

Results of ω -order reversing partial contraction mapping generating a differential operator

Annotate Highlight

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Abstract. In this paper, we presents some partial differential operators defined on suitably chosen function spaces such as $H^{-1}(\Omega)$, $L^p(\Omega)$, with $p \in [1, +\infty)$. Laplace operator on a domain Ω in $\mathbb{R}^n$ subject to the Dirichlet boundary condition was established by generating a C_0 -semigroup, which is generated by an infinitesimal generator ω -order reversing partial contraction (ω -ORCP_n).

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1. Introduction and Background

Considering heat equation in a domain Ω in $\mathbb{R}^3$

$$\begin{cases} v_s = \Delta v & (s, x) \in Q_\infty \\ v = 0 & (s, x) \in \Sigma_\infty \\ v(0, x) = v_0(x) & x \in \Omega, \end{cases} \quad (1.1)$$

where Δ is the Laplace operator, $Q_\infty = \mathbb{R}_+ \times \Omega$ and $\Sigma_\infty = \mathbb{R}_+ \times \Gamma$. We rewrite this partial differential equation as an ordinary differential equation of the form

$$\begin{cases} v' = Av \\ v(0) = v_0 \end{cases} \quad (1.2)$$

in an infinite-dimensional Banach space X which is chosen suitably, so that the unbounded linear operator $A : D(A) \subseteq X \rightarrow X$ generate a C_0 -Semigroup of contractions.

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Let X be a Banach space, $X_n \subseteq X$ be a finite set, H is a Hilbert space, $(T(s))_{s \geq 0}$ is a C_0 -semigroup, $\omega - ORCP_n$ is the ω -order reversing partial contraction mapping, M_m be matrix, P_n is a partial transformation semigroup, $L(X)$ is a bounded linear operator on X , $\rho(A)$ is a resolvent set, $\sigma(A)$ is the spectrum and $A \in \omega - ORCP_n$ is a generator of C_0 -semigroup.

Akinyele *et al.* [1], introduced some results on perturbation of infinitesimal generator in semigroup and also in [2], Akinyele *et al.* obtained infinitesimal generator of Mean Ergodic theorem in semigroup of linear operator. Amann [3], established and solved some linear quasilinear parabolic problems and also in [4], Amann introduced measures to a linear parabolic problems. Arendt [5], introduced some Laplace transform in vector-valued and Cauchy problems. Balakrishnan [6], obtained an operator in infinitesimal generator of semigroup. Banach [7], established and introduced the concept of Banach spaces. Barbu [8], deduced some boundary problems for partial differential equation. Carja and Vrabie [9], obtained some results on new viability for semilinear differential insertion. Rauf and Akinyele [10], obtained ω -order-preserving partial contraction mapping and established the properties, also in [11], Rauf *et al.* established some stability and spectra properties on semigroup of linear operator. Vrabie [12], deduced some results of C_0 -semigroup and its applications. Yosida [13], established made a representation and differentiability of one-parameter semigroup.

2. Preliminaries

Definition 2.1 (ω -ORCP $_n$) [10]

A transformation $\alpha \in P_n$ is called ω -order-reversing partial contraction mapping if $\forall x, y \in \text{Dom} \alpha : x \leq y \implies \alpha x \geq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(s+t) = T(s)T(t)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 2.2 (C_0 -semigroup) [12]

A C_0 -Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

Definition 2.3 (C_0 -semigroup of contraction)[12]

A C_0 -semigroup $\{T(s); s \geq 0\}$ is called of type (ζ, ω) with $\zeta \geq 1$ and $\omega \in \mathbb{R}$, if for each $t \geq 0$, we have

$$\|T(s)\|_{L(X)} \leq \zeta e^{t\omega}.$$

A C_0 -semigroup $\{T(s); s \geq 0\}$ is called a C_0 -semigroup of contraction or non expansive operator, if it is of type $1 < \alpha < 0$ for all $\alpha \in \mathbb{R}$, and for each $s \geq 0$, we have

$$\|T(s)\|_{L(X)} \leq 1.$$

Definition 2.4 (Differential operator) [8]

A differential operator is an operator defined as a function of the differentiation operator.

