\frac{10\eta}{3}}a_{10}-\frac{1331}{27}e^{-\frac{11\eta}{3}}a_{11}-64e^{-4\eta}a_{12} \\
& -\frac{2197}{27}e^{-\frac{13\eta}{3}}a_{13}-\frac{2744}{27}e^{-\frac{14\eta}{3}}a_{14}-125e^{-5\eta}a_{15}-\frac{4096}{27}e^{-\frac{16\eta}{3}}a_{16}),
\end{aligned}$$

$$\begin{aligned}
R_{\theta_3}(a_i, b_k, c_k, \eta) = & \text{EcFs}(-\frac{1}{3}e^{-\frac{\eta}{3}}a_1-\frac{2}{3}e^{-\frac{2\eta}{3}}a_2-e^{-\eta}a_3-\frac{4}{3}e^{-\frac{4\eta}{3}}a_4-\frac{5}{3}e^{-\frac{5\eta}{3}}a_5-2e^{-2\eta}a_6 \\
& -\frac{7}{3}e^{-\frac{7\eta}{3}}a_7-\frac{8}{3}e^{-\frac{8\eta}{3}}a_8-3e^{-3\eta}a_9-\frac{10}{3}e^{-\frac{10\eta}{3}}a_{10}-\frac{11}{3}e^{-\frac{11\eta}{3}}a_{11}-4e^{-4\eta}a_{12}-\frac{13}{3}e^{-\frac{13\eta}{3}}a_{13} \\
& -\frac{14}{3}e^{-\frac{14\eta}{3}}a_{14}-5e^{-5\eta}a_{15}-\frac{16}{3}e^{-\frac{16\eta}{3}}a_{16})^3 + \text{EcM}(-\frac{1}{3}e^{-\frac{\eta}{3}}a_1-\frac{2}{3}e^{-\frac{2\eta}{3}}a_2-e^{-\eta}a_3-\frac{4}{3}e^{-\frac{4\eta}{3}}a_4 \\
& -\frac{5}{3}e^{-\frac{5\eta}{3}}a_5-2e^{-2\eta}a_6-\frac{7}{3}e^{-\frac{7\eta}{3}}a_7-\frac{8}{3}e^{-\frac{8\eta}{3}}a_8-3e^{-3\eta}a_9-\frac{10}{3}e^{-\frac{10\eta}{3}}a_{10}-\frac{11}{3}e^{-\frac{11\eta}{3}}a_{11}-4e^{-4\eta}a_{12} \\
& -\frac{13}{3}e^{-\frac{13\eta}{3}}a_{13}-\frac{14}{3}e^{-\frac{14\eta}{3}}a_{14}-5e^{-5\eta}a_{15}-\frac{16}{3}e^{-\frac{16\eta}{3}}a_{16})^2 + \text{EcPp}(1+\frac{1}{\beta})(-\frac{1}{3}e^{-\frac{\eta}{3}}a_1-\frac{2}{3}e^{-\frac{2\eta}{3}}a_2 \\
& -e^{-\eta}a_3-\frac{4}{3}e^{-\frac{4\eta}{3}}a_4-\frac{5}{3}e^{-\frac{5\eta}{3}}a_5-2e^{-2\eta}a_6-\frac{7}{3}e^{-\frac{7\eta}{3}}a_7-\frac{8}{3}e^{-\frac{8\eta}{3}}a_8-3e^{-3\eta}a_9-\frac{10}{3}e^{-\frac{10\eta}{3}}a_{10} \\
& -\frac{11}{3}e^{-\frac{11\eta}{3}}a_{11}-4e^{-4\eta}a_{12}-\frac{13}{3}e^{-\frac{13\eta}{3}}a_{13}-\frac{14}{3}e^{-\frac{14\eta}{3}}a_{14}-5e^{-5\eta}a_{15}-\frac{16}{3}e^{-\frac{16\eta}{3}}a_{16})^2 + \text{Ec} \cdot \\
& (1+\frac{1}{\beta})(\frac{1}{9}e^{-\frac{\eta}{3}}a_1+\frac{4}{9}e^{-\frac{2\eta}{3}}a_2+e^{-\eta}a_3+\frac{16}{9}e^{-\frac{4\eta}{3}}a_4+\frac{25}{9}e^{-\frac{5\eta}{3}}a_5+4e^{-2\eta}a_6+\frac{49}{9}e^{-\frac{7\eta}{3}}a_7 \\
