



Numerical Integration of Seventh order Boundary Value Problems by Standard Collocation Method via Four Orthogonal Polynomials

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ABSTRACT

Based on standard collocation technique, four (4) different orthogonal polynomials were used as basis functions in the numerical treatment of seventh (7th) order boundary value problems in Ordinary Differential Equations. The performance of each of these polynomials as basis function in the trial solution was then compared. The results obtained from three examples showed that Chebyshev polynomial is the best in term of performance, and closely followed by Hermite polynomial, which was followed by Legendre polynomial while the least in performance is Laguerre polynomial.

1. INTRODUCTION

Seventh order boundary value problems appear in several branches of applied mathematics and engineering sciences. For example, problems that arise in Mathematics modeling of induction motors with two rotor circuits, induction motor model contains fifth order differential equation model, which includes two stator

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variables, two rotor state variables and shaft speed (Ghazala and Hamood, 2014). For the effect of a starting cage, deep bars or rotor distributed parameter, additional rotor circuit is required so that two more state variables and two equations may be added. Additional state variable create computational burden. To avoid this, models are often limited to the fifth order and rotor impedance is algebraically altered as function of rotor speed to account for torque discrepancies at startup. This is done by the assumption of frequency of rotor currents depends on rotor speed. Fifth order model running near full speed subjected to sudden voltage dip, would not reproduce the transient drag torque produced by stationary flux linkage trapped in the stator windings, although such behavior would show up in the seventh order model. Many researchers have solved seventh order BVPs using different methods. Siddiqi *et. al.* (2012) solved these using Variation of parameters method, Differential transformation method and Variation iteration technique. Reproducing kernel space method was used by Haar (2017). Siddiqi and Muzammal (2013) also used Adomian Decomposition Method for solution of seventh order boundary Value problems. The pitfalls of the above mentioned methods is that they take more time in terms of computation even the Adomian package program is very expensive. The researcher is still using standard collocation method because of its simplicity in terms of computation and better CPU.

Orthogonal polynomials play an important role in mathematics and in physics, often as solution to differential equation or eigen functions of differential operators. The widely used orthogonal polynomials are the classical orthogonal polynomials, consisting of the Hermite polynomials, Laguerre polynomials, the Jacobi polynomials together with their special cases, the Gegenbauer polynomials, Chebyshev polynomials, Legendre polynomial (Yisa, 2015a).

The usefulness of various type of orthogonal polynomials like Hermite, Chebyshev, Legendre and Laguerre polynomials can never be over emphasized. This is evidenced in the works of many scientists like, Gauss in his popular integration method called Gauss Quadrature formula, where he derived the nodes for any given problem by equating to zero the polynomial of equivalent order, mostly Legendre. Lanczos (1938, 1956) [8] elucidated more of on the properties of these polynomials and came to the conclusion that chebyshev polynomial is the best, considering the mini-max property of the latter in the error propagation.

Here, we use four frequently encountered orthogonal polynomials namely, Hermite, Chebyshev, Legendre and Laguerre polynomial as basis functions. We compare their results to determine the order of their responsiveness.

Here, we consider the seventh order BVPs of the form

$$y^{(7)}(x) + f(x)y(x) = r(x), \quad x \in (a, b)$$

subject to the following conditions

$$y(x_0) = \alpha, \quad y(x_n) = \beta, \quad y^{(1)}(x_0) = \alpha,$$

$$y^{(1)}(x_n) = \beta', \quad y^{(2)}(x_0) = \alpha'$$

$$y^{(2)}(x_n) = \beta'', \quad y^{(3)}(x_0) = \alpha''$$

This article is organized as follows: Section 2 deals with orthogonal polynomials, Section 3 is on the methodology, numerical examples are considered in section 4 while Section 5 concludes the work.

2. ORTHOGONAL POLYNOMIALS

Consider the polynomial

$$Q_N(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_Nx^N$$

where $N = 0, 1, 2, \dots$ and $a_n \in \mathbb{R}$

Two functions $Q_n(x)$ and $Q_m(x)$ are orthogonal if the inner product

$$\langle Q_n(x), Q_m(x) \rangle = 0, \quad n \neq m$$

That is, if

$$\int_a^b w(x) Q_n(x) Q_m(x) dx = 0$$

where $w(x)$ is the weight function and a and b , are constants.