Example 1

Consider the 3×3 matrix $[M_m(\mathbb{C})]$, and for each $\beta > 0$ such that $\beta \in \rho(A)$, where $\rho(A)$ is a resolvent set on X . Suppose

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$

and assume $T(t) = e^{tA_\beta}$, then

$$e^{tA_\beta} = \begin{pmatrix} e^{3t\beta} & e^{2t\beta} & e^{t\beta} \\ e^{2t\beta} & e^{2t\beta} & e^{t\beta} \\ e^{3t\beta} & e^{2t\beta} & e^{2t\beta} \end{pmatrix}.$$

Example 2

In the $H^{-1}(\Omega)$ setting, assume Ω be a nonempty and open subset in $\mathbb{R}^n$, let $X = H^{-1}(\Omega)$, and suppose we

define $A : D(A) \subseteq X \rightarrow X$ by

$$\begin{cases} D(A) = H_0^1(\Omega) \\ Av = \Delta v, \end{cases} \quad (2.1)$$

for each $v \in D(A)$ and $A \in \omega - ORCP_n$. It follows that $H_0^1(\Omega)$ is equipped with the usual norm on $H^{-1}(\Omega)$ defined by

$$\|v\|_{H^1(\Omega)} = (\|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2)^{\frac{1}{2}}.$$

Example 3

In the $L^2(\Omega)$ setting, suppose Ω be a nonempty and open subset in $\mathbb{R}^n$ and assume $X = L^2(\Omega)$. Consider the operator A on X , defined by

$$\begin{cases} D(A) = \{x \in H_0^1(\Omega); \Delta v \in L^2(\Omega)\} \\ Av = \Delta v, \end{cases} \quad (2.2)$$

for each $x \in D(A)$ and $A \in \omega - ORCP_n$.

Theorem 2.1

Suppose Ω is a nonempty, open and bounded subset in $\mathbb{R}^n$ whose boundary is of class C^1 , $r \in \mathbb{N}$ and $p, q \in [1, +\infty)$. Then,

- i. if $rp < n$ and $q < \frac{np}{n-rp}$, we have that $W^{r,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$;
- ii. if $rp = n$ and $q \in [1, +\infty)$ is compactly imbedded in $L^q(\Omega)$; and
- iii. if $rp > n$, then $W^{r,p}(\Omega)$ is compactly imbedded in $C(\overline{\Omega})$.

Theorem 2.2

Assume H is a Hilbert space and $\{A, D(A)\}$ a densely defined operator. Then we have,

- i. if $(I - A)^{-1} \in \mathcal{L}(H)$, then A is self-adjoint if and only if A is symmetric; and
- ii. if $(I \pm A)^{-1} \in \mathcal{L}(H)$, then A is skew - adjoint if and only if A is skew - symmetric.

Theorem 2.3

For any $\beta > 0$ and $f \in H^{-1}(\Omega)$, the equation $\beta v - \Delta v = f$ has a unique solution $v \in H_{0(\Omega)}^1$.

Theorem 2.4

Suppose Ω is a nonempty open and bounded subset in $\mathbb{R}^n$ whose boundary Γ is of class C^1 . Then $\|\cdot\| : H^1(\Omega) \rightarrow \mathbb{R}_+$ defined by

$$\|v\| = (\|\nabla v\|_{L^2(\Omega)}^2 + \|v_\Gamma\|_{L^2(\Gamma)}^2)^{\frac{1}{2}}$$

for each $v \in H^1(\Omega)$, is a norm on $H^1(\Omega)$ and equivalent with the usual one. In particular, the restriction of this norm to $H_0^1(\Omega)$, i.e. $\|\cdot\| : H_0^1(\Omega) \rightarrow \mathbb{R}_+$ defined by

$$\|v\|_0 = \|\nabla v\|_{L^2(\Omega)},$$

for each $v \in H_0^1(\Omega)$, is a norm on $H_0^1(\Omega)$ (called the gradient norm) equivalent with the usual one. In respect with this norm the application $D : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$, defined by

$$\langle u, \Delta v \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} = \int_{\Omega} \nabla v \nabla u \, dw,$$

is a canonical isomorphism between $H_0^1(\Omega)$ and its dual H^{-1} . The restriction of this application to H^2 coincides with $-\Delta$, where Δ is the Laplace operator in the sense of distributions over $\Delta(\Omega)$.