& +\frac{64}{9}e^{-\frac{8\eta}{3}}a_8+9e^{-3\eta}a_9+\frac{100}{9}e^{-\frac{10\eta}{3}}a_{10}+\frac{121}{9}e^{-\frac{11\eta}{3}}a_{11}+16e^{-4\eta}a_{12}+\frac{169}{9}e^{-\frac{13\eta}{3}}a_{13}+\frac{196}{9}e^{-\frac{14\eta}{3}}a_{14} \\
& +25e^{-5\eta}a_{15}+\frac{256}{9}e^{-\frac{16\eta}{3}}a_{16})^2 + \text{Nt}(-\frac{1}{3}e^{-\frac{\eta}{3}}b_1-\frac{2}{3}e^{-\frac{2\eta}{3}}b_2-e^{-\eta}b_3-\frac{4}{3}e^{-\frac{4\eta}{3}}b_4-\frac{5}{3}e^{-\frac{5\eta}{3}}b_5 \\
& -2e^{-2\eta}b_6-\frac{7}{3}e^{-\frac{7\eta}{3}}b_7-\frac{8}{3}e^{-\frac{8\eta}{3}}b_8-3e^{-3\eta}b_9-\frac{10}{3}e^{-\frac{10\eta}{3}}b_{10}-\frac{11}{3}e^{-\frac{11\eta}{3}}b_{11}-4e^{-4\eta}b_{12}-\frac{13}{3}e^{-\frac{13\eta}{3}}b_{13} \\
& -\frac{14}{3}e^{-\frac{14\eta}{3}}b_{14}-5e^{-5\eta}b_{15}-\frac{16}{3}e^{-\frac{16\eta}{3}}b_{16})^2 + (a_0+e^{-\frac{\eta}{3}}a_1+e^{-\frac{2\eta}{3}}a_2+e^{-\eta}a_3+e^{-\frac{4\eta}{3}}a_4+e^{-\frac{5\eta}{3}}a_5
\end{aligned}$$

$$\begin{aligned}
& +e^{-2\eta}a_6 + e^{-\frac{7\eta}{3}}a_7 + e^{-\frac{8\eta}{3}}a_8 + e^{-3\eta}a_9 + e^{-\frac{10\eta}{3}}a_{10} + e^{-\frac{11\eta}{3}}a_{11} + e^{-4\eta}a_{12} + e^{-\frac{13\eta}{3}}a_{13} + e^{-\frac{14\eta}{3}}a_{14} \\
& + e^{-5\eta}a_{15} + e^{-\frac{16\eta}{3}}a_{16})(-\frac{1}{3}e^{-\frac{\eta}{3}}b_1 - \frac{2}{3}e^{-\frac{2\eta}{3}}b_2 - e^{-\eta}b_3 - \frac{4}{3}e^{-\frac{4\eta}{3}}b_4 - \frac{5}{3}e^{-\frac{5\eta}{3}}b_5 - 2e^{-2\eta}b_6 \\
& - \frac{7}{3}e^{-\frac{7\eta}{3}}b_7 - \frac{8}{3}e^{-\frac{8\eta}{3}}b_8 - 3e^{-3\eta}b_9 - \frac{10}{3}e^{-\frac{10\eta}{3}}b_{10} - \frac{11}{3}e^{-\frac{11\eta}{3}}b_{11} - 4e^{-4\eta}b_{12} - \frac{13}{3}e^{-\frac{13\eta}{3}}b_{13} \\
& - \frac{14}{3}e^{-\frac{14\eta}{3}}b_{14} - 5e^{-5\eta}b_{15} - \frac{16}{3}e^{-\frac{16\eta}{3}}b_{16}) + \text{Nb}(-\frac{1}{3}e^{-\frac{\eta}{3}}b_1 - \frac{2}{3}e^{-\frac{2\eta}{3}}b_2 - e^{-\eta}b_3 - \frac{4}{3}e^{-\frac{4\eta}{3}}b_4 \\
& - \frac{5}{3}e^{-\frac{5\eta}{3}}b_5 - 2e^{-2\eta}b_6 - \frac{7}{3}e^{-\frac{7\eta}{3}}b_7 - \frac{8}{3}e^{-\frac{8\eta}{3}}b_8 - 3e^{-3\eta}b_9 - \frac{10}{3}e^{-\frac{10\eta}{3}}b_{10} - \frac{11}{3}e^{-\frac{11\eta}{3}}b_{11} - 4e^{-4\eta}b_{12} \\