2.1 Properties of Orthogonal Polynomials.

- (1) Any polynomial $f(x)$ of degree n can be expanded in terms of $p_0(x), p_1(x), \dots, p_n(x)$, that is, there exist coefficients a_i such that

$$f(x) = \sum_{i=0}^n a_i p_i(x)$$

- (2) Given an orthogonal set of polynomial $\{p_0(x), p_1(x), \dots\}$, each polynomial, $p_k(x)$ is orthogonal to any polynomial of degree $< k$
- (3) Any orthogonal set of polynomials $\{p_0(x), p_1(x), \dots\}$, has a recurrence formula that relates any three consecutive polynomial in the sequence, that is, the relation $p_{n+1} = (a_nx + b_n)p_n - c_np_{n-1}$ exists, where the coefficients a, b and c depend on n . Such a recurrence formula is often used to generate higher order members in the set
- (4) Each polynomial in $\{p_0(x), p_1(x), \dots\}$ has all n of its roots real, distinct and strictly within the interval of orthogonality (i.e not on its ends). This is an extremely unusual property! It is particularly important when considering the classes of polynomials that arise as quantum mechanical solutions to a given Hamiltonian (or other Hermitian) operator

- (5) Furthermore the roots of the n^{th} degree polynomial, p_n lie strictly inside the roots of the $(n+1)^{th}$ degree polynomial p_{n+1}
 In what now follows, we shall briefly define the Orthogonal polynomials adopted in this work.

2.2 SOME ORTHOGONAL POLYNOMIALS

2.2.1 HERMITE POLYNOMIAL [1]

Hermite polynomial was named in honour of Charles Hermite (1822 - 1901). The Hermite polynomials are orthogonal in the interval $(-\infty, \infty)$ with respect to the weight function $w(x) = e^{-x^2}$. They are defined by means of their Rodrigue formula

$$H_n(x) = (-1)^n e^{-x^2} \frac{d^n}{dx^n} e^{-x^2}$$

and the recurrence formula is given as

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \text{ where } n = 1, 2, \dots$$

2.2.2 LAGUERRE POLYNOMIAL [2]

Laguerre polynomial was named in honour of Edmond Laguerre (1834 - 1885). Laguerre polynomials are orthogonal in the interval $(0, \infty)$ with respect to the weight function $w(x) = e^{-x}$. It is given by Rodrigue formula

$$L_n(x) = e^x \frac{d^n}{dx^n} x^n e^{-x} \text{ where } n = 1, 2, \dots$$

and the recurrence formula is given as

$$L_{n+1}(x) = (x - 2n - 1)L_n(x) + n^2 L_{n-1}(x)$$

2.2.3 CHEBYSHEV POLYNOMIAL [8]

The Chebyshev polynomials named after a Russian Mathematician, Pafnuty Lvovich Chebyshev (1794 - 1869). The Chebyshev polynomials of the first kind denoted by $T_n(x)$ are sets of polynomials degree n defined as the solution to the first kind Chebyshev differential equation

$$(1 - x^2)y'' - xy' + n^2y = 0$$

A Chebyshev polynomial of degree n of the first kind is defined as

$$T_n(x) = \cos(n \cos^{-1} x, \quad x \in [-1, 1]$$

and the recurrence relation is given by

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1$$

Also, in $a \leq x \leq b$, the Chebyshev polynomial is defined as

$$T_n(x) = \cos \left[n \cos \left(\frac{2x - a - b}{b - a} \right) \right], \quad x \in [a, b]$$

and satisfies the recurrence relation;

$$T_{n+1}(x) = 2 \left(\frac{2x - a - b}{b - a} \right) T_n(x) - T_{n-1}(x), \quad n \geq 0, \quad x \in [a, b]$$