Theorem 2.5

The application $I - \Delta : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is the canonical isomorphism between $H_0^1(\Omega)$, endowed with the

usual norm on $H^1(\Omega)$ and its dual $H^{-1}(\Omega)$, equipped with the usual dual norm. In addition, for each $v \in H_0^1$ and each $u \in L^2(\Omega)$, we have

$$\langle v, u \rangle_{L^2(\Omega)} = \langle v, u \rangle_{H_0^1(\Omega), H^{-1}(\Omega)}.$$

Theorem 2.6(Hille-Yoshida)[12]

A linear operator $\{A, D(A)\}$ is the infinitesimal generator for a C_0 -semigroup of contraction if and only if

- i. A is densely defined and closed; and
- ii. $(0, +\infty) \subseteq \rho(A)$ and for each $\beta > 0$, we have

$$\|R(\beta, A)\|_{L(X)} \leq \frac{1}{\beta}. \quad (2.3)$$

Theorem 2.7

Assume $\{A, D(A)\}$ is the infinitesimal generator of a C_0 -semigroup and let $\|\cdot\|_{D(A)} : D(A) \rightarrow \mathbb{R}_+$ and $|\cdot|_{D(A)} : D(A) \rightarrow \mathbb{R}_+$ be defined by $\|x\|_{D(A)} = \|x\| + \|Ax\|$, and respectively by $|x|_{D(A)} = \|x - Ax\|$, for each $x \in D(A)$. Then:

- i. $\|\cdot\|_{D(A)}$ is a norm on $D(A)$, called the graph norm, with respect to which $D(A)$ is a Banach space;
- ii. $D(A)$ endowed with the norm $\|\cdot\|_{D(A)}$ is continuously imbedded in X ;
- iii. $A \in L(D(A), X)$ where $D(A)$ is endowed with $\|\cdot\|_{D(A)}$;
- iv. $|\cdot|_{D(A)}$ is a norm on $D(A)$ equivalent with $\|\cdot\|_{D(A)}$;
- v. $I - A$ is an isometry from $(D(A), |\cdot|_{D(A)})$ to $(X, \|\cdot\|)$; and
- vi. for each $x \in D(A)$, $S(\cdot)x \in C[0, +\infty)$; $D(A) \cup C^1([0, +\infty); X)^1$.

3. Main Results

This section presents results of ω -ORCP_n on Laplace operator with respect to the Dirichlet boundary condition by generating a C_0 -semigroup of contractions:

Theorem 3.1

The operator $A \in \omega$ -ORCP_n defined by

$$\|v\|_{H^1(\Omega)} = (\|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2)^{\frac{1}{2}}.$$

is the generator of a C_0 -semigroup of contractions. In addition, A is self-adjoint and $\|\cdot\|_{D(A)}$ is equivalent with the norm of the space $H^{-1}(\Omega)$.

Proof:

By virtue of Theorem 2.5, we know that $I - \Delta$ is the canonical isomorphism between $H_0^1(\Omega)$, endowed with usual norm of $H^1(\Omega)$, and its dual $H^{-1}(\Omega)$. Let us denote that $F = (I - \Delta)^{-1}$ is an isometry joining $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Consequently

$$\langle v, u \rangle_{H^{-1}(\Omega)} = \langle Fv, Fu \rangle_{H_0^1(\Omega)} \quad (3.1)$$

for each $u, v \in H^{-1}(\Omega)$. Let $u, v \in H_0^1(\Omega)$, then we have

$$\begin{aligned} \langle v, Fu \rangle_{H_0^1(\Omega)} &= \int_{\Omega} \nabla v \nabla (Fu) dw + \int_{\Omega} u Fv dw \\ &= \int_{\Omega} v(-\Delta(Fu)) dw + \int_{\Omega} v Fudw \\ &= \int_{\Omega} v(I - \Delta)F(u) dw = \langle v, u \rangle_{L^2(\Omega)}. \end{aligned} \quad (3.2)$$

From (3.1), taking into account that $F(I - \Delta) = I$, we deduce

$$\begin{aligned} \langle -\Delta v, u \rangle_{H^{-1}(\Omega)} &= \langle v - \Delta v, u \rangle_{H^{-1}(\Omega)} - \langle v, u \rangle_{H^{-1}(\Omega)} \\ &= \langle F(v - \Delta v), Fu \rangle_{H_0^1(\Omega)} - \langle v, u \rangle_{H^{-1}(\Omega)} \\ &= \langle v, Fv \rangle_{H^{-1}(\Omega)} - \langle v, u \rangle_{H^{-1}(\Omega)}. \end{aligned}$$