& - \frac{13}{3}e^{-\frac{13\eta}{3}}b_{13} - \frac{14}{3}e^{-\frac{14\eta}{3}}b_{14} - 5e^{-5\eta}b_{15} - \frac{16}{3}e^{-\frac{16\eta}{3}}b_{16})(-\frac{1}{3}e^{-\frac{\eta}{3}}c_1 - \frac{2}{3}e^{-\frac{2\eta}{3}}c_2 - e^{-\eta}c_3 \\
& - \frac{4}{3}e^{-\frac{4\eta}{3}}c_4 - \frac{5}{3}e^{-\frac{5\eta}{3}}c_5 - 2e^{-2\eta}c_6 - \frac{7}{3}e^{-\frac{7\eta}{3}}c_7 - \frac{8}{3}e^{-\frac{8\eta}{3}}c_8 - 3e^{-3\eta}c_9 - \frac{10}{3}e^{-\frac{10\eta}{3}}c_{10} - \frac{11}{3}e^{-\frac{11\eta}{3}}c_{11} \\
& - 4e^{-4\eta}c_{12} - \frac{13}{3}e^{-\frac{13\eta}{3}}c_{13} - \frac{14}{3}e^{-\frac{14\eta}{3}}c_{14} - 5e^{-5\eta}c_{15} - \frac{16}{3}e^{-\frac{16\eta}{3}}c_{16})
\end{aligned}$$

and

$$\begin{aligned}
R_{\phi_3}(a_i, b_k, c_k, \eta) = & -\exp(-E((\text{Tr}-1)(e^{-\frac{\eta}{3}}b_1 + e^{-\frac{2\eta}{3}}b_2 + e^{-\eta}b_3 + e^{-\frac{4\eta}{3}}b_4 + e^{-\frac{5\eta}{3}}b_5 + e^{-2\eta}b_6 \\
& + e^{-\frac{7\eta}{3}}b_7 + e^{-\frac{8\eta}{3}}b_8 + e^{-3\eta}b_9 + e^{-\frac{10\eta}{3}}b_{10} + e^{-\frac{11\eta}{3}}b_{11} + e^{-4\eta}b_{12} + e^{-\frac{13\eta}{3}}b_{13} + e^{-\frac{14\eta}{3}}b_{14} + e^{-5\eta}b_{15} \\
& + e^{-\frac{16\eta}{3}}b_{16}) + 1)^{-1})\text{Sc}(e^{-\frac{\eta}{3}}c_1 + e^{-\frac{2\eta}{3}}c_2 + e^{-\eta}c_3 + e^{-\frac{4\eta}{3}}c_4 + e^{-\frac{5\eta}{3}}c_5 + e^{-2\eta}c_6 + e^{-\frac{7\eta}{3}}c_7 + e^{-\frac{8\eta}{3}}c_8 \\
& + e^{-3\eta}c_9 + e^{-\frac{10\eta}{3}}c_{10} + e^{-\frac{11\eta}{3}}c_{11} + e^{-4\eta}c_{12} + e^{-\frac{13\eta}{3}}c_{13} + e^{-\frac{14\eta}{3}}c_{14} + e^{-5\eta}c_{15} + e^{-\frac{16\eta}{3}}c_{16})\sigma^* \cdot \\
& ((\text{Tr}-1)(e^{-\frac{\eta}{3}}b_1 + e^{-\frac{2\eta}{3}}b_2 + e^{-\eta}b_3 + e^{-\frac{4\eta}{3}}b_4 + e^{-\frac{5\eta}{3}}b_5 + e^{-2\eta}b_6 + e^{-\frac{7\eta}{3}}b_7 + e^{-\frac{8\eta}{3}}b_8 + e^{-3\eta}b_9 \\
& + e^{-\frac{10\eta}{3}}b_{10} + e^{-\frac{11\eta}{3}}b_{11} + e^{-4\eta}b_{12} + e^{-\frac{13\eta}{3}}b_{13} + e^{-\frac{14\eta}{3}}b_{14} + e^{-5\eta}b_{15} + e^{-\frac{16\eta}{3}}b_{16}) + 1)^m \\