2.2.4 LEGENDRE POLYNOMIAL [2]

The Legendre Polynomial was introduced in 1784 by a French Mathematician, Legendre A.M. (1752 - 1833). The Legendre Polynomial is defined and denoted by

$$p_{n+1}(x) = \frac{1}{n+1} \{ (2n+1)xp_n(x) - np_{n-1}(x) \}$$

and $p_n(x)$ is expressed by Rodrigue's formula

$$p_n(x) = \frac{d^n}{2^n n! dx^n} (x^2 - 1)^n; \quad n = 0, 1, 2, \dots$$

2.3 CO-ORDINATE TRANSFORMATION AND SHIFTED ORTHOGONAL POLYNOMIAL

In order to shift from the interval $[-1, 1]$ to the interval $[a, b]$, let

$$y = mx + n$$

$$a = -m + n$$

$$b = m + n$$

Add equation (18) and (19) we have

$$2n = a + b$$

$$n = \frac{a + b}{2}$$

Putting (21) in (19) and simplify to obtain

$$m = \frac{a - b}{2}$$

Substituting (21) and (22) in (17), we have

$$x = \frac{2y - (a + b)}{b - a}$$

2.3.1 SHIFTED CHEBYSHEV POLYNOMIAL [9]

Shifed Chebyshev Polynomial are orthogonal on $[0, 1]$. And it has rodrigue formula

$$T_n^*(x) = \cos\{n \cos^{-1}(2x - 1)\}$$

And recurrence relation

$$T_{n+1}^*(x) = 2(2x - 1)T_n(x) - T_{n-1}(x)$$

2.3.2 SHIFTED LEGENDRE POLYNOMIAL [5]

Shifted Legendre Polynomial are orthogonal on interval $[0, 1]$. They are define by means of their rodrigue formula

$$p_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - x)^n \text{ where } n = 1, 2, \dots$$

and the recurrence formula is given as

$$p_n(x) = \frac{(2x - 1)(2n + 1)}{n + 1} p_n(x) - \frac{n}{n + 1} p_{n-1}(x), \text{ where } n = 1, 2, \dots, n$$

3 METHODOLOGY

Here, we assumed an approximate solution of the form

$$y(x) \approx y_n(x) = \sum_{i=0}^n a_i Q_i(x)$$

where $Q_i(x)$ is orthogonal polynomial.

Substituting (28) in (1) and (2) to obtain

$$\frac{d^7}{dx^7} \left\{ \sum_{i=0}^n a_i Q_i(x) \right\} + f(x) \left\{ \sum_{i=0}^n a_i Q_i(x) \right\} = r(x)$$

and

$$\begin{aligned} \sum_{i=0}^n a_i Q_i(x_0) = \alpha, \quad \sum_{i=0}^n a_i Q_i(x_n) = \beta, \quad \sum_{i=0}^n a_i Q_i'(x_0) = \alpha', \quad \sum_{i=0}^n a_i Q_i'(x_n) = \beta' \\ \sum_{i=0}^n a_i Q_i''(x_0) = \alpha'', \quad \sum_{i=0}^n a_i Q_i''(x_n) = \beta'', \quad \sum_{i=0}^n a_i Q_i'''(x_0) = \alpha''' \end{aligned}$$

Equation (29) is simplified to get

$$\begin{aligned} \frac{d^7}{dx^7} (a_0 Q_0(x) + a_1 Q_1(x) + a_2 Q_2(x) + \dots + a_n Q_n(x)) \\ + f(x) (a_0 Q_0(x) + a_1 Q_1(x) + a_2 Q_2(x) + \dots + a_n Q_n(x)) = r(x) \end{aligned}$$

and

$$\left[\left(\frac{d^7}{dx^7} + f(x) \right) Q_0(x) \right] a_0 + \left[\left(\frac{d^7}{dx^7} + f(x) \right) Q_1(x) \right] a_1 + \left[\left(\frac{d^7}{dx^7} + f(x) \right) Q_2(x) \right] a_2 + \cdots + \left[\left(\frac{d^7}{dx^7} + f(x) \right) Q_n(x) \right] a_n = r(x)$$

Equation(32) is collocated at x_j as

$$\left[\left(\frac{d^7}{dx^7} + f(x_j) \right) Q_0(x_j) \right] a_0 + \left[\left(\frac{d^7}{dx^7} + f(x_j) \right) Q_1(x_j) \right] a_1 + \left[\left(\frac{d^7}{dx^7} + f(x_j) \right) Q_2(x_j) \right] a_2 + \cdots + \left[\left(\frac{d^7}{dx^7} + f(x_j) \right) Q_n(x_j) \right] a_n = r(x_j)$$