From (3.2), we have

$$\langle \Delta v, u \rangle_{H^{-1}(\Omega)} = \langle v, u \rangle_{H^{-1}(\Omega)} - \langle v, u \rangle_{L^2(\Omega)}. \quad (3.3)$$

Therefore A is symmetric. But $(I - A)^{-1} \in \mathcal{L}(H^{-1}(\Omega))$, and therefore by Theorem 2.2, it follows that A is self-adjoint. Taking $u = v$ in (3.3), we obtain

$$\langle Av, v \rangle_{H^{-1}(\Omega)} = \|v\|_{H^{-1}(\Omega)}^2 - \|v\|_{L^2(\Omega)}^2 \leq 0. \quad (3.4)$$

Theorem 2.3 shows that, for $\beta > 0$, we have $(\beta I - A)^{-1} \in \mathcal{L}(H^{-1}(\Omega))$, while (3.4) implies that, for $\beta > 0$, we have

$$\langle \lambda v - Av, v \rangle_{H^{-1}(\Omega)} \geq \lambda \|v\|_{H^{-1}(\Omega)}^2.$$

Hence $\|R(\beta; A)\|_{\mathcal{L}(H^{-1}(\Omega))} \leq \frac{1}{\beta}$. Since $H_0^1(\Omega)$ is dense in $H^{-1}(\Omega)$, we are in the hypothesis of Theorem 2.6, from where it follows that A generates a C_0 -semigroup of contractions on $H^{-1}(\Omega)$. Finally by (iv) in Theorem 2.7 and (3.4), it follows that $\|\cdot\|_{D(A)}$ is equivalent with the norm of the space H^Ω and this complete the proof.

Theorem 3.2

The linear operator $A \in \omega - ORCP_n$ defined by

$$\begin{cases} D(A) = \{v \in H_0^1(\Omega); \Delta v \in L^2(\Omega)\} \\ Av = \Delta v, \end{cases} \quad (3.5)$$

for each $v \in D(A)$ is the infinitesimal generator of a C_0 -semigroup of contractions. Moreover, A is self-adjoint and $(D(A), \|\cdot\|_{D(A)})$ is continuously included in $H_0^1(\Omega)$. Suppose Ω is bounded with C^1 boundary, then $(D(A), \|\cdot\|_{D(A)})$ is compactly imbedded in $L^2(\Omega)$.

Proof:

Assume $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, and $C_0^\infty(\Omega) \subseteq D(A)$, it follows that A is densely defined. Let $\lambda > 0$ and $f \in L^2(\Omega)$. Since $L^2(\Omega)$ is continuously imbedded in $H^{-1}(\Omega)$, and $-\Delta : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is the duality mapping with respect to the gradient norm on $H_0^1(\Omega)$, we have

$$\langle Av, u \rangle_{L^2(\Omega)} = \langle \nabla v, \nabla u \rangle_{L^2(\Omega)} = \langle u, \Delta v \rangle_{H_0^1(\Omega), H^1(\Omega)}. \quad (3.6)$$

By Theorem 3.1, we know that for any $\lambda > 0$ and $f \in L^2(\Omega)$ (notice that $L^2(\Omega) \subset H^{-1}(\Omega)$), the equation

$$\lambda v - \Delta v = f \quad (3.7)$$

has a unique solution $v_\lambda \in H_0^1(\Omega) \subset L^2(\Omega)$. So, $\Delta v_\lambda = \lambda v_\lambda - f$ is in $L^2(\Omega)$, which shows that $v_\lambda \in D(A)$ and $\lambda v_\lambda - Av_\lambda = f$. Taking the L^2 inner product on both sides of (3.7) above by v_λ and taking into account that by (3.6), we have $\langle Av, v \rangle_{L^2(\Omega)} \leq 0$ for each $v \in D(A)$, then we deduce that

$$\lambda \|v_\lambda\|_{L^2(\Omega)}^2 \leq \langle f, v_\lambda \rangle_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|v_\lambda\|_{L^2(\Omega)},$$

which shows that $\|R(\lambda; A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}$. Finally from (3.6) and Theorem 2.2, it follows that A is self-adjoint. Considering both inclusions, then $D(A) \subset H_0^1 \subset L^2(\Omega)$ are continuous, and the latter is compact whenever Ω is bounded by Theorem 2.1. Hence the proof is achieved.