& + \frac{\text{Nt}}{\text{Nb}}((\frac{1}{9}e^{-\frac{\eta}{3}}b_1 + \frac{4}{9}e^{-\frac{2\eta}{3}}b_2 + e^{-\eta}b_3 + \frac{16}{9}e^{-\frac{4\eta}{3}}b_4 + \frac{25}{9}e^{-\frac{5\eta}{3}}b_5 + 4e^{-2\eta}b_6 + \frac{49}{9}e^{-\frac{7\eta}{3}}b_7 + \frac{64}{9}e^{-\frac{8\eta}{3}}b_8 \\
& + 9e^{-3\eta}b_9 + \frac{100}{9}e^{-\frac{10\eta}{3}}b_{10} + \frac{121}{9}e^{-\frac{11\eta}{3}}b_{11} + 16e^{-4\eta}b_{12} + \frac{169}{9}e^{-\frac{13\eta}{3}}b_{13} + \frac{196}{9}e^{-\frac{14\eta}{3}}b_{14} + 25e^{-5\eta}b_{15} \\
& + \frac{256}{9}e^{-\frac{16\eta}{3}}b_{16})) + \frac{1}{9}e^{-\frac{\eta}{3}}c_1 + \frac{4}{9}e^{-\frac{2\eta}{3}}c_2 + e^{-\eta}c_3 + \frac{16}{9}e^{-\frac{4\eta}{3}}c_4 + \frac{25}{9}e^{-\frac{5\eta}{3}}c_5 + 4e^{-2\eta}c_6 + \frac{49}{9}e^{-\frac{7\eta}{3}}c_7
\end{aligned}$$

$$\begin{aligned}
& + \frac{64}{9} e^{-\frac{8\eta}{3}} c_8 + 9e^{-3\eta} c_9 + \frac{100}{9} e^{-\frac{10\eta}{3}} c_{10} + \frac{121}{9} e^{-\frac{11\eta}{3}} c_{11} + 16e^{-4\eta} c_{12} + \frac{169}{9} e^{-\frac{13\eta}{3}} c_{13} \\
& + \frac{196}{9} e^{-\frac{14\eta}{3}} c_{14} + 25e^{-5\eta} c_{15} + \frac{256}{9} e^{-\frac{16\eta}{3}} c_{16} + \text{Sc}(a_0 + e^{-\frac{\eta}{3}} a_1 + e^{-\frac{2\eta}{3}} a_2 + e^{-\eta} a_3 + e^{-\frac{4\eta}{3}} a_4 \\
& + e^{-\frac{5\eta}{3}} a_5 + e^{-2\eta} a_6 + e^{-\frac{7\eta}{3}} a_7 + e^{-\frac{8\eta}{3}} a_8 + e^{-3\eta} a_9 + e^{-\frac{10\eta}{3}} a_{10} + e^{-\frac{11\eta}{3}} a_{11} + e^{-4\eta} a_{12} + e^{-\frac{13\eta}{3}} a_{13} \\
& + e^{-\frac{14\eta}{3}} a_{14} + e^{-5\eta} a_{15} + e^{-\frac{16\eta}{3}} a_{16}) (-\frac{1}{3} e^{-\frac{\eta}{3}} c_1 - \frac{2}{3} e^{-\frac{2\eta}{3}} c_2 - e^{-\eta} c_3 - \frac{4}{3} e^{-\frac{4\eta}{3}} c_4 - \frac{5}{3} e^{-\frac{5\eta}{3}} c_5 \\
& - 2e^{-2\eta} c_6 - \frac{7}{3} e^{-\frac{7\eta}{3}} c_7 - \frac{8}{3} e^{-\frac{8\eta}{3}} c_8 - 3e^{-3\eta} c_9 - \frac{10}{3} e^{-\frac{10\eta}{3}} c_{10} - \frac{11}{3} e^{-\frac{11\eta}{3}} c_{11} - 4e^{-4\eta} c_{12} - \frac{13}{3} e^{-\frac{13\eta}{3}} c_{13} \\
& - \frac{14}{3} e^{-\frac{14\eta}{3}} c_{14} - 5e^{-5\eta} c_{15} - \frac{16}{3} e^{-\frac{16\eta}{3}} c_{16})
\end{aligned}$$