where

$$x_j = a_j + \frac{(b-a)j}{N+1}, \quad j = 1, 2, 3, \dots, N$$

Equation (33), after collocation, is put in matrix form as

$$\begin{pmatrix} \left(\frac{d^7}{dx^7} + f(x_1) \right) Q_0(x_1) & \left(\frac{d^7}{dx^7} + f(x_1) \right) Q_1(x_1) & \cdots & \left(\frac{d^7}{dx^7} + f(x_1) \right) Q_n(x_1) \\ \left(\frac{d^7}{dx^7} + f(x_2) \right) Q_0(x_2) & \left(\frac{d^7}{dx^7} + f(x_2) \right) Q_1(x_2) & \cdots & \left(\frac{d^7}{dx^7} + f(x_2) \right) Q_n(x_2) \\ \vdots & \vdots & & \vdots \\ \left(\frac{d^7}{dx^7} + f(x_n) \right) Q_0(x_n) & \left(\frac{d^7}{dx^7} + f(x_n) \right) Q_1(x_n) & \cdots & \left(\frac{d^7}{dx^7} + f(x_n) \right) Q_n(x_n) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} r(x_1) \\ r(x_2) \\ \vdots \\ r(x_n) \end{pmatrix}$$

Equation (35) will give $(N - (n - 1))$ system of algebraic equations.

Also N equations are derived from the seven initial conditions, thus,

$$\begin{aligned} y_n(x_0) &= a_0 Q_0(x_0) + a_1 Q_1(x_0) + a_2 Q_2(x_0) + \cdots + a_n Q_n(x_0) = \alpha \\ y_n(x_n) &= a_0 Q_0(x_n) + a_1 Q_1(x_n) + a_2 Q_2(x_n) + \cdots + a_n Q_n(x_n) = \beta \\ y_n^{(1)}(x_0) &= a_0 Q_0^{(1)}(x_0) + a_1 Q_1^{(1)}(x_0) + a_2 Q_2^{(1)}(x_0) + \cdots + a_n Q_n^{(1)}(x_0) = \alpha' \\ y_n^{(1)}(x_n) &= a_0 Q_0^{(1)}(x_n) + a_1 Q_1^{(1)}(x_n) + a_2 Q_2^{(1)}(x_n) + \cdots + a_n Q_n^{(1)}(x_n) = \beta' \\ y_n^{(2)}(x_0) &= a_0 Q_0^{(2)}(x_0) + a_1 Q_1^{(2)}(x_0) + a_2 Q_2^{(2)}(x_0) + \cdots + a_n Q_n^{(2)}(x_0) = \alpha'' \\ y_n^{(2)}(x_n) &= a_0 Q_0^{(2)}(x_n) + a_1 Q_1^{(2)}(x_n) + a_2 Q_2^{(2)}(x_n) + \cdots + a_n Q_n^{(2)}(x_n) = \beta'' \\ y_n^{(3)}(x_0) &= a_0 Q_0^{(3)}(x_0) + a_1 Q_1^{(3)}(x_0) + a_2 Q_2^{(3)}(x_0) + \cdots + a_n Q_n^{(3)}(x_0) = \alpha''' \end{aligned}$$

Equation (36) are also put in matrix form as

$$\begin{pmatrix} Q_0(x_0) & Q_1(x_0) & \cdots & Q_n(x_0) \\ Q_0(x_n) & Q_1(x_n) & \cdots & Q_n(x_n) \\ Q_0^{(1)}(x_0) & Q_1^{(1)}(x_0) & \cdots & Q_n^{(1)}(x_0) \\ Q_0^{(1)}(x_n) & Q_1^{(1)}(x_n) & \cdots & Q_n^{(1)}(x_n) \\ Q_0^{(2)}(x_0) & Q_1^{(2)}(x_0) & \cdots & Q_n^{(2)}(x_0) \\ Q_0^{(2)}(x_n) & Q_1^{(2)}(x_n) & \cdots & Q_n^{(2)}(x_n) \\ \vdots & \vdots & & \vdots \\ Q_0^{(n)}(x_0) & Q_1^{(n)}(x_0) & \cdots & Q_n^{(n)}(x_0) \\ Q_0^{(n)}(x_n) & Q_1^{(n)}(x_n) & \cdots & Q_n^{(n)}(x_n) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ \vdots \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \alpha' \\ \beta' \\ \alpha'' \\ \beta'' \\ \vdots \\ \alpha^{(n)} \\ \beta^{(n)} \end{pmatrix}$$