Theorem 3.3

Let $A \in \omega - ORCP_n$ be the Laplace operator with the Dirichlet boundary condition in $H^{-1}(\Omega)$, let $\lambda > 0$ and $1 \leq p < +\infty$. Then:

(1.) There exists a unique $\mathcal{R}_\lambda \in \mathcal{L}(L^p(\Omega))$ so that $\mathcal{R}_\lambda u = R(\lambda; A)u$ for all $u \in H^{-1}(\Omega) \cap L^p(\Omega)$ and $\mathcal{R}_\lambda$ satisfies:

- i. $\|\mathcal{R}_\lambda u\|_{L^p(\Omega)} \leq \frac{1}{\lambda} \|u\|_{L^p(\Omega)}$;
- ii. for each $f \in L^p(\Omega)$, $A\mathcal{R}_\lambda f \in L^p(\Omega)$ and $\lambda\mathcal{R}_\lambda f - A\mathcal{R}_\lambda f = f$; and
- iii. for each $\lambda > 0$ and $\mu > 0$, $\mathcal{R}_\lambda(L^p(\Omega)) = \mathcal{R}_\mu(L^p(\Omega))$.

(2.) Let $\mathcal{R}_1 \in \mathcal{L}(L^p(\Omega))$ for each $u \in \mathcal{R}(L^p(\Omega))$, we have $\Delta u \in L^p(\Omega)$, and the operator $A : D(A) \subseteq L^p(\Omega) \rightarrow L^p(\Omega)$, defined by

$$\begin{cases} D(A) = \mathcal{R}_1(L^p(\Omega)) \\ Au = \Delta u \text{ for } u \in D(A), \end{cases}$$

is the generator of a C_0 -semigroup of contractions.

Proof:

Since $H^{-1}(\Omega) \cap L^p(\Omega)$ is dense in $L^p(\Omega)$, then it follows that $R(\lambda; A)$ has a unique extension $\mathcal{R}_\lambda \in \mathcal{L}(L^p(\Omega))$ satisfying (i). Next, let $(f_k)_{k \in \mathbb{N}}$ be a sequence in $D(\Omega)$ convergent to f in $L^p(\Omega)$. As $\mathcal{R}_\lambda f_k - \lambda A\mathcal{R}_\lambda f_k = f_k$ in $H^{-1}(\Omega)$, we have $\mathcal{R}_\lambda f - \lambda A\mathcal{R}_\lambda f = f$ in $D^1(\Omega)$, from there we get (ii). Finally, let $f \in H^{-1}(\Omega) \cap L^p(\Omega)$, and $u = \mathcal{R}_\lambda f \in H^{-1}(\Omega) \cap L^p(\Omega)$. For each $\mu > 0$, we have

$$\mu u - \Delta u = f + (\mu - \lambda)\mathcal{R}_\lambda f \quad (3.8)$$

Let us denote by g the right-hand side of (3.8), i.e.

$$g = f + (\mu - \lambda)\mathcal{R}_\lambda f$$

and let us observe that $\mathcal{R}_\lambda f = u = \mathcal{R}_\mu g \in H^{-1}(\Omega) \cap L^p(\Omega)$ and therefore

$$\mathcal{R}_\lambda(H^{-1}(\Omega) \cap L^p(\Omega)) \subseteq \mathcal{R}_\mu(H^{-1}(\Omega) \cap L^p(\Omega)).$$

Analogously

$$\mathcal{R}_\mu(H^{-1}(\Omega) \cap L^p(\Omega)) \subseteq (\mathcal{R}_\lambda(H^{-1}(\Omega) \cap L^p(\Omega))),$$

and so

$$\mathcal{R}_\lambda(H^{-1}(\Omega) \cap L^p(\Omega)) = (\mathcal{R}_\mu(H^{-1}(\Omega) \cap L^p(\Omega))).$$

Since $H^{-1}(\Omega) \cap L^p(\Omega)$ is dense in $L^p(\Omega)$, and $\mathcal{R}_\lambda, \mathcal{R}_\mu$ are linear and continuous operators in $L^p(\Omega)$, then we deduce (iii). And this complete the proof of (1). To prove (2), for each $u \in \mathcal{R}_1(L^p(\Omega))$ and $A \in \omega - ORCP_n$, we have $\Delta u \in L^p(\Omega)$, follows from (ii) in (1) above. So let $u \in D(A)$, $\lambda > 0$, $A \in \omega - ORCP_n$ and denote that $f = \lambda u - \Delta u$. From (iii) in (1), there exists $g \in L^p(\Omega)$ such that $u = \mathcal{R}_\lambda g$. We then conclude that $g = \lambda u - \Delta u$ and so $f = g$. Then $\lambda \in \rho(A)$ and $R(\lambda; A) = (\lambda I - A)^{-1} = \mathcal{R}_\lambda$. This relation in (i) from (1) above show that $\|R(\lambda; A)\|_{L^p(\Omega)} \leq \frac{1}{\lambda} \|u\|_{L^p(\Omega)}$. Thus A satisfies (ii) in Theorem 2.6. To complete the proof, we have to merely to show that $D(A)$ is dense in $L^p(\Omega)$. To this aim, let $u \in D(\Omega)$ and $f = u - \Delta u \in D(\Omega)$. Obviously $u = \mathcal{R}_1 f$ and therefore $D(\Omega) \subseteq D(A)$. Hence $D(A)$ is dense in $L^p(\Omega)$ which complete the proof.

Theorem 3.4

Let Ω be a nonempty and open subset in $\mathbb{R}^n$ with C^1 boundary Γ , let $X = [H^{-1}(\Omega)]^*$ then:

(i.) operator $A : D(A) \subseteq X \rightarrow X$, defined by

$$\begin{cases} D(A) = H^1(\Omega) \\ \langle Au, v \rangle_{H^1(\Omega), [H^1(\Omega)]^*} = \langle \nabla u, \nabla v \rangle_{L^2(\Omega)} \end{cases}$$

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for each $u, v \in H^1(\Omega)$ and $A \in \omega - ORCP_n$ is the generator of a C_0 -semigroup of contraction on X ; and
(ii.) the operator $\{B, D(B)\}$, defined by

$$\begin{cases} D(B) = \{u \in H^2(\Omega); u_v = 0 \text{ on } \Gamma \\ Bu = \Delta, \text{ for } u \in D(B) \end{cases}$$

is the generator of a C_0 -semigroup of contraction on X .

Proof:

Since $H^1(\Omega)$ is densely imbedded in $[H^1(\Omega)]^*$, in view of Theorem 2.6, we have merely to show that for each $\lambda > 0$, the operator $\lambda I - A : D(A) \subseteq X \rightarrow X$, where A is defined as above is one to one onto and

$$\|(\lambda I - A)^{-1}\|_{L(X)} \leq \frac{1}{\lambda}. \quad (3.9)$$

But this simply follows from the obvious identity

$$\langle \lambda u - Au, u \rangle_{[H^1(\Omega)]^*, H^1(\Omega)} = \lambda \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2$$

and this achieves the proof of (i). To prove (ii), let $u \in D(B)$. Then, for each $v \in H^1(\Omega)$, we have

$$\langle Au, v \rangle_{H^1(\Omega), [H^1(\Omega)]^*} = \langle \nabla u, \nabla v \rangle_{L^2(\Omega)} = \langle \Delta u, v \rangle_{L^2(\Omega)}$$

and thus, $Au = Bu$ for each $u \in D(B)$ and $A, B \in \omega - ORCP_n$. In addition

$$\langle Bu, v \rangle_{L^2(\Omega)} = - \langle \nabla u, \nabla v \rangle_{L^2(\Omega)}$$

for each $u, v \in D(B)$ and $B \in \omega - ORCP_n$. Thus B is symmetric and for each $\lambda > 0$, $\lambda I - B$ is bijective from $D(B)$ to $L^2(\Omega)$ and

$$\|(\lambda I - B)^{-1}\|_{L(X)} \leq \frac{1}{\lambda}.$$

If $D(B)$ is dense in $X = L^2(\Omega)$, then we are in the hypothesis of the Theorem 2.6 and this complete the proof.

4. Conclusion

This paper have established that $\omega - ORCP_n$ generates a C_0 -semigroup of contractions which was obtained by a Laplace operator with Dirichlet boundary condition.

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