Equation (35) and (37) are combined to give $(N + 1)$ system of equations. This system of $N+1$ equations are then solved simultaneously to obtained the unknown constants a_j , $j = 0, 1, 2, \dots, n$ which are substituted back into the trial solution to obtain the required approximate solutions

4 NUMERICAL EXAMPLES

Example 1

Consider the linear seventh order boundary value problem

$$\begin{aligned} \frac{d^7 y}{dx^7} &= xy(x) + e^x(x^2 - 2x - 6) \\ y(0) &= 1, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = -2, \quad y(1) = 0, \\ y'(1) &= -e, \quad y''(1) = -2e \end{aligned}$$

The exact solution of the problem is given as

$$y(x) = (1 - x)e^x$$

Solution. Here, we demonstrate the methodology using Hermite polynomial

Let

$$y \cong y_N(x) = \sum_{n=0}^N a_n H_n(x)$$

where H_n are Hermite polynomials

Substituting (41) into (38) and (39) respectively, then we have;

$$\begin{aligned} \frac{d^7}{dx^7} \left(\sum_{n=0}^N a_n H_n(x) \right) &= x \left(\sum_{n=0}^N a_n H_n(x) \right) + e^x(x^2 - 2x - 6) \\ y_N(0) &= 1, \quad y'_N(0) = 0, \quad y''_N(0) = -1, \quad y'''_N(0) = -2, \quad y_N(1) = 0, \\ y'_N(1) &= -e, \quad y''_N(1) = -2e \end{aligned}$$

Choosing $N = 7$, then (41) becomes

$$y_7(x) = a_0H_0(x) + a_1H_1(x) + a_2H_2(x) + a_3H_3(x) + a_4H_4(x) + a_5H_5(x) + a_6H_6(x) + a_7H_7(x)$$

i.e

$$y_7(x) = a_0 + 2xa_1 + a_2(4x^2 - 2) + a_3(8x^3 - 12x) + a_4(16x^4 - 48x^2 + 12) + a_5(32x^5 - 160x^3 + 120x) + a_6(64x^6 - 480x^4 + 720x^2 - 120) + a_7(128x^7 - 1344x^5 + 3360x^3 - 1680x)$$

Substituting (44) into (38) and after simplification we obtained

$$645120a_7 = x(a_0 + 2a_1(x) + a_2(4x^2 - 2x) + a_3(8x^3 - 12x) + a_4(16x^4 - 48x^2 + 12) + a_5(32x^5 - 160x^3 + 120x) + a_6(64x^6 - 480x^4 + 720x^2 - 120) + a_7(128x^7 - 1344x^5 + 3360x^3 - 1680x)) + 2.718281828x(x^2 - 2x - 6)$$

now collocating (45) using

$$x_i = a + \frac{(b-a)i}{N+1}$$

$$i = 1, 2, \dots, [N - (n-1)]$$

when $i=1$, $x_1 = \frac{1}{8}$ we obtain;

$$645120a_7 = 0.125a_0 + 0.031250a_1 - 0.023437500a_2 - 0.1855468750a_3 + 1.406738281a_4 + 1.836059570a_5 - 13.60836792a_6 - 25.43480682a_7 - 7.06447238$$

(46) is written in matrix form as

$$(0.125 \quad 0.031250 \quad -0.023437500 \quad -0.1855468750 \quad 1.406738281 \quad 1.836059570 \quad -13.60836792 \quad -645145.43480682) \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix}$$

$$(1) \quad = 7.06447238$$

which is one algebraic equations with 8 unknowns. The remaining 7 equations are derived from the 7 initial conditions. i.e.,

$$y_7'(x) = 2a_1 + 8xa_2 + a_3(24x^2 - 12) + a_4(64x^3 - 96x) + a_5(160x^4 - 480x^2 + 120) + a_6(384x^5 - 1920x^3 + 1440x) + a_7(896x^6 - 6720x^4 + 10080x^2 - 1680)$$

$$y_7''(x) = 8a_2 + 48xa_3 + a_4(192x^2 - 96) + a_5(640x^3 - 960x) \\ + a_6(1920x^4 - 5760x^2 + 1440) + a_7(5376x^5 - 26880x^3 + 20160x)$$

$$y_7'''(x) = 48a_3 + 384xa_4 + a_5(1280x^2 - 960) + a_6(7680x^3 - 11520x) \\ + a_7(26880x^4 - 80640x^2 + 20160)$$

(2)

$$y_7^{iv}(x) = 384a_4 + 2560xa_5 + a_6(23040x^2 - 11520) + a_7(107520x^3 - 161280x)$$

$$(3) \quad y_7^v(x) = 2560a_5 + 46080xa_6 + a_7(322560x^2 - 161280)$$

$$(4) \quad y_7^{vi}(x) = 46080a_6 + 645120xa_7$$

$$(5) \quad y_7^{vii}(x) = 645120a_7$$

Thus,

(6)

$$y(0) = 1; \Rightarrow a_0 + 12a_4 - 120a_6 = 1$$

(7)

$$y'(0) = 0; \Rightarrow 2a_1 - 2a_2 - 12a_3 + 120a_5 - 1680a_7 = 0$$

(8)

$$y'(0) = -1; \Rightarrow 8a_2 - 96a_4 + 1440a_6 = -1$$

(9)

$$y'''(0) = 2; \Rightarrow 48a_3 - 960a_5 + 20160a_7 = -2$$

(10)

$$y(1) = 0; \Rightarrow a_0 + 2a_1 + 2a_2 - 4a_3 - 20a_4 - 8a_5 + 184a_6 + 464a_7 = 0$$

(11)

$$y'(1) = -e; \Rightarrow 2a_1 + 6a_2 + 12a_3 - 32a_4 - 200a_5 - 96a_6 + 2576a_7 = -2.718281828$$

(12)

$$y''(1) = -2e; \Rightarrow 8a_2 + 48a_3 + 96a_4 - 320a_5 - 2400a_6 - 1344a_7 = -5.436563657$$

Also, putting (51) to (65) in matrix form, we obtained,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 120 & 0 & -120 & 0 \\ 0 & 2 & -2 & -12 & 0 & 120 & 0 & -1680 \\ 0 & 0 & 80 & 0 & -96 & 0 & 1440 & 0 \\ 0 & 0 & 0 & 48 & 0 & -960 & 0 & 20160 \\ 1 & 2 & 2 & -4 & -20 & -8 & 184 & 464 \\ 0 & 2 & 6 & 12 & -32 & -200 & -96 & 2576 \\ 0 & 0 & 2 & 48 & 96 & -320 & -2400 & 1344 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \\ 0 \\ -2.718281828 \\ -5.436563657 \end{pmatrix}$$

solving the system (62) and (47) simultaneously using MAPPLE software we get,

$$\begin{aligned} a_0 &= 1.122958872, a_1 = -0.5599818551, a_2 = -0.2407495784, \\ a_3 &= -0.06625028456, a_4 = -0.01144812150, a_5 = -0.001455109745, \\ a_6 &= -0.0001201548869, a_7 = -0.00001075851669 \end{aligned}$$

substituting the values of (63) into (41), we obtain our approximate solution;

$$\begin{aligned} y(x) &= 1 + 5 \times 10^{-10}x - 0.5x^2 - 0.333333334x^3 - 0.1254955983x^4 \\ &\quad - 0.03210406541x^5 - 0.007689912762x^6 - 0.001377090136x^7 \end{aligned}$$

Following the procedure of the methodology, the approximate solution obtained for Laguerre, Chebyshev and Legendre Polynomial respectively are as follows:

$$\begin{aligned} y_L(x) &= 0.9999999997 - 1.1 \times 10^{-7}x - 0.50000034x^2 - 0.33333319x^3 \\ &\quad - 0.12549582x^4 - 0.032103433x^5 - 0.00769054484x^6 - 0.001376879442x^7 \end{aligned}$$

$$\begin{aligned} y_C(x) &= 0.9999999999 - 3.9154 \times 10^{10}x - 0.5000000004x^2 - 0.333333334x^3 \\ &\quad - 0.1254958091x^4 - 0.0321034330x^5 - 0.00769054484x^6 - 0.001376879442x^7 \end{aligned}$$

$$\begin{aligned} y_P(x) &= 0.9999999994 - 4.5596 \times 10^{-10}x - 0.4999999996x^2 - 0.3333333333x^3 \\ &\quad - 0.128298960x^4 - 0.02369397926x^5 - 0.01609999890x^6 + 0.001426271911x^7 \end{aligned}$$

Example 2

Consider the linear seventh order boundary value problem

$$\frac{d^7 y}{dx^7} = -y(x) - e^x(2x^2 + 12x + 35)$$

$$y(0) = 1, y'(0) = 1, y''(0) = 0, y'''(0) = -3, y(1) = 0, y'(1) = -e, y''(1) = -4e$$

The exact solution of the problem is given as

$$y(x) = x(1 - x)e^x$$

The approximate solution obtained are as follows:

$$\begin{aligned} y_h(x) &= 1 \times 10^{-10} + 1.0000000001x - 0.5x^3 - 0.336916565x^4 - 0.1161229914x^5 \\ &\quad - 0.03872249365x^6 - 0.008237949783x^7 \end{aligned}$$

$$\begin{aligned} y_l(x) &= -4 \times 10^{-7} + 1.000x - 5 \times 10^7 x^2 - 0.500000x^3 - 0.33691656x^4 - 0.116122988x^5 \\ &\quad - 0.0387224936x^6 - 0.008237949801x^7 \end{aligned}$$

$$\begin{aligned} y_c(x) &= -4.4756 \times 10^{-11} + 1.00x - 1.25 \times 10^{-10}x^2 - 0.5000000002x^3 - 0.3369165652x^4 \\ &\quad - 0.11612299914x^5 - 0.03872249365x^6 - 0.008237949780x^7 \end{aligned}$$

$$\begin{aligned} y_p(x) &= -5.882 \times 10^{-12} + 1.000x - 6.06 \times 10^{-10}x^2 - 0.4999999990x^3 - 0.3369165654x^4 \\ &\quad - 0.1161229913x^5 - 0.03872249365x^6 - 0.008237949782x^7 \end{aligned}$$

Example 3

Consider the linear seventh order boundary value problem

$$\frac{d^7 y}{dx^7} = y(x) - 7e^x$$

$$y(0) = 1, y'(0) = 0, y''(0) = -1, y'''(0) = -2, y(1) = 0, y'(1) = -e, y''(1) = -2e$$

The exact solution of the problem is given as

$$y(x) = (1 - x)e^x$$

The approximate solution obtained are as follows:

$$y_h(x) = 0.999999995 + 2 \times 10^{-10}x - 0.5x^2 - 0.333333332x^3 - 0.1254955983x^4 - 0.03210406545x^5 \\ - 0.007689912717x^6 - 0.001377090150x^7$$

$$y_l(x) = 0.9999999914 - 7.4 \times 10^{-7}x - 0.4999997x^2 - 0.33333330x^3 - 0.125538571x^4 - 0.031975154x^5 \\ - 0.00781882387x^6 - 0.001334119769x^7$$

$$y_c(x) = 1.00 - 1.1088 \times 10^{-10}x - 0.5x^2 - 0.333333335x^3 - 0.1255385687x^4 \\ - 0.03197515433x^5 - 0.007818823847x^6 - 0.001334119772x^7$$

$$y_p(x) = -2.6799102 \times 10^{-9}x + 1.000000000x^2 + 129.8381176x^3 - 387.6411679x^4 \\ + 386.4862070x^5 - 128.6830991x^6 - 0.00005765024021x^7$$

6 CONCLUSION

In this paper, we solved seventh order BVPs arising from Mathematical modeling of induction motors with two rotor circuits to test the efficiency of four orthogonal polynomials using standard collocation method. We have taken three test problems of linear type only; the results obtained from the three examples show that Hermite and Chebyshev perform best followed by Legendre polynomial and then the least in performance is Laguerre polynomial.

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TABLE 1. Table of Results for Example 1

width=1.4center					
x	Exact	$H_n(x)$	$L_n(x)$	$T_n(x)$	$P_n(x)$
0.00	1.0000	0.000000000000	3.870000×10^{-7}	1.00000×10^{-10}	6.000000×10^{-10}
0.10	0.9947	4.6217700×10^{-5}	4.399839×10^{-4}	4.621190×10^{-5}	4.641680×10^{-5}
0.20	0.9771	2.1757200×10^{-5}	2.752580×10^{-3}	2.17571×10^{-5}	1.946010×10^{-5}
0.30	0.9449	4.3430000×10^{-7}	6.173840×10^{-3}	4.34900×10^{-7}	8.223600×10^{-6}
0.40	0.8951	$8.52031000 \times 10^{-6}$	9.208010×10^{-3}	8.521600×10^{-6}	2.402250×10^{-5}
0.50	0.8244	4.430700×10^{-5}	1.037330×10^{-2}	4.430870×10^{-5}	6.620910×10^{-5}
0.60	0.7288	4.202810×10^{-5}	9.259490×10^{-3}	4.202630×10^{-5}	1.877510×10^{-5}
0.70	0.6041	2.132330×10^{-5}	6.197460×10^{-3}	2.132200×10^{-5}	3.149300×10^{-6}
0.80	0.4451	5.814800×10^{-6}	2.739390×10^{-3}	5.814100×10^{-6}	3.371800×10^{-6}
0.90	0.2459	5.981560×10^{-5}	5.496220×10^{-4}	5.981530×10^{-5}	5.797555×10^{-5}
1.00	0.0000	8.000000×10^{-12}	4.818000×10^{-6}	1.08460×10^{-10}	6.000000×10^{-10}

TABLE 2. Table of Results for Example 2

width=1.4center					
x	Exact	$H_n(x)$	$L_n(x)$	$T_n(x)$	$P_n(x)$
0.00	1.0000	1.00000×10^{-10}	-4.00000×10^{-7}	-4.4756×10^{-11}	-5.88200×10^{-12}
0.10	0.9947	3.48922×10^{-5}	3.52878×10^{-5}	3.48925×10^{-5}	3.48925×10^{-5}
0.20	0.9771	2.11908×10^{-5}	2.08073×10^{-5}	2.11904×10^{-5}	2.110905×10^{-5}
0.30	0.9449	4.12330×10^{-5}	4.15991×10^{-5}	4.12334×10^{-5}	4.12335×10^{-5}
0.40	0.8951	1.37326×10^{-5}	1.33867×10^{-5}	1.37321×10^{-5}	1.37321×10^{-5}
0.50	0.8244	5.55262×10^{-5}	5.58514×10^{-5}	5.55268×10^{-5}	5.55269×10^{-5}
0.60	0.7288	3.13565×10^{-5}	3.16627×10^{-5}	3.13573×10^{-5}	3.13573×10^{-5}
0.70	0.6041	4.45510×10^{-5}	4.48421×10^{-5}	4.45518×10^{-5}	4.45519×10^{-5}
0.80	0.4451	3.06987×10^{-5}	3.09812×10^{-5}	3.06996×10^{-5}	3.06996×10^{-5}
0.90	0.2459	3.93292×10^{-5}	3.96114×10^{-5}	3.93301×10^{-5}	3.93300×10^{-5}
1.00	0.0000	1.06700×10^{-9}	-2.91400×10^{-7}	1.00000×10^{-10}	2.68000×10^{-10}

TABLE 3. Table of Results for Example 3

width=1.4center					
x	Exact	$H_n(x)$	$L_n(x)$	$T_n(x)$	$P_n(x)$
0.00	1.0000	5.0000×10^{-5}	8.60000×10^{-9}	0.00000000	0.00000000
0.10	0.9947	4.62122×10^{-5}	4.63723×10^{-5}	4.62149×10^{-5}	4.62149×10^{-5}
0.20	0.9771	2.17567×10^{-5}	2.14988×10^{-5}	2.17221×10^{-5}	2.17221×10^{-5}
0.30	0.9449	4.34700×10^{-7}	8.37400×10^{-7}	5.53700×10^{-7}	5.53700×10^{-7}
0.40	0.8951	8.52070×10^{-6}	9.06300×10^{-6}	8.75800×10^{-6}	8.75790×10^{-6}
0.50	0.8244	4.43073×10^{-5}	5.50297×10^{-5}	4.46426×10^{-5}	4.46426×10^{-5}
0.60	0.7288	4.20277×10^{-5}	4.12422×10^{-5}	4.16718×10^{-5}	4.16718×10^{-5}
0.70	0.6041	2.13230×10^{-5}	2.05792×10^{-5}	2.10449×10^{-5}	2.10450×10^{-5}
0.80	0.4451	5.81460×10^{-6}	5.17860×10^{-6}	5.67400×10^{-6}	5.67420×10^{-6}
0.90	0.2459	5.98154×10^{-5}	5.98154×10^{-5}	5.97872×10^{-5}	5.97874×10^{-5}
1.00	0.0000	1.1700×10^{-10}	0.00000000	0.00000000	2.69000×10^{-